A NORMALIZATION THEOREM IN ASYMPTOTIC DIFFERENTIAL ALGEBRA

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ABSTRACT. We define the universal exponential extension of an algebraically closed differential field and investigate its properties in the presence of a nice valuation and in connection with linear differential equations. Next we prove normalization theorems for algebraic differential equations over H-fields, as a tool in solving such equations in suitable extensions. The results in this monograph are essential in our work on Hardy fields in [6].

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[The] field that is normally classified as algebra really consists of two quite separate fields. Let us call them algebra one and algebra two, for lack of a better language. Algebra one is the algebra whose bottom lines are algebraic geometry or algebraic number theory. Algebra one has by far a better pedigree than algebra two, and has reached a high degree of sophistication and breadth. [...] Algebra two has had a more accidented history. It can be traced back to George Boole, who was the initiator of three well-known branches of algebra two, namely: in the first place, Boolean algebra, in the second place, the operational calculus that views the derivative as an operator D, on which Boole wrote two books of great beauty, and finally, invariant theory [...] G. H. Hardy subtly condemned algebra two in England in the latter half of the nineteenth century, with the exclamation 'Too much $f(D)$!' G. H. Hardy must be turning in his grave now.

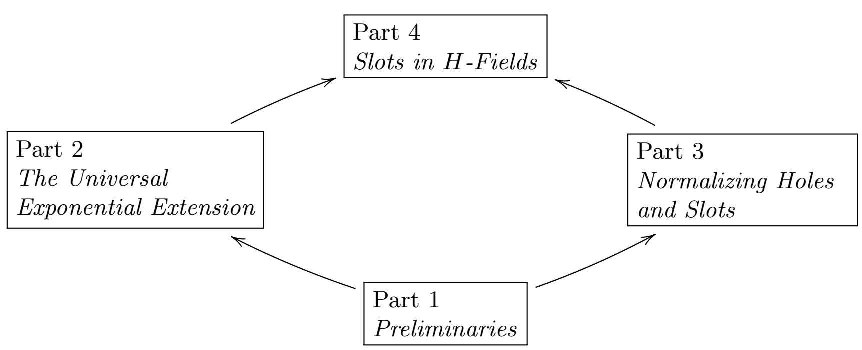
— Gian-Carlo Rota, *Combinatorics, representation theory and invariant theory: the story of a ménage à trois* [26].

Introduction

This monograph follows up on our book [ADH]. That book contains a model-theoretic analysis of the ordered differential field $\mathbb{T}$ of transseries (introduced by Écalle [12] in connection with his work on Dulac's Problem), including an explicit complete axiomatization and a quantifier elimination.

In 2021 we settled one of the main open problems left in [ADH] (see also [4]), by proving that all maximal Hardy fields are elementarily equivalent, as ordered differential fields, to $\mathbb{T}$. Here, a *Hardy field* is a differential field of germs at $+\infty$ of differentiable one-variable real-valued functions defined on intervals $(a, +\infty)$. Hardy fields were introduced by Bourbaki [10], revived in the 1980s by Boshernitzan and Rosenlicht [9, 25], and became important in the study of o-minimal expansions of the real field [18]. A Hardy field is *maximal* if it has no proper Hardy field extension. By Zorn, every Hardy field extends to a maximal one. (There are in fact very many maximal Hardy fields.)

The full solution of this problem takes a lot of space and is in the unwieldy manuscript [5]. In the present monograph we have extracted the main differential-algebraic (that is, non-analytic) parts, because we intend to use this material also for other purposes. It has four parts, each including its own introduction, with the following dependencies:



The application to Hardy fields requires in addition a good dose of analysis and some model theory, and is presented in [6].

To explain the overall aim of this monograph, consider an arbitrary algebraic differential equation with transseries coefficients, for example

$$y^7 - \mathrm{e}^{\mathrm{e}^x}\, y''y^3 + 7\,\Gamma(x^x)\, y'y''' - \frac{\zeta(x)}{x + \log x + 1} \;=\; 0 \qquad (x > \mathbb{R}).$$

By the general theory from [ADH] this equation happens to have a solution $y \in \mathbb{T}$. (Indeed, this follows from the differential polynomial on the left-hand side of this equation having odd degree, see [ADH, pp. 17–18].) However, a concrete solution cannot be derived in a transparent and direct way from the equation. It is even less clear why there should also be a solution in a Hardy field.

Our main goal is to reduce algebraic differential equations such as the one above to normal forms that can be solved more easily and explicitly in various contexts. Some of this can be accomplished using existing tools from [ADH], but for [6] we need more precision. After laying the groundwork in Parts 1, 2, 3 we prove the main results of this monograph in Part 4. Some ideas from Parts 3 and 4 also occur in [16] in the more restricted context of grid-based transseries solutions to algebraic differential equations over $\mathbb{R}$. However, several shortcuts apply in this case that are not available for [6].

In the rest of this introduction we illustrate some key ideas informally by a few examples, before stating the main result in detail. We make frequent reference to [ADH]; the section *Concepts and Results from* [ADH] at the end of this introduction should help in consulting this source.

Solving Quasilinear Equations

Our book [ADH] contains a differential analogue of the Newton diagram method to reduce arbitrary univariate algebraic differential equations with asymptotic side conditions (*asymptotic equations*, for short) over certain valued differential fields such as $\mathbb{T}$ to quasilinear equations, which can be considered as a first, crude kind of normal form for asymptotic equations. An example of a *quasilinear* equation is

$$\text{(E)} \qquad y' - y \;=\; x^{-1}yy'' - \mathrm{e}^{-x}\,y^2 + x^{-1}, \qquad y \prec 1.$$

"Quasilinear" means that the *linear part* of the equation (that is, the homogeneous part of degree 1, placed here on the left-hand side) dominates the rest in a certain sense. It is straightforward to compute a formal transseries solution of a quasilinear equation such as (E) by transfinite recursion: we first determine its *dominant term,* in this case $f := -x^{-1}$, and then substitute $f + y$ for y, with asymptotic side condition $y \prec f$, in the equation. This yields a new quasilinear equation

$$\text{(E}_{+f}\text{)} \quad x^{-2}\,y'' + y' + (-1 + 2x^{-4} - 2x^{-1}\,\mathrm{e}^{-x})\,y \;= \\ x^{-1}\,y\,y'' - \mathrm{e}^{-x}\,y^2 - x^{-2} + 2x^{-5} - x^{-2}\,\mathrm{e}^{-x}, \qquad y \prec x^{-1}.$$

Such an asymptotic equation obtained by an additive change of variables together with the imposition of a possibly stricter asymptotic constraint is called a *refinement* of (E). Applying this method recursively, we obtain a particular solution

$$y_* \;:=\; -x^{-1} + x^{-2} - 2x^{-3} + 6x^{-4} - 26x^{-5} + \cdots + x^{-2}\,\mathrm{e}^{-x} - 2x^{-3}\,\mathrm{e}^{-x} + \cdots$$

to (E). This process is really transfinite: computing the first ω terms of y_* gives

$$f \;:=\; -x^{-1} + x^{-2} - 2x^{-3} + 6x^{-4} - 26x^{-5} + \cdots$$

and then we refine the original (E) by substituting $f + y$ for y, with the new side condition $y \prec x^{\mathbb{N}}$. Likewise for other limit ordinals. In order to obtain the general solution to (E), we refine (E) by substituting $y_* + y$ for y, with asymptotic side condition $y \prec 1$, which yields

$$\text{(E}_*\text{)} \quad (x^{-2} - x^{-3} + 2x^{-4} - 6x^{-5} + \cdots)y'' + y' + (-1 + 2x^{-4} - 6x^{-5} + \cdots)y \;= \\ x^{-1}\,yy'' - \mathrm{e}^{-x}\,y^2, \qquad y \prec 1.$$

The first term of a non-zero solution to this asymptotic equation is necessarily the first term of a zero of the linear part of this equation. Further computations show that such a dominant term is of the form $c\,\mathfrak{m}$ with $c \in \mathbb{R}^\times$ and $\mathfrak{m} := x^{-5}\,\mathrm{e}^{-x^3/3 - x^2/2}$. It follows that the general solution of (E) has the form

$$y_c \;=\; y_* + c\,\mathfrak{m} + \varepsilon_c \qquad \text{where } c \in \mathbb{R} \text{ and } \varepsilon_c \prec \mathfrak{m}.$$

However, this form of the general solution cannot be read off directly from the original equation (E), and not even from (E$_{+f}$). This is due to the fact that the shape of the linear part of the equation may drastically change under refinement.

Only after the computation of the first three terms of y_* do we obtain a refinement that is sufficiently similar to (E_*) to allow us to safely determine the form of the general solution. Here we note that a particular solution y_* is generally not yet available during the resolution process. This prevents us from working directly with (E_*). We thus aim to always operate with sufficiently good approximations to y_* and to (E_*) instead.

Things become even worse if we introduce a parameter ε to perturb (E) to

$$(\mathrm{E}_\varepsilon) \qquad y' - y \;=\; x^{-1}\, yy'' - \mathrm{e}^{-x}\, y^2 + \varepsilon, \qquad y \prec 1,$$

This equation is quasilinear for every $\varepsilon \prec 1$. For $\varepsilon = x^{-1}$ it has continuum many solutions in $\mathbb{T}$, as explained above. However, (E_ε) has only one solution in $\mathbb{T}$ for $\varepsilon = -x^{-1}$. Indeed, in the latter case a particular solution is given by

$$y_{**} \;=\; x^{-1} - x^{-2} + 2x^{-3} - 6x^{-4} + 22x^{-5} + \cdots + x^{-2}\,\mathrm{e}^{-x} - 2x^{-3}\,\mathrm{e}^{-x} + \cdots,$$

and replacing y by $y_{**} + y$ in (E_ε) leads to the following counterpart of (E_*):

$$\begin{aligned}(-x^{-2} + x^{-3} - 2x^{-4} + 6x^{-5} - 26x^{-6} + \cdots)y'' + y' + (-1 - 2x^{-4} + 6x^{-5} + \cdots)y \;=&\\ x^{-1}yy'' - \mathrm{e}^{-x}\, y^2, \qquad y \prec 1.&\end{aligned}$$

One can show that $y = 0$ is the only solution to this refinement of (E_ε), again by examining its linear part. That *the general solution of a quasilinear equation is not transparent from the equation* and that *the resolution process is not uniform under small perturbations* are two major reasons why it is desirable to normalize asymptotic equations beyond quasilinearity.

The Role of Factoring Linear Differential Operators

Here is an example of the kind of nicely normalized equation that we are after:

$$y' \;=\; \mathrm{e}^{-x} + x^{-1}y^2, \qquad y \prec 1.$$

This equation can formally be solved by iterated integration:

$$\begin{aligned} y \;&=\; \textstyle\int \mathrm{e}^{-x} + \int x^{-1}y^2 \\ &=\; \textstyle\int \mathrm{e}^{-x} + \int x^{-1}\big(\int \mathrm{e}^{-x} + \int x^{-1}y^2\big)^2 \\ &=\; \cdots \\ &=\; \textstyle\int \mathrm{e}^{-x} + \int x^{-1}\left(\int \mathrm{e}^{-x}\right)^2 + 2\int x^{-1}\left(\int \mathrm{e}^{-x}\right)\left(\int x^{-1}\left(\int \mathrm{e}^{-x}\right)^2\right) + \cdots. \end{aligned}$$

It turns out that this expansion converges in the formal transseries setting, but also analytically to a germ that belongs to a Hardy field. In the formal setting, this requires so-called *distinguished integration*, where all integration constants are taken to be zero. To obtain a Hardy field solution, we systematically integrate from $+\infty$. For details we refer respectively to [15, Section 6.5] and [6].

Let us consider more generally a quasilinear equation

$$(\mathrm{Q}) \qquad L(y) \;=\; R(y), \qquad y \prec 1,$$

where $L \in \mathbb{T}[\partial]^{\neq}$ is a monic linear differential operator and $R \in \mathbb{T}\{Y\}$ is a differential polynomial. (For greater flexibility, R is allowed to have a non-zero linear part.) First of all, this requires the (distinguished) integration operator $\partial^{-1} := \int$ to be replaced by a more general right inverse L^{-1} to the $\mathbb{R}$-linear map $y \mapsto L(y)\colon \mathbb{T} \to \mathbb{T}$

defined by L. For $L = \partial - f$ ($f \in \mathbb{T}$) of order 1, this is easy: identifying each $g \in \mathbb{T}$ with the $\mathbb{R}$-linear map $y \mapsto gy\colon \mathbb{T} \to \mathbb{T}$, we may take

$$L^{-1} \;=\; (\partial - f)^{-1} \;:=\; \mathrm{e}^{\int f} \circ \partial^{-1} \circ \mathrm{e}^{-\int f}.$$

More generally, if L splits over $\mathbb{T}$, that is, if

$$L \;=\; (\partial - f_1)\cdots(\partial - f_r) \qquad \text{where } f_1,\dots,f_r \in \mathbb{T},$$

then we may take

$$L^{-1} \;:=\; (\partial - f_r)^{-1} \circ \cdots \circ (\partial - f_1)^{-1}.$$

Now L might not split over $\mathbb{T}$, but it does factor into a product of order 1 and order 2 operators over $\mathbb{T}$. And if $\partial^2 + a\partial + b \in \mathbb{T}[\partial]$ ($a, b \in \mathbb{T}$) is irreducible, it splits over the complexification $\mathbb{T}[i]$ of $\mathbb{T}$: there are $f, g \in \mathbb{T}$ with $g \neq 0$ and

$$\begin{aligned}
\partial^2 + a\partial + b \;&=\; \big(\partial - (f - gi + g^\dagger)\big)\big(\partial - (f + gi)\big),\\
a \;&=\; -(2f + g^\dagger),\\
b \;&=\; f^2 + g^2 - f' + fg^\dagger,
\end{aligned}$$

so that we can formally invert this second-order operator as

$$\begin{aligned}
(\partial^2 + a\partial + b)^{-1} \;&:=\; \big(\mathrm{e}^{\int (f+gi)} \circ \partial^{-1}\, \mathrm{e}^{-\int (f+gi)}\big) \circ \big(\mathrm{e}^{\int (f-gi)}\, g \circ \partial^{-1} \circ \mathrm{e}^{-\int (f-gi)}\, g^{-1}\big)\\
&=\; \mathrm{e}^{\int (f+gi)} \circ \partial^{-1} \circ \mathrm{e}^{-2\int gi}\, g \circ \mathrm{e}^{-\int (f-gi)}\, g^{-1}.
\end{aligned}$$

This discussion is to suggest that factoring linear differential operators plays a key role in our normalization program, and that it will involve transseries in $\mathbb{T}[i]$, and even *oscillatory transseries* in $\mathbb{T}[i][\mathrm{e}^{\mathbb{T}i}]$ when inverting such operators.

The framework of asymptotic differential algebra in [ADH] was introduced in anticipation of this kind of developments, but the Hardy fields in [6] require a very detailed and explicit treatment. Part 2 of this monograph is dedicated to this task in the abstract setting of H-fields as in [ADH], while relying on Part 1 for miscellaneous preliminary material. (An *H-field* is an ordered valued differential field subject to certain first-order laws. Ordered differential subfields of $\mathbb{T}$ extending $\mathbb{R}$ as well as Hardy field extensions of $\mathbb{R}$ are H-fields.)

In Part 2, we introduce the *universal exponential extension* of an algebraically closed differential field and investigate its connection to linear differential equations, especially in the presence of a compatible valuation on the field. The universal exponential extension of $\mathbb{T}[i]$ can be identified with $\mathbb{T}[i][\mathrm{e}^{i\mathbb{T}}]$: see Section 2.4 as well as [15, Sections 7.7 and 7.8] for this and the connection with factorization of linear differential operators over $\mathbb{T}[i]$. For a Liouville closed Hardy field $H \supseteq \mathbb{R}$ one can identify the universal exponential extension of its algebraic closure $K = H[i]$ with $K[\mathrm{e}^{iH}]$, an integral domain of germs of $\mathbb{C}$-valued functions. This is a key point in [6] and is proved there. (An H-field H is said to be *Liouville closed* if H is real closed and for all $f, g \in H$, there exists $y \in H^\times$ with $y' + fy = g$.)

Desiderata for a Normal Form

Returning to the quasilinear equation (Q), assume now that the coefficients of L and R lie in an arbitrary H-field H (instead of $\mathbb{T}$). Ideally, when should we consider this equation to be in normal form? Three natural requirements are:

(R1) *L does not significantly change under refinement.*

(R2) *L splits over $K := H[i]$.*

(R3) *The equation* (Q) *has a solution which is as unique as possible.*

The first requirement amounts to L *strongly dominating* R. What *significantly* and *strongly dominating* mean here is made precise in terms of the *span* $\mathfrak{v} = \mathfrak{v}(L)$ of L, which measures how far L is from being "regular singular". (For example, the span of the operator on the left-hand side of (E_{+f}) is equal to the quotient of the coefficients of y'' and y, which is asymptotic to x^{-2}.) It turns out that (R1) holds if R is dominated by $\mathfrak{v}^m L$ for a large enough m, under some additional assumptions such as $\operatorname{order} R \leqslant \operatorname{order} L$ and $y \prec 1$ as the asymptotic side condition in (Q); more general side conditions $y \prec \mathfrak{m}$ where $\mathfrak{m} \in H^>$ can be reduced to this case via a multiplicative change of variables (replacing y by $y\mathfrak{m}$).

Concerning (R2), the condition on H that *all* monic linear operators in $H[\partial]^{\neq}$ split over K is very strong. Fortunately, for our main application in [6] and the particular operators L needed there, such factorizations come almost for free. To sketch this, consider an immediate (in the sense of valued fields) differentially algebraic H-field extension $\widehat{H}$ of H and an H-field embedding ι of H into a Hardy field. Our main goal in [6] is to extend ι to an embedding of $\widehat{H}$ into a Hardy field. Suppose $\widehat{y} \in \widehat{K} \setminus K$, where $\widehat{K} := \widehat{H}[i]$, and we want to extend ι to an embedding of $H\langle \operatorname{Re}\widehat{y}, \operatorname{Im}\widehat{y}\rangle \subseteq \widehat{H}$ into some Hardy field. Now the trick is to choose $\widehat{H}$ and $\widehat{y}$ such that the minimal annihilator $P \in K\{Y\}$ of $\widehat{y}$ over K is of *minimal complexity*. This means in particular that P is of minimal order over K, say r, and so K contains all zeros of differential polynomials over K of order $< r$, as long as these zeros live in an extension of K of the form $E[i]$ for some immediate H-field extension E of H. Consequently, all $A \in K[\partial]^{\neq}$ of order $\leqslant r$ split over K, since their associated Riccati polynomials are of order $< r$.

This minimality argument relies on working over K instead of H. The interplay between H and K is subtle: we need minimal elements to be taken in K, but actual extensions to be done on the level of H-fields. The length of this monograph is partly due to the fact that some of the material applies to differential-valued fields, and thus to both H and K, whereas other results need to be developed separately for H and K, often with minor though crucial differences.

In the present monograph we do not set ourselves the task to actually solve any asymptotic equations. Instead, we wish to prepare them as much as possible in the purely algebraic and abstract setting of H-fields. The resulting normalized asymptotic equations should then be easier to deal with in suitable contexts (Hardy fields, various kinds of transseries, surreal numbers, etc.).

On a technical level, this is implemented using the notion of a *hole* in H, that is, an asymptotic equation $P(y) = 0$, $y \prec \mathfrak{m}$ over H that comes with a solution $\widehat{y}$ in an immediate H-field extension $\widehat{H}$ of H, but outside H itself; notation: $(P, \mathfrak{m}, \widehat{y})$. For our purposes, this asymptotic equation can be arranged to be quasilinear, and even non-singular in a certain sense. (This corresponds to the notion of *deep* holes.) Therefore, roughly speaking, a hole in H is a witness that H is not (yet) newtonian. (In [ADH] we defined H to be *newtonian* if every quasilinear equation (Q) has a solution in H; this is really the most fundamental first-order property of $\mathbb{T}$, given by an axiom scheme.) We then try to modify such a hole $(P, \mathfrak{m}, \widehat{y})$ further to bring the associated asymptotic equation into a normal form $P(f + y) = 0$, $y \prec \mathfrak{n}$, for some choice of $f \in H$ and $\mathfrak{n} \preccurlyeq \mathfrak{m}$. This new equation should satisfy versions of (R1), (R2), (R3) above, and hence be more amenable to solving in the kind of extension we are really interested in, rather than in the abstractly given extension $\widehat{H}$

where a solution $\widehat{y}-f$ is already given. For example, if $H \supseteq \mathbb{R}$ is a Hardy field, we wish to solve this equation in an immediate Hardy field extension of H.

Turning now to the last requirement (R3), we saw that in $\mathbb{T}$, the equation (E_ε) has the unique solution y_{**} for $\varepsilon = -x^{-1}$ and an infinite family (y_c) of solutions if $\varepsilon = x^{-1}$. There are cases in which the existence of more than one solution is unavoidable. For instance, assume that we start with an H-field $H = \mathbb{R}(x, \mathrm{e}^x)$ with constant field $\mathbb{R}$ over which the equation (E) makes sense. Then H contains no element f with $f - y_* \prec x^{-\mathbb{N}}$. (This follows from the fact that the first ω terms of y_* form a divergent power series in x^{-1}, as can be checked using techniques from [11, 12, 14].) Now consider an immediate H-field extension $\widehat{H}$ of H that contains all solutions y_c. Each y_c realizes the same cut in the ordered set H and also satisfies the same algebraic differential equations over H. In model theoretic terms, this means that all y_c realize exactly the same quantifier-free 1-type over H (in the language of valued ordered differential fields). Therefore no quantifier-free first-order condition over H can distinguish between two distinct solutions and there is consequently no first-order normalization process over H for which the normal form would have a particular y_c as its unique solution in $\widehat{H}$. This kind of indistinguishability does not necessarily survive under H-field extensions. For instance, consider an H-field extension $H_1 := H\langle f, \mathrm{e}^{-x^3/3 - x^2/2}\rangle$ of H where $f \prec 1$ satisfies $f' - f = x^{-1} f f'' + x^{-1}$. Then for $c_1 \neq c_2$, the solutions y_{c_1} and y_{c_2} realize distinct quantifier-free 1-types over H_1.

In view of the above discussion, a more precise formulation of (R3) would be to require all solutions of (Q) to realize the same quantifier-free 1-type over H. In this monograph, we introduce an even more stringent requirement. The idea is to investigate closely how distinguishable and indistinguishable solutions arise. The linear part of the equation is again the key here, as we already saw when obtaining the general solution y_c from the linear part of (E_*). Under the assumption (R1), the asymptotic behavior of the non-zero solutions to the linear differential equation $L(h) = 0$ does not change under refinement. Now assume that $\widehat{y} \in \widehat{H} \setminus H$ satisfies (Q). Roughly speaking, (R3) holds if $\widehat{y}$ and $\widehat{y} + |h|$ realize the same cut in H, for all $h \prec 1$ satisfying $L(h) = 0$. Here h typically belongs to the universal exponential extension of $H[\mathrm{i}]$.

The Main Theorem

Before we state our main result we introduce a few more concepts and motivate some hypotheses of this theorem. Again, we only do this on an informal level; the precise definitions are given at the appropriate places in the monograph.

First, in light of (R2), our H-field H better be sufficiently rich from the outset, say Liouville closed and $\upomega$-free. Here, the property of $\upomega$-*freeness* is a certain first-order condition about linear differential equations of order 2, satisfied by $\mathbb{T}$, which plays a crucial role in [ADH]. With an eye towards factorizations over the complexification $K = H[\mathrm{i}]$ of H, we also assume that K is 1-*linearly newtonian*: each quasilinear equation $y' + fy = g$ $(f, g \in K)$ has a solution $y \prec 1$ in K. Minor assumptions are that the constant field of H is archimedean and the derivation of H is *small* in the sense that $h \prec 1 \Rightarrow h' \prec 1$ for all $h \in H$. Both conditions hold for $H = \mathbb{T}$ and any Hardy field.

Since our aim is to solve quasilinear equations that do not already have a solution in H, we will further assume that H is *not* newtonian. This hypothesis yields a quasilinear hole $(Q,1,\widehat{b})$ in H, that is, a quasilinear equation $Q(y)=0$, $y\prec 1$ and a solution $\widehat{b}\notin H$ thereof in some immediate extension of H. We call such a hole *minimal* if Q has minimal complexity (among all holes in H), and we say that it is *special* if $\widehat{b}$ is a pseudo-limit of a pseudo-cauchy sequence in H exhibiting a power-type rate of pseudo-convergence.

One important feature of the general theory of H-fields from [ADH] is that many important asymptotic properties are *eventual*. This means that they hold upon replacing the derivation ∂ of H by a small derivation $\phi^{-1}\partial$ with sufficiently small $\phi\in H$, $\phi>0$. This corresponds to replacing the underlying "time" variable x by faster times, like e^x or $\mathrm{e}^{\mathrm{e}^x}$. We say that $\phi\in H$ with $\phi>0$ is *active* in H if the derivation $\phi^{-1}\partial$ remains small. For such ϕ the ordered field H equipped with the derivation $\phi^{-1}\partial$ is again an H-field, denoted by H^ϕ, and rewriting a differential polynomial P over H in terms of the new derivation $\phi^{-1}\partial$, we obtain an equivalent differential polynomial P^ϕ over H^ϕ. Thus any hole $(P,\mathfrak{m},\widehat{y})$ in H naturally gives rise to a hole $(P^\phi,\mathfrak{m},\widehat{y})$ in H^ϕ.

The precise formalization of the requirements (R1), (R2), (R3) from above is done stepwise, via the introduction of progressively stronger normalization properties of holes. We actually work also with *slots*: a bit more general than holes and useful intermediate stages in the normalization of holes. Likewise, instead of minimal holes we often deal with somewhat more general Z-*minimal* holes and slots.

At the end of the day, the requirements (R1), (R2) and (R3) are formalized through the concepts of strongly repulsive-normal and ultimate holes. *Normality* corresponds to the requirement (R1). The notion *repulsive* takes care of (R2) and part of (R3); the terminology is motivated by analytic considerations in [6]. The qualifier *strongly* and the property *ultimate* indicate further contributions to (R3).

Let us now state our main result with all the necessary fine print:

Normalization Theorem. *Let H be an $\upomega$-free Liouville closed H-field with small derivation, archimedean ordered constant field C, and 1-linearly newtonian algebraic closure $H[\mathrm{i}]$. Suppose H is not newtonian. Then for some Z-minimal special hole $(Q,1,\widehat{b})$ in H with* $\operatorname{order} Q\geqslant 1$ *and some active $\phi>0$ in H with $\phi\preccurlyeq 1$, the hole $(Q^\phi,1,\widehat{b})$ in H^ϕ is deep, strongly repulsive-normal, and ultimate.*

The assumption on H that C is archimedean is not first-order in the logical sense; it can perhaps be dropped. (Corollary 4.3.41 characterizes newtonianity is a less sharp way, but without this hypothesis on C.) As already mentioned, this Normalization Theorem is an essential tool in [6]. The example (E_ε) also illustrates that stronger normal forms are mandatory for the uniform resolution of asymptotic equations depending on parameters. We expect this to play an important role for a deeper understanding of definable functions and better effective versions of our quantifier elimination result from [ADH].

Concepts and Results from [ADH]

This section includes notation and terminology used throughout this monograph. Thus m, n always range over the set $\mathbb{N}=\{0,1,2,\dots\}$ of natural numbers. Given an additively written abelian group A we set $A^{\neq}:=A\setminus\{0\}$. Rings (usually, but not always, commutative) are associative with identity 1. For a ring R we let $R^\times$ be the multiplicative group of units of R (consisting of the $a\in R$ such that $ab=ba=1$

for some $b \in R$). Let S be a totally ordered set and $A \subseteq S$. We set

$$A^{\downarrow} := \{s \in S : s \leqslant a \text{ for some } a \in A\}, \quad A^{\uparrow} := \{s \in S : s \geqslant a \text{ for some } a \in A\}$$

where $\leqslant$ is the ordering of S. We say that A is *downward closed* in S if $A = A^{\downarrow}$ and *upward closed* in S if $A = A^{\uparrow}$. For $a \in S$ we also let $S^{>a} := \{s \in S : s > a\}$. Ordered abelian groups and ordered fields are totally ordered, by convention. Let Γ be an ordered abelian group, written additively. Then $\Gamma^{>} := \Gamma^{>0}$, and likewise with $\geqslant$, $<$, or $\leqslant$ in place of $>$. The ordered divisible hull of Γ is denoted by $\mathbb{Q}\Gamma$.

Differential rings and fields. Let R be a *differential ring*, that is, a commutative ring R containing (an isomorphic copy of) $\mathbb{Q}$ as a subring and equipped with a derivation $\partial\colon R \to R$. When its derivation ∂ is clear from the context, then for $a \in R$ we denote $\partial(a), \partial^2(a), \dots, \partial^n(a), \dots$ by $a', a'', \dots, a^{(n)}, \dots$. If $a \in R^{\times}$, then $a^{\dagger} := a'/a$ denotes the *logarithmic derivative* of a, so $(ab)^{\dagger} = a^{\dagger} + b^{\dagger}$ for all $a, b \in R^{\times}$. We have a subring $C_R := \ker \partial$ of R, called the ring of constants of R, with $\mathbb{Q} \subseteq C_R$. A *differential field* is a differential ring K whose underlying ring is a field. In this case C_K is a subfield of K, and if K is understood from the context we often write C instead of C_K. Note that a differential field has characteristic 0.

Often we are given a differential field H in which -1 is not a square, and then $H[\mathrm{i}]$ is a differential field extension with $\mathrm{i}^2 = -1$: the derivation ∂ on H uniquely extends to a derivation on $H[\mathrm{i}]$, and this extension has i in its constant field. For $z = a + bi \in H[\mathrm{i}]$ $(a, b \in H)$ we set $\operatorname{Re} z := a$, $\operatorname{Im} z := b$, and $\overline{z} := a - bi$. Then $z \mapsto \overline{z}$ is an automorphism of the differential field $H[\mathrm{i}]$. If there is also given a differential field extension F of H in which -1 is not a square, we always tacitly arrange i to be such that $H[\mathrm{i}]$ is a differential subfield of the differential field extension $F[\mathrm{i}]$ of F.

Differential polynomials. Let R be a differential ring. We have the differential ring $R\{Y\} = R[Y, Y', Y'', \dots]$ of differential polynomials in a differential indeterminate Y over R. Let $P = P(Y) \in R\{Y\}$. The *order* of P, denoted by $\operatorname{order}(P)$, is the least $r \in \mathbb{N}$ such that $P \in R[Y, Y', \dots, Y^{(r)}]$. Let $\operatorname{order}(P) \leqslant r$. Then

$$P = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$$

with $\boldsymbol{i}$ ranging over tuples $(i_0, \dots, i_r) \in \mathbb{N}^{1+r}$, $Y^{\boldsymbol{i}} := Y^{i_0}(Y')^{i_1} \cdots (Y^{(r)})^{i_r}$, coefficients $P_{\boldsymbol{i}}$ in R, and $P_{\boldsymbol{i}} \neq 0$ for only finitely many $\boldsymbol{i}$. For such $\boldsymbol{i}$ we set

$$|\boldsymbol{i}| \;:=\; i_0 + i_1 + \cdots + i_r, \qquad \|\boldsymbol{i}\| \;:=\; i_1 + 2i_2 + \cdots + ri_r.$$

The *multiplicity of* P (at 0) is

$$\operatorname{mul} P \;:=\; \min\big\{|\boldsymbol{i}| : P_{\boldsymbol{i}} \neq 0\big\} \in \mathbb{N} \text{ if } P \neq 0, \qquad \operatorname{mul} P \;:=\; +\infty \text{ if } P = 0,$$

the *degree* of P is

$$\deg P \;:=\; \max\big\{|\boldsymbol{i}| : P_{\boldsymbol{i}} \neq 0\big\} \in \mathbb{N} \text{ if } P \neq 0, \qquad \deg P \;:=\; -\infty \text{ if } P = 0,$$

and the *weight* of P is

$$\operatorname{wt} P \;:=\; \max\big\{\|\boldsymbol{i}\| : P_{\boldsymbol{i}} \neq 0\big\} \in \mathbb{N} \text{ if } P \neq 0, \qquad \operatorname{wt} P \;:=\; -\infty \text{ if } P = 0.$$

For $d \in \mathbb{N}$ we set $P_d := \sum_{|\boldsymbol{i}|=d} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$ (the *homogeneous part* of degree d of P), so $P = \sum_d P_d$, and if $P \neq 0$, then

$$\operatorname{mul} P \;=\; \min\{d : P_d \neq 0\}, \qquad \deg P \;=\; \max\{d : P_d \neq 0\}.$$

For $a \in R$ we let

$$P_{+a} := P(a+Y) \quad \text{and} \quad P_{\times a} := P(aY)$$

be the *additive conjugate* and the *multiplicative conjugate* of P by a, respectively. For $\phi \in R^{\times}$ we let R^{ϕ} be the *compositional conjugate of R by ϕ*: the differential ring with the same underlying ring as R but with derivation $\phi^{-1}\partial$ (usually denoted by δ) instead of ∂. We have an R-algebra isomorphism $P \mapsto P^{\phi}\colon R\{Y\} \to R^{\phi}\{Y\}$ such that $P^{\phi}(y) = P(y)$ for all $y \in R$; see [ADH, 5.7].

A *differentially algebraic* (for short: d-*algebraic*) extension of a differential field K is a differential field extension L of K such that for all $y \in L$ we have $P(y) = 0$ for some differential polynomial $P \in K\{Y\}^{\neq}$. See [ADH, 4.1] for more on this.

Complexity and the separant. Let K be a differential field and $P \in K\{Y\} \setminus K$, and set $r = \operatorname{order} P$, $s = \deg_{Y^{(r)}} P$, and $t = \deg P$. Then the *complexity* of P is the triple $\mathrm{c}(P) = (r,s,t) \in \mathbb{N}^3$; we order $\mathbb{N}^3$ lexicographically. Let $a \in K$. Then $\mathrm{c}(P_{+a}) = \mathrm{c}(P)$, and $\mathrm{c}(P_{\times a}) = \mathrm{c}(P)$ if $a \neq 0$. The differential polynomial $S_P := \frac{\partial P}{\partial Y^{(r)}}$ is called the *separant* of P; thus $\mathrm{c}(S_P) < \mathrm{c}(P)$ (giving complexity $(0,0,0)$ to elements of K), and $S_{aP} = aS_P$ if $a \neq 0$. Moreover,

$$(0.1) \qquad S_{P_{+a}} = (S_P)_{+a}, \quad S_{P_{\times a}} = a \cdot (S_P)_{\times a}, \quad S_{P^{\phi}} = \phi^r (S_P)^{\phi} \text{ for } \phi \in K^{\times}.$$

(For $S_{P_{+a}}$ and $S_{P_{\times a}}$ this is from [ADH, p. 216]; for $S_{P^{\phi}}$, express P as a polynomial in $Y^{(r)}$ and use $(Y^{(r)})^{\phi} = \phi^r Y^{(r)} + \text{lower order terms}$.)

Linear differential operators. Let R be a differential ring. We associate to R the ring $R[\partial]$ of linear differential operators over R; see [ADH, 5.1]. This is the ring extension of R generated over R by an element ∂: we use here the same symbol that denotes the derivation of R, impose $\partial^m \neq \partial^n$ for all $m \neq n$, require $R[\partial]$ to be free as a left R-module with basis $1 = \partial^0, \partial = \partial^1, \partial^2, \partial^3, \dots$, and impose $\partial a = a\partial + a'$ (in $R[\partial]$) for $a \in R$. Each $A \in R[\partial]$ has accordingly the form

$$(0.2) \qquad A = a_0 + a_1\partial + \cdots + a_n\partial^n \qquad (a_0, \dots, a_n \in R),$$

and for such A and $y \in R$ we put

$$A(y) := a_0 y + a_1 y' + \cdots + a_n y^{(n)} \in R.$$

Then $(AB)(y) = A\big(B(y)\big)$ for all $A, B \in R[\partial]$ and $y \in R$. The kernel of $A \in R[\partial]$ is the C_R-submodule

$$\ker A = \big\{y \in R : A(y) = 0\big\}$$

of R. If we want to stress the dependence on R, we also write $\ker_R A$ for $\ker A$. For $A \in R[\partial]^{\neq}$ there are unique elements $a_0, \dots, a_n$ of R with $a_n \neq 0$ such that (0.2) holds. Then $\operatorname{order}(A) := n$ is the *order* of A, and we say that A is *monic* if $a_n = 1$. Let $u \in R^{\times}$. For $A \in R[\partial]$ we set $A_{\ltimes u} := u^{-1}Au \in R[\partial]$, the *twist* of A by u. If A is monic, then so is $A_{\ltimes u}$, and $A \mapsto A_{\ltimes u}$ is an automorphism of the ring $R[\partial]$ which is the identity on R [ADH, p. 243]; its inverse is $B \mapsto B_{\ltimes u^{-1}}$). Let $\phi \in R^{\times}$. Then we have the ring $R^{\phi}[\delta]$ of linear differential operators over the differential ring R^{ϕ} (with derivation $\delta = \phi^{-1}\partial$), and we have a ring isomorphism $A \mapsto A^{\phi}\colon R[\partial] \to R^{\phi}[\delta]$; it is the identity on R, with $\partial^{\phi} = \phi\delta$.

The *linear part* of $P \in R\{Y\}$ is the linear differential operator

$$L_P := \sum_n \frac{\partial P}{\partial Y^{(n)}}(0)\partial^n \in R[\partial], \qquad \text{so} \quad L_{P_{+a}} = \sum_n \frac{\partial P}{\partial Y^{(n)}}(a)\partial^n \text{ for } a \in R.$$

We have $L_P(y) = P_1(y)$ for all $y \in R$.

Suppose now K is a differential field. Then $A \in K[\partial]$ is said to *split* over K if $A = c(\partial - f_1)\cdots(\partial - f_r)$ for some $c \in K^\times$, $f_1, \dots, f_r \in K$; cf. [ADH, 5.1]. If A splits over K, then so does aAb for $a, b \in K^\times$, and A^ϕ splits over K^ϕ for $\phi \in K^\times$. In [ADH, 5.2] we defined the functions

$$\omega\colon K \to K, \qquad \omega(z) \ := \ -(2z' + z^2) \tag{0.3}$$

and

$$\sigma\colon K^\times \to K, \qquad \sigma(y) \ := \ \omega(z) + y^2 \quad \text{where } z := -y^\dagger. \tag{0.4}$$

Then for $A = 4\partial^2 + f$ $(f \in K)$ we have

$$A \text{ splits over } K \quad \Longleftrightarrow \quad f \in \omega(K) \tag{0.5}$$

and if -1 is not a square in K, then

$$A \text{ splits over } K[\mathrm{i}] \quad \Longleftrightarrow \quad f \in \omega(K) \cup \sigma(K^\times). \tag{0.6}$$

We say that K is *linearly closed* if every $A \in K[\partial]^{\neq}$ splits over K. By [ADH, 5.8.9] this holds if K is *weakly differentially closed*: each $P \in K\{Y\} \setminus K$ has a zero in K (hence K is also *linearly surjective*, that is, $A(K) = K$ for all $A \in K[\partial]^{\neq}$).

Valued fields. For a field K we have $K^\times = K^{\neq}$, and a (Krull) valuation on K is a surjective map $v\colon K^\times \to \Gamma$ onto an ordered abelian group Γ (additively written) satisfying the usual laws, and extended to $v\colon K \to \Gamma_\infty := \Gamma \cup \{\infty\}$ by $v(0) = \infty$, where the ordering on Γ is extended to a total ordering on Γ_∞ by $\gamma < \infty$ for all $\gamma \in \Gamma$. A *valued field* K is a field (also denoted by K) together with a valuation ring $\mathcal{O}$ of that field, and the corresponding valuation $v\colon K^\times \to \Gamma$ on the underlying field is such that $\mathcal{O} = \{a \in K : va \geqslant 0\}$ as explained in [ADH, 3.1].

Let K be a valued field with valuation ring $\mathcal{O}_K$ and valuation $v\colon K^\times \to \Gamma_K$. Then $\mathcal{O}_K$ is a local ring with maximal ideal $\smallo_K = \{a \in K : va > 0\}$ and residue field $\operatorname{res}(K) = \mathcal{O}_K/\smallo_K$. If $\operatorname{res}(K)$ has characteristic zero, then K is said to be of equicharacteristic zero. When, as here, we use the capital K for the valued field under consideration, then we denote Γ_K, $\mathcal{O}_K$, $\smallo_K$, by Γ, $\mathcal{O}$, $\smallo$, respectively. A very handy alternative notation system in connection with the valuation is as follows. With a, b ranging over K, set

$$\begin{array}{lll} a \asymp b \ :\Leftrightarrow \ va = vb, & a \preccurlyeq b \ :\Leftrightarrow \ va \geqslant vb, & a \prec b \ :\Leftrightarrow \ va > vb, \\ a \succcurlyeq b \ :\Leftrightarrow \ b \preccurlyeq a, & a \succ b \ :\Leftrightarrow \ b \prec a, & a \sim b \ :\Leftrightarrow \ a - b \prec a. \end{array}$$

It is easy to check that if $a \sim b$, then $a, b \neq 0$ and $a \asymp b$, and that $\sim$ is an equivalence relation on $K^\times$. Given a valued field extension L of K, we identify in the usual way $\operatorname{res}(K)$ with a subfield of $\operatorname{res}(L)$, and Γ with an ordered subgroup of Γ_L. Such a valued field extension is called *immediate* if $\operatorname{res}(K) = \operatorname{res}(L)$ and $\Gamma = \Gamma_L$. We use *pc-sequence* to abbreviate *pseudocauchy sequence*, and $a_\rho \rightsquigarrow a$ indicates that (a_ρ) is a pc-sequence pseudoconverging to a; here the a_ρ and a lie in a valued field understood from the context, see [ADH, 2.2, 3.2].

Next we summarize the complementary processes of *coarsening* and *specialization* of a valued field, which play an important role in Parts 3 and 4. (For more details see [ADH, 3.4].) Let K be a valued field and Δ a convex subgroup of its value group Γ. Equip $\dot{\Gamma} := \Gamma/\Delta$ with the unique ordering making it an ordered abelian group such that the residue morphism $\Gamma \to \dot{\Gamma}$ is increasing. Then the map $\dot{v} = v_\Delta \colon K^\times \to \dot{\Gamma}$ given by $\dot{v}f := vf + \Delta$ is a valuation on K, called the *coarsening* of v by Δ, or just the Δ-*coarsening* of v. We denote the asymptotic relations associated with $\dot{v}$ by a subscript Δ, so $\preccurlyeq_\Delta$, $\prec_\Delta$, etc., or just by $\dot{\preccurlyeq}$, $\dot{\prec}$, etc., if Δ is clear from the context. The valuation ring of $\dot{v}$ is $\dot{\mathcal{O}} := \big\{a \in K : va \in \Delta^{\uparrow}\big\}$ with maximal ideal $\dot{o} := \big\{a \in K : va > \Delta\big\}$. The valued field $(K, \dot{\mathcal{O}})$ is called the *coarsening* of the valued field K by Δ, or simply the Δ-*coarsening* of K. Let $\dot{K} := \dot{\mathcal{O}}/\dot{o}$ be the residue field of $(K, \dot{\mathcal{O}})$, with residue morphism $a \mapsto a + \dot{o} \colon \dot{\mathcal{O}} \to \dot{K}$. Then for $a \in \dot{\mathcal{O}} \setminus \dot{o}$, the value va only depends on $\dot{a}$, and we obtain a valuation $v \colon \dot{K}^\times \to \Delta$ on $\dot{K}$ with $v\dot{a} := va$ for $a \in \dot{\mathcal{O}} \setminus \dot{o}$. The valuation ring of this valuation on $\dot{K}$ is $\mathcal{O}_{\dot{K}} := \{\dot{a} : a \in \mathcal{O}\}$. The valued field $(\dot{K}, \mathcal{O}_{\dot{K}})$, called the Δ-*specialization* of K, is also denoted by $\dot{K}$. The composed map $\mathcal{O} \to \mathcal{O}_{\dot{K}} \to \operatorname{res}(\dot{K})$ has kernel o, and thus induces a field isomorphism $\operatorname{res}(K) \to \operatorname{res}(\dot{K})$; we use it to identify $\operatorname{res}(K)$ with $\operatorname{res}(\dot{K})$.

Valued differential fields. As in [ADH], a *valued differential field* is a valued field of equicharacteristic zero together with a derivation, generally denoted by ∂, on the underlying field. (Unlike [3] we do not assume in this definition that ∂ is continuous with respect to the valuation topology.) Let K be a valued differential field and $K\langle Y\rangle$ the fraction field of $K\{Y\}$. We extend the valuation $v \colon K \to \Gamma_\infty$ and the corresponding relations $\prec, \preccurlyeq$, etc., first to $K\{Y\}$ by

$$v(P) \ := \ \min\big\{v(P_{\boldsymbol{i}}) : \boldsymbol{i} \in \mathbb{N}^{1+r}\big\} \qquad \text{for } P \in K\{Y\} \text{ of order} \leqslant r,$$

and then (uniquely) to a valuation $v : K\langle Y\rangle \to \Gamma_\infty$ (the *gaussian extension*). We also define the *dominant degree* $\operatorname{ddeg} P$ for $P \in K\{Y\}$ by

$$\operatorname{ddeg} P \ := \ \max\big\{d : v(P_d) = v(P)\big\} \in \mathbb{N} \ \text{ if } P \neq 0, \qquad \operatorname{ddeg} P \ := \ -\infty \text{ if } P = 0.$$

The *dominant weight* $\operatorname{dwt} P$ of P is defined in the same way, and so is the *dominant multiplicity* $\operatorname{dmul} P$ of P (at 0), with $\operatorname{dmul} P = +\infty$ if $P = 0$.

The derivation ∂ of a valued differential field K is said to be *small* if $\partial o \subseteq o$; then ∂ is continuous with respect to the valuation topology of K, and $\partial\mathcal{O} \subseteq \mathcal{O}$ [ADH, 4.4.2], so ∂ induces a derivation on $\operatorname{res}(K)$ making the residue map $\mathcal{O} \to \operatorname{res}(K)$ into a morphism of differential rings.

We say that K is *differential-henselian* (d-*henselian* for short) if its derivation is small, the differential residue field $\operatorname{res}(K)$ is linearly surjective, and for every $P \in \mathcal{O}\{Y\}$ with $P_0 \prec P_1 \asymp 1$ there exists $y \prec 1$ in K such that $P(y) = 0$. See [ADH, Chapter 7] for more, and for the weaker notions of r-d-*henselian*, $r \in \mathbb{N}$.

Suppose now the derivation of K is small, and $A = \sum_i a_i \partial^i \in K[\partial]$ (all $a_i \in K$). In [ADH, 5.6] we defined $v(A) := \min_i v(a_i) \in \Gamma_\infty$, extending accordingly the relations $\prec, \preccurlyeq$, etc., on K to $K[\partial]$. For $A \neq 0$ we set

$$\operatorname{dwm} A \ := \ \min\big\{i : v(a_i) = v(A)\big\} \in \mathbb{N}, \qquad \operatorname{dwt} A \ := \ \max\big\{i : v(a_i) = v(A)\big\} \in \mathbb{N},$$

the *dominant weighted multiplicity* of A and the *dominant weight* of A, respectively. Let y range over $K^\times$. Then $v(Ay)$ only depends on vy, not on y, and we can thus define $v_A(\gamma) := v(Ay)$ for $\gamma := vy$. The quantity $\operatorname{dwt}(Ay)$ also only depends on vy, so we can define $\operatorname{dwt}_A(\gamma) := \operatorname{dwt}(Ay)$ for $\gamma = vy$. Likewise, if $\partial\mathcal{O} \subseteq o$, then $\operatorname{dwm}(Ay)$ only depends on vy, and we define $\operatorname{dwm}_A(\gamma) := \operatorname{dwm}(Ay)$ for $\gamma = vy$. The set of *exceptional values* of A is

$$\mathscr{E}(A) \ = \ \mathscr{E}_K(A) \ := \ \big\{vy : \operatorname{dwm}(Ay) > 0\big\} \ \subseteq \ \Gamma;$$

it contains $v(\ker^{\neq} A)$, where $\ker^{\neq} A := (\ker A)^{\neq}$; cf. [ADH, 5.6.7].

Ordered differential fields. An *ordered differential field* is a differential field K with an ordering on K making K an ordered field. Likewise, an *ordered valued differential field* is a valued differential field K equipped an ordering on K making K an ordered field (no relation between derivation, valuation, or ordering being assumed). Let K be an ordered differential field. Then we have the convex subring

$$\mathcal{O} \ := \ \{g \in K : |g| \leqslant c \text{ for some } c \in C\},$$

which is a valuation ring of K and has maximal ideal

$$\mathcal{o} \ = \ \{g \in K : |g| < c \text{ for all positive } c \in C\}.$$

We call K an *H-field* if for all $f \in K$ with $f > C$ we have $f' > 0$, and $\mathcal{O} = C + \mathcal{o}$. We view such an H-field K as an ordered valued differential field with its valuation given by $\mathcal{O}$. *Pre-H-fields* are the ordered valued differential subfields of H-fields. See [ADH, 10.5] for basic facts about (pre-)H-fields. An H-field K is said to be *Liouville closed* if K is real closed and for all $f, g \in K$ there exists $y \in K^{\times}$ with $y' + fy = g$. Every H-field extends to a Liouville closed one; see [ADH, 10.6].

Let K be a pre-H-field. In [ADH, p. 520] we singled out the following subsets:

$$\Gamma(K) \ := \ (K^{\succ 1})^{\dagger}, \quad \Lambda(K) \ := \ -\big[(K^{\succ 1})^{\dagger\dagger}\big], \quad \Delta(K) \ := \ -\big[(K^{\neq, \prec 1})'^{\dagger}\big].$$

If K is Liouville closed, then the restriction of ω to $\Lambda(K)$ is strictly increasing with $\omega\big(\Lambda(K)\big) = \omega\big(\Delta(K)\big)$. The restriction of σ to $\Gamma(K)$ is strictly increasing, and $\omega\big(\Lambda(K)\big) < \sigma\big(\Gamma(K)\big)$; see [ADH, 11.8]. We call K *Schwarz closed* if K is Liouville closed and $K = \omega\big(\Lambda(K)\big) \cup \sigma\big(\Gamma(K)\big)$ [ADH, 11.8.33].

We alert the reader that in a few places we refer to the Liouville closed H-field $\mathbb{T}_{\mathrm{g}}$ of grid-based transseries from [15], which is denoted there by $\mathbb{T}$. Here we adopt the notation of [ADH] where $\mathbb{T}$ is the larger field of logarithmic-exponential series, which is also a Liouville closed H-field.

Asymptotic fields. In their capacity as valued differential fields, H-fields and pre-H-fields are among so-called *asymptotic fields*, which also include the algebraic closure of any pre-H-field (where this algebraic closure is equipped with the unique derivation extending the derivation of the pre-H-field and any valuation extending the valuation of the pre-H-field). That is one of the reasons (not the only one) to consider this notion more closely: An *asymptotic field* is a valued differential field K such that for all nonzero $f, g \prec 1$ in K we have: $f \prec g \Longleftrightarrow f' \prec g'$. Let K be an asymptotic field. Then K^{ϕ} with $\phi \in K^{\times}$ is also asymptotic. We associate to K its *asymptotic couple* (Γ, ψ), where $\psi \colon \Gamma^{\neq} \to \Gamma$ is given by

$$\psi(vg) \ = \ v(g^{\dagger}) \ \text{ for } g \in K^{\times} \text{ with } vg \neq 0.$$

So the asymptotic couple of K is the ordered abelian value group Γ with a function on it that is induced by the derivation of K; it serves a similar purpose as the value group of a mere valued field. We put $\Psi := \psi(\Gamma^{\neq})$, and if we want to stress the dependence on K we also write (Γ_K, ψ_K) and Ψ_K instead of (Γ, ψ) and Ψ, respectively. See [ADH, 9.1, 9.2] for more on asymptotic couples, in particular the taxonomy of asymptotic fields introduced via their asymptotic couples: having a *gap,* being *grounded,* having *asymptotic integration,* and having *rational asymptotic integration.* We now consider various conditions on our asymptotic field K, some mentioned in the last sentence. To define those conditions, let f, g range over K, and c over the constant field C of K:

H-asymptotic (or *of H-type*): $0 \neq f \prec g \prec 1 \Rightarrow f^\dagger \succcurlyeq g^\dagger$. (Pre-$H$-fields and their algebraic closures are H-asymptotic. If K is H-asymptotic and $\phi \in K^\times$, then K^ϕ is H-asymptotic.)

Differential-valued (or d-*valued*): for all $f \asymp 1$ there exists c with $f \sim c$. (H-fields and their algebraic closures are d-valued.)

Pre-differential-valued (or *pre*-d-*valued*): $f \preccurlyeq 1$ & $0 \neq g \prec 1 \Rightarrow f' \prec g^\dagger$. (Pre-$H$-fields are pre-d-valued. Every pre-d-valued field has a canonical d-valued extension, its d-valued hull $\operatorname{dv}(K)$, by [ADH, 10.3].)

Grounded: there is a nonzero $f \not\asymp 1$ such that for all nonzero $g \not\asymp 1$ we have $g^\dagger \succcurlyeq f^\dagger$.

Asymptotic integration: for all $f \neq 0$ there exists $g \not\asymp 1$ with $g' \asymp f$. (If K has asymptotic integration, then K is ungrounded with $\Gamma \neq \{0\}$. Liouville closed H-fields have asymptotic integration.)

λ-*free*: H-asymptotic, ungrounded, and for all f there exists $g \succ 1$ with $f - g^{\dagger\dagger} \succcurlyeq g^\dagger$.

ω-*free*: H-asymptotic, ungrounded, and for all f there exists $g \succ 1$ such that $f - \omega(g^{\dagger\dagger}) \succcurlyeq (g^\dagger)^2$, where ω is as in (0.3).

Here we note that ω-freeness is very robust, and powerful, and that $\mathbb{T}$ is ω-free. For more on this, see [ADH, 13.6] and Section 1.3 below.

Flattening. This is from [ADH, 9.4]. Suppose K is H-asymptotic with asymptotic couple (Γ, ψ). Then we have a convex subgroup $\Gamma^\flat := \{\gamma \in \Gamma : \psi(\gamma) > 0\}$ of Γ, and the $\Gamma^\flat$-coarsening $v^\flat \colon K^\times \to \Gamma/\Gamma^\flat$ of v is called the *flattening* of v. The differential field K together with the valuation ring of $v^\flat$ is H-asymptotic. We denote the relations $\asymp$, $\sim$, $\preccurlyeq$, $\prec$ associated to $v^\flat$ by $\asymp^\flat$, $\sim^\flat$, $\preccurlyeq^\flat$, $\prec^\flat$, respectively. For $\phi \in K^\times$ we denote the flattened objects $v^\flat_\phi$, $\Gamma^\flat_\phi$, $\asymp^\flat_\phi$, $\sim^\flat_\phi$, $\preccurlyeq^\flat_\phi$, $\prec^\flat_\phi$ associated to K^ϕ by a subscript ϕ. In particular, $\Gamma^\flat_\phi = \{\gamma \in \Gamma : \psi(\gamma) > v\phi\}$.

Newtonianity. In order to define *newtonian*, assume the asymptotic field K is ungrounded with $\Gamma \neq \{0\}$. Then an element $\phi \in K$ is said to be *active in K* if $\phi \neq 0$ and $\phi \succcurlyeq f^\dagger$ for some nonzero $f \not\asymp 1$. If ϕ is active in K, then the derivation $\phi^{-1}\partial$ of K^ϕ is small. Let ϕ range over the active elements of K. A property $S(\phi)$ of (active) elements ϕ is said to hold *eventually* if there is an active ϕ_0 in K such that $S(\phi)$ holds for all $\phi \preccurlyeq \phi_0$; cf. [ADH, p. 479]. The way to understanding Liouville closed H-fields such as $\mathbb{T}$ involves often eventual behavior of this kind. For example, in [ADH, 11.1] we showed that for $P \in K\{Y\}$, $\operatorname{dmul} P^\phi$, $\operatorname{ddeg} P^\phi$, and $\operatorname{dwt} P^\phi$ are eventually constant. The eventual values of these quantities are denoted by $\operatorname{nmul} P$, $\operatorname{ndeg} P$, and $\operatorname{nwt} P$, respectively, and are called the *Newton multiplicity* of P, the *Newton degree* of P, and the *Newton weight of P*. We call P *quasilinear* if $\operatorname{ndeg} P = 1$.

Definition 0.1. An asymptotic field K is said to be *newtonian* if K is ungrounded of H-type with $\Gamma \neq \{0\}$, and every quasilinear $P \in K\{Y\}$ has a zero in $\mathcal{O}$.

We now list some properties of this notion from [ADH] that we shall frequently use. For this, assume as before that K is an ungrounded H-asymptotic field with $\Gamma \neq \{0\}$. First a consequence of [ADH, 14.0.1 and the remarks after it]:

(0.7) *If K is ω-free, then K has an immediate* d*-algebraic extension which is newtonian and ω-free.* [ADH, 14.0.1]

For d-valued K we often need the following:

(0.8) *If K is* d*-valued, ω-free, newtonian, and algebraically closed, then K is weakly differentially closed.* [ADH, 14.5.3]

For more on d-valued K, see [ADH, 14.5.4] and [21, Theorem B].

(0.9) *If K is newtonian, then K is linearly surjective.* [ADH, 14.4.2]

Next, suppose K is ω-free and d-valued with divisible value group, and L is a valued differential field extension of K and algebraic over K. Then:

(0.10) *If K is newtonian, then so is L.* [ADH, 14.5.7]

(0.11) *K is newtonian if L is newtonian and $L = K(C_L)$.* [ADH, 14.5.6]

For example, the valued differential field $\mathbb{T}$ is newtonian by [ADH, 15.0.2], hence its algebraic closure $\mathbb{T}[i]$ is also newtonian by (0.10), and thus $\mathbb{T}[i]$ is linearly closed by (0.8) and linearly surjective by (0.9).

See also [ADH, 14.2] for *r-newtonian* and *r-linearly newtonian*, useful weakenings of *newtonian* that allow for induction on $r \in \mathbb{N}$.

Closed H-fields. A *closed H-field* (or *H-closed field*) is a Liouville closed, ω-free, and newtonian H-field. A fundamental fact from [ADH] about the elementary theory of $\mathbb{T}$ as an ordered valued differential field is that it is completely axiomatized by the requirements of being a closed H-field with small derivation. Moreover, the closed H-fields are exactly the existentially closed models of the theory of H-fields.

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Part 1. Preliminaries

After generalities on linear differential operators and differential polynomials in Section 1.1, we investigate the group of logarithmic derivatives in valued differential fields of various kinds (Section 1.2). We also assemble some basic preservation theorems for $\uplambda$-*freeness* and $\upomega$-*freeness* (Section 1.3) and continue the study of linear differential operators over H-asymptotic fields initiated in [ADH, 5.6, 14.2] (Section 1.4). In our analysis of the solutions of algebraic differential equations over H-asymptotic fields in Part 3, special pc-sequences in the sense of [ADH, 3.4] play an important role; Section 1.5 explains why. A cornerstone of [ADH] is the concept of *newtonianity*, an analogue of henselianity appropriate for H-asymptotic fields with asymptotic integration [ADH, Chapter 14]. Related to this is *differential-henselianity* [ADH, Chapter 7], which makes sense for a broader class of valued differential fields. In Sections 1.6 and 1.7 we further explore these notions. Among other things, we study their persistence under taking the completion of a valued differential field with small derivation (as defined in [ADH, 4.4]).

1.1. Linear Differential Operators and Differential Polynomials

This section gathers miscellaneous facts of a general nature about linear differential operators and differential polynomials, sometimes in a valued differential setting. We discuss splittings and least common left multiples of linear differential operators, and then prove some useful estimates for derivatives of exponential terms and for Riccati transforms.

Splittings. In this subsection K is a differential field. Let $A \in K[\partial]^{\neq}$ be monic of order $r \geqslant 1$. A **splitting of** A **over** K is a tuple $(g_1,\dots,g_r) \in K^r$ such that $A = (\partial - g_1)\cdots(\partial - g_r)$. If $(g_1,\dots,g_r)$ is a splitting of A over K and $\mathfrak{n} \in K^\times$, then $(g_1 - \mathfrak{n}^\dagger,\dots,g_r - \mathfrak{n}^\dagger)$ is a splitting of $A_{\ltimes\mathfrak{n}} = \mathfrak{n}^{-1}A\mathfrak{n}$ over K.

Suppose $A = A_1\cdots A_m$ where every $A_i \in K[\partial]$ is monic of positive order r_i (so $r = r_1 + \cdots + r_m$). Given any splittings

$$(g_{11},\dots,g_{1r_1}),\ \dots,\ (g_{m1},\dots,g_{mr_m})$$

of $A_1,\dots,A_m$, respectively, we obtain a splitting

$$\big(g_{11},\dots,g_{1r_1},\ \dots,\ g_{m1},\dots,g_{mr_m}\big)$$

of A by concatenating the given splittings of $A_1,\dots,A_m$ in the order indicated, and call it a splitting of A **induced** by the factorization $A = A_1\cdots A_m$. For $B \in K[\partial]$ of order $r \geqslant 1$ we have $B = bA$ with $b \in K^\times$ and monic $A \in K[\partial]$, and then a **splitting of** B **over** K is by definition a splitting of A over K. A splitting of B over K remains a splitting of aB over K, for any $a \in K^\times$. Thus:

Lemma 1.1.1. *If $B \in K[\partial]$ has order $r \geqslant 1$, and $(g_1,\dots,g_r)$ is a splitting of B over K and $\mathfrak{n} \in K^\times$, then $(g_1 - \mathfrak{n}^\dagger,\dots,g_r - \mathfrak{n}^\dagger)$ is a splitting of $B_{\ltimes\mathfrak{n}}$ over K and a splitting of $B\mathfrak{n}$ over K.*

Let $A \in K[\partial]^{\neq}$ and $r := \operatorname{order}(A)$. From [ADH, 5.1, 5.7] we know that if A splits over K, then for any $\phi \in K^\times$ the operator $A^\phi \in K^\phi[\updelta]$ splits over K^ϕ; here is how a splitting of A over K transforms into a splitting of A^ϕ over K^ϕ:

Lemma 1.1.2. *Let $\phi \in K^\times$ and*

$$A \;=\; c(\partial - a_1)\cdots(\partial - a_r) \quad \text{ with } c \in K^\times \text{ and } a_1,\dots,a_r \in K.$$

Then in $K^\phi[\delta]$ we have

$$A^\phi \;=\; c\phi^r(\delta - b_1)\cdots(\delta - b_r) \quad \text{ where } b_j := \phi^{-1}\big(a_j - (r-j)\phi^\dagger\big)\ (j=1,\dots,r).$$

Proof. Induction on r. The case $r=0$ being obvious, suppose $r \geqslant 1$, and set $B := (\partial - a_2)\cdots(\partial - a_r)$. By inductive hypothesis

$$B^\phi = \phi^{r-1}(\delta - b_2)\cdots(\delta - b_r) \quad \text{ where } b_j := \phi^{-1}\big(a_j - (r-j)\phi^\dagger\big) \text{ for } j = 2,\dots,r.$$

Then

$$A^\phi \;=\; c\phi\,\big(\delta - (a_1/\phi)\big)\,B^\phi \;=\; c\phi^r\,\big(\delta - (a_1/\phi)\big)_{\ltimes \phi^{r-1}}\,(\delta - b_2)\cdots(\delta - b_r)$$

with

$$\big(\delta - (a_1/\phi)\big)_{\ltimes \phi^{r-1}} = \delta - (a_1/\phi) + (r-1)\phi^\dagger/\phi$$

by [ADH, p. 243]. $\square$

A different kind of factorization, see for example [19], reduces the process of solving the differential equation $A(y)=0$ to repeated multiplication and integration:

Lemma 1.1.3. *Let $A \in K[\partial]^{\neq}$ be monic of order $r \geqslant 1$. If $b_1,\dots,b_r \in K^\times$ and*

$$A \;=\; b_1\cdots b_{r-1}b_r(\partial b_r^{-1})(\partial b_{r-1}^{-1})\cdots(\partial b_1^{-1}),$$

then $(a_r,\dots,a_1)$, where $a_j := (b_1\cdots b_j)^\dagger$ for $j=1,\dots,r$, is a splitting of A over K. Conversely, if $(a_r,\dots,a_1)$ is a splitting of A over K and $b_1,\dots,b_r \in K^\times$ are such that $b_j^\dagger = a_j - a_{j-1}$ for $j=1,\dots,r$ with $a_0 := 0$, then A is as in the display.

This follows easily by induction on r.

Real splittings. Let H be a differential field in which -1 is not a square. Then we let i denote an element in a differential field extension of H with $\mathrm{i}^2 = -1$, and consider the differential field $K = H[\mathrm{i}]$. Suppose $A \in H[\partial]$ is monic of order 2 and splits over K, so

$$A \;=\; (\partial - f)(\partial - g), \qquad f,g \in K.$$

Then

$$A \;=\; \partial^2 - (f+g)\partial + fg - g',$$

and thus $f \in H$ iff $g \in H$. One checks easily that if $g \notin H$, then there are unique $a,b \in H$ with $b \neq 0$ such that

$$f \;=\; a - b\mathrm{i} + b^\dagger, \qquad g \;=\; a + b\mathrm{i},$$

and thus

$$A \;=\; \partial^2 - (2a + b^\dagger)\partial + a^2 + b^2 - a' + ab^\dagger.$$

Conversely, if $a,b \in H$ and $b \neq 0$, then for $f := a - b\mathrm{i} + b^\dagger$ and $g := a + b\mathrm{i}$ we have $(\partial - f)(\partial - g) \in H[\partial]$.

Let now $A \in H[\partial]$ be monic of order $r \geqslant 1$. Recall: A is *irreducible* iff there are no monic $A_1, A_2 \in K[\partial]$ of positive order with $A = A_1A_2$; cf. [ADH, p. 250].

Lemma 1.1.4. *Suppose A splits over K. Then $A = A_1\cdots A_m$ for some $A_1,\dots,A_m$ in $H[\partial]$ that are monic and irreducible of order 1 or 2 and split over K.*

Proof. By [ADH, 5.1.35], $A = A_1\cdots A_m$, where every $A_i \in H[\partial]$ is monic and irreducible of order 1 or 2. By [ADH, 5.1.22], such A_i split over K. $\square$

Definition 1.1.5. A **real splitting of** A (over K) is a splitting of A over K that is induced by a factorization $A = A_1 \cdots A_m$ where every $A_i \in H[\partial]$ is monic of order 1 or 2 and splits over K. (Note that we do not require the A_i of order 2 to be irreducible in $H[\partial]$.)

Thus if A splits over K, then A has a real splitting over K by Lemma 1.1.4. Note that if $(g_1, \dots, g_r)$ is a real splitting of A and $\mathfrak{n} \in H^\times$, then $(g_1 - \mathfrak{n}^\dagger, \dots, g_r - \mathfrak{n}^\dagger)$ is a real splitting of $A_{\ltimes \mathfrak{n}}$.

It is convenient to extend the above slightly: for $B \in H[\partial]$ of order $r \geqslant 1$ we have $B = bA$ with $b \in H^\times$ and monic $A \in H[\partial]$, and then a **real splitting of** B (over K) is by definition a real splitting of A (over K).

In later use, H is a valued differential field with small derivation such that -1 is not a square in the differential residue field $\operatorname{res}(H)$. For such H, let $\mathcal{O}$ be the valuation ring of H. We make K a valued differential field extension of H with small derivation by taking $\mathcal{O}_K = \mathcal{O} + \mathcal{O}i$ as the valuation ring of K. We have the residue map $a \mapsto \operatorname{res} a \colon \mathcal{O}_K \to \operatorname{res}(K)$, so $\operatorname{res}(K) = \operatorname{res}(H)[i]$, writing here i for $\operatorname{res} i$. We extend this map to a ring morphism $B \mapsto \operatorname{res} B \colon \mathcal{O}_K[\partial] \to \operatorname{res}(K)[\partial]$ by sending $\partial \in \mathcal{O}[\partial]$ to $\partial \in \operatorname{res}(K)[\partial]$.

Lemma 1.1.6. *Suppose $(g_1, \dots, g_r) \in \operatorname{res}(K)^r$ is a real splitting of a monic operator $D \in \operatorname{res}(H)[\partial]$ of order $r \geqslant 1$. Then there are $b_1, \dots, b_r \in \mathcal{O}_K$ such that*

$$B \ := \ (\partial - b_1) \cdots (\partial - b_r) \in \mathcal{O}[\partial],$$

$(b_1, \dots, b_r)$ is a real splitting of B, and $\operatorname{res} b_j = g_j$ for $j = 1, \dots, r$.

Proof. We can assume $r \in \{1, 2\}$. The case $r = 1$ is obvious, so let $r = 2$. Then the case where $g_1, g_2 \in \operatorname{res}(H)$ is again obvious, so let $g_1 = \operatorname{res}(a) - \operatorname{res}(b)i + (\operatorname{res} b)^\dagger$, $g_2 = \operatorname{res}(a) + \operatorname{res}(b)i$ where $a, b \in \mathcal{O}$, $\operatorname{res} b \neq 0$. Set $b_1 := a - bi + b^\dagger$, $b_2 := a + bi$. Then $b_1, b_2 \in \mathcal{O}_K$ with $\operatorname{res} b_1 = g_1$, $\operatorname{res} b_2 = g_2$, and $B := (\partial - b_1)(\partial - b_2) \in \mathcal{O}[\partial]$ have the desired properties. $\square$

Least common left multiples and complex conjugation. In this subsection H is a differential field. Recall from [ADH, 5.1] the definition of the *least common left multiple* $\operatorname{lclm}(A_1, \dots, A_m)$ of operators $A_1, \dots, A_m \in H[\partial]^{\neq}$, $m \geqslant 1$: this is the monic operator $A \in H[\partial]$ such that $H[\partial]A_1 \cap \cdots \cap H[\partial]A_m = H[\partial]A$. For $A, B \in H[\partial]^{\neq}$ we have:

$$\max\big\{\operatorname{order}(A), \operatorname{order}(B)\big\} \ \leqslant \ \operatorname{order}\big(\operatorname{lclm}(A,B)\big) \ \leqslant \ \operatorname{order}(A) + \operatorname{order}(B).$$

For the inequality on the right, note that the natural $H[\partial]$-module morphism

$$H[\partial] \ \to \ \big(H[\partial]/H[\partial]A\big) \times \big(H[\partial]/H[\partial]B\big)$$

has kernel $H[\partial]\operatorname{lclm}(A,B)$, and for $D \in H[\partial]^{\neq}$, the H-linear space $H[\partial]/K[\partial]D$ has dimension order D.

We now assume that -1 is not a square in H; then we have a differential field extension $H[i]$ where $i^2 = -1$. The automorphism $a + bi \mapsto \overline{a + bi} := a - bi$ $(a, b \in H)$ of the differential field $H[i]$ extends uniquely to an automorphism $A \mapsto \overline{A}$ of the ring $H[i][\partial]$ with $\overline{\partial} = \partial$. Let $A \in H[i][\partial]$; then $\overline{A} = A \Longleftrightarrow A \in H[\partial]$. Hence if $A \neq 0$ is monic, then $L := \operatorname{lclm}(A, \overline{A}) \in H[\partial]$ and thus $L = BA = \overline{B}\,\overline{A}$ where $B \in H[i][\partial]$.

Example 1.1.7. Let $A = \partial - a$ where $a \in H[i]$. If $a \in H$, then $\operatorname{lclm}(A, \overline{A}) = A$, and if $a \notin H$, then $\operatorname{lclm}(A, \overline{A}) = (\partial - b)(\partial - a) = (\partial - \overline{b})(\partial - \overline{a})$ where $b \in H[i] \setminus H$.

Let now F be a differential field extension of H in which -1 is not a square; we assume that i is an element of a differential ring extension of F.

Lemma 1.1.8. *Let $A \in H[\mathrm{i}][\partial]^{\neq}$ be monic, $b \in H[\mathrm{i}]$, and $f \in F[\mathrm{i}]$ such that $A(f) = b$. Let $B \in H[\mathrm{i}][\partial]$ be such that $L := \operatorname{lclm}(A, \overline{A}) = BA$. Then $L(f) = B(b)$ and hence $L\big(\operatorname{Re}(f)\big) = \operatorname{Re}\big(B(b)\big)$ and $L\big(\operatorname{Im}(f)\big) = \operatorname{Im}\big(B(b)\big)$.*

In [6] we shall need a slight extension of this lemma:

Remark 1.1.9. Let F be a differential ring extension of H in which -1 is not a square and let i be an element of a commutative ring extension of F such that $\mathrm{i}^2 = -1$ and the F-algebra $F[\mathrm{i}] = F + F\mathrm{i}$ is a free F-module with basis $1, \mathrm{i}$. For $f = g + h\mathrm{i} \in F[\mathrm{i}]$ with $g, h \in F$ we set $\operatorname{Re}(f) := g$ and $\operatorname{Im}(f) := h$. We make $F[\mathrm{i}]$ into a differential ring extension of F in the only way possible (which has $\mathrm{i}' = 0$). Then Lemma 1.1.8 goes through.

Estimates for derivatives of exponential terms. In this subsection K is an asymptotic differential field with small derivation, and $\phi \in K$. We also fix $\mathfrak{m} \in K^{\times}$ with $\mathfrak{m} \prec 1$. Here is a useful bound:

Lemma 1.1.10. *Let $r \in \mathbb{N}$ and $y \in K$ satisfy $y \prec \mathfrak{m}^{r+m} \prec 1$. Then $P(y) \prec \mathfrak{m}^{m\mu} P$ for all $P \in K\{Y\}^{\neq}$ of order at most r with $\mu = \operatorname{mul}(P) \geqslant 1$.*

Proof. Note that $0 \neq \mathfrak{m} \prec 1$ and $r + m \geqslant 1$. Hence

$$y' \ \prec\ (\mathfrak{m}^{r+m})' \ =\ (r+m)\mathfrak{m}^{r+m-1}\mathfrak{m}' \ \prec\ \mathfrak{m}^{r-1+m},$$

so by induction $y^{(i)} \prec \mathfrak{m}^{r-i+m}$ for $i = 0, \dots, r$. Hence $y^{\boldsymbol{i}} \prec \mathfrak{m}^{(r+m)|\boldsymbol{i}| - \|\boldsymbol{i}\|} \preccurlyeq \mathfrak{m}^{m|\boldsymbol{i}|}$ for nonzero $\boldsymbol{i} = (i_0, \dots, i_r) \in \mathbb{N}^{1+r}$, which yields the lemma. □

Corollary 1.1.11. *If $f \in K$ and $f \prec \mathfrak{m}^n$, then $f^{(k)} \prec \mathfrak{m}^{n-k}$ for $k = 0, \dots, n$.*

Proof. This is a special case of Lemma 1.1.10. □

Corollary 1.1.12. *Let $f \in K^{\times}$ and $n \geqslant 1$ be such that $f \preccurlyeq \mathfrak{m}^n$. Then $f^{(k)} \prec \mathfrak{m}^{n-k}$ for $k = 1, \dots, n$.*

Proof. Note that $\mathfrak{m}^n \neq 0$, so $f' \preccurlyeq (\mathfrak{m}^n)' = n\mathfrak{m}^{n-1}\mathfrak{m}' \prec \mathfrak{m}^{n-1}$ [ADH, 9.1.3]. Now apply Corollary 1.1.11 with f', $n-1$ in place of f, n. □

In the remainder of this subsection we let $\xi \in K^{\times}$ and assume $\xi \succ 1$ and $\zeta := \xi^{\dagger} \succcurlyeq 1$.

Lemma 1.1.13. *The elements $\xi, \zeta \in K$ have the following asymptotic properties:*

(i) *$\zeta^n \prec \xi$ for all n;*
(ii) *$\zeta^{(n)} \preccurlyeq \zeta^2$ for all n.*

Thus for each $P \in \mathcal{O}\{Z\}$ there is an $N \in \mathbb{N}$ with $P(\zeta) \preccurlyeq \zeta^N$, and hence $P(\zeta) \prec \xi$.

Proof. Part (i) follows from [ADH, 9.2.10(iv)] for $\gamma = v(\xi)$. As to (ii), if $\zeta' \preccurlyeq \zeta$, then $\zeta^{(n)} \preccurlyeq \zeta$ by [ADH, 4.5.3], and we are done. Suppose $\zeta' \succ \zeta$ and set $\gamma := v(\zeta)$. Then $\gamma, \gamma^{\dagger} < 0$, so $\gamma^{\dagger} = o(\gamma)$ by [ADH, 9.2.10(iv)] and hence $v(\zeta^{(n)}) = \gamma + n\gamma^{\dagger} > 2\gamma = v(\zeta^2)$ by [ADH, 6.4.1(iv)]. □

Recall from [ADH, 5.8] that for a homogeneous differential polynomial $P \in K\{Y\}$ of degree $d \in \mathbb{N}$ the *Riccati transform* $\operatorname{Ri}(P) \in K\{Z\}$ of P satisfies

$$\operatorname{Ri}(P)(z) = P(y)/y^d \quad \text{for } y \in K^{\times},\ z = y^{\dagger}.$$

We put $R_n := \operatorname{Ri}(Y^{(n)}) \in \mathbb{Q}\{Z\}$, so

$$R_0 = 1, \quad R_1 = Z, \quad R_2 = Z^2 + Z', \quad \ldots.$$

For $A = a_0 + a_1\partial + \cdots + a_n\partial^n \in K[\partial]$ $(a_0, \ldots, a_n \in K)$ we also let

$$\operatorname{Ri}(A) := a_0R_0 + a_1R_1 + \cdots + a_nR_n \in K\{Z\}.$$

In the next two corollaries, $l \in \mathbb{Z}$, $\xi = \phi'$, and e^{ϕ} denotes a unit of a differential ring extension of K with multiplicative inverse $\mathrm{e}^{-\phi}$ and such that $(\mathrm{e}^{\phi})' = \phi'\,\mathrm{e}^{\phi}$.

Corollary 1.1.14. $(\xi^l\,\mathrm{e}^{\phi})^{(n)} \;=\; \xi^{l+n}(1+\varepsilon)\,\mathrm{e}^{\phi}$ *where* $\varepsilon \in K$, $\varepsilon \prec 1$.

Proof. By Lemma 1.1.13(i) we have $l\zeta + \xi \sim \xi \succ 1$. Now use $(\xi^l\,\mathrm{e}^{\phi})^{(n)}/(\xi^l\,\mathrm{e}^{\phi}) = R_n(l\zeta+\xi)$ for $R_n = \operatorname{Ri}(Y^{(n)})$ in combination with [ADH, 11.1.5]. □

Applying the corollary above with ϕ, ξ replaced by $-\phi$, $-\xi$, respectively, we obtain:

Corollary 1.1.15. $(\xi^l\,\mathrm{e}^{-\phi})^{(n)} \;=\; (-1)^n\xi^{l+n}(1+\delta)\,\mathrm{e}^{-\phi}$ *where* $\delta \in K$, $\delta \prec 1$.

Estimates for Riccati transforms. In this subsection K is a valued differential field with small derivation. For later use we prove variants of [ADH, 11.1.5].

Lemma 1.1.16. *If* $z \in K^{\succ 1}$, *then* $R_n(z) = z^n(1+\varepsilon)$ *with* $v\varepsilon \geqslant v(z^{-1}) + o(vz) > 0$.

Proof. This is clear for $n=0$ and $n=1$. Suppose $z \succ 1$, $n \geqslant 1$, and $R_n(z) = z^n(1+\varepsilon)$ with ε as in the lemma. As in the proof of [ADH, 11.1.5],

$$R_{n+1}(z) \;=\; z^{n+1}\left(1 + \varepsilon + n\frac{z^{\dagger}}{z}(1+\varepsilon) + \frac{\varepsilon'}{z}\right).$$

Now $v(z^{\dagger}) \geqslant o(vz)$: this is obvious if $z^{\dagger} \preccurlyeq 1$, and follows from $\nabla(\gamma) = o(\gamma)$ for $\gamma \neq 0$ if $z^{\dagger} \succ 1$ [ADH, 6.4.1(iii)]. This gives the desired result in view of $\varepsilon' \prec 1$. □

Lemma 1.1.17. *Suppose* $\partial\mathcal{O} \subseteq \smallo$. *If* $z \in K^{\succcurlyeq 1}$, *then* $R_n(z) = z^n(1+\varepsilon)$ *with* $\varepsilon \prec 1$.

Proof. The case $z \succ 1$ follows from Lemma 1.1.16. For $z \asymp 1$, proceed as in the proof of that lemma, using $\partial\mathcal{O} \subseteq \smallo$. □

By [ADH, 9.1.3 (iv)] the condition $\partial\mathcal{O} \subseteq \smallo$ is satisfied if K is d-valued, or asymptotic with $\Psi \cap \Gamma^{>} \neq \emptyset$. The following observation is not used later:

Lemma 1.1.18. *Suppose K is asymptotic, and* $z \in K$ *with* $0 \neq z \preccurlyeq z' \prec 1$. *Then* $R_n(z) \sim z^{(n-1)}$ *for* $n \geqslant 1$.

Proof. Induction on n gives $z \preccurlyeq z' \preccurlyeq \cdots \preccurlyeq z^{(n)} \prec 1$ for all n. We now show $R_n(z) \sim z^{(n-1)}$ for $n \geqslant 1$, also by induction. The case $n=1$ is clear from $R_1 = Z$, so suppose $n \geqslant 1$ and $R_n(z) \sim z^{(n-1)}$. Then

$$R_{n+1}(z) \;=\; zR_n(z) + R_n(z)'$$

where $R_n(z)' \sim z^{(n)}$ by [ADH, 9.1.4(ii)] and $zR_n(z) \asymp zz^{(n-1)} \prec z^{(n-1)} \preccurlyeq z^{(n)}$. Hence $R_{n+1}(z) \sim z^{(n)}$. □

1.2. The Group of Logarithmic Derivatives

Let K be a differential field. The map $y \mapsto y^\dagger \colon K^\times \to K$ is a morphism from the multiplicative group of K to the additive group of K, with kernel $C^\times$. Its image

$$(K^\times)^\dagger \;=\; \{y^\dagger : y \in K^\times\}$$

is an additive subgroup of K, which we call the **group of logarithmic derivatives** of K. The morphism $y \mapsto y^\dagger$ induces an isomorphism $K^\times/C^\times \to (K^\times)^\dagger$. To shorten notation, set $0^\dagger := 0$, so $K^\dagger = (K^\times)^\dagger$. For $\phi \in K^\times$ we have $\phi(K^\phi)^\dagger = K^\dagger$. The group $K^\times$ is divisible iff both $C^\times$ and $K^\dagger$ are divisible. If K is algebraically closed, then $K^\times$ and hence $K^\dagger$ are divisible, making $K^\dagger$ a $\mathbb{Q}$-linear subspace of K. Likewise, if K is real closed, then the multiplicative subgroup $K^>$ of $K^\times$ is divisible, so $K^\dagger = (K^>)^\dagger$ is a $\mathbb{Q}$-linear subspace of K.

Lemma 1.2.1. *Suppose $K^\dagger$ is divisible, L is a differential field extension of K with $L^\dagger \cap K = K^\dagger$, and M is a differential field extension of L and algebraic over L. Then $M^\dagger \cap K = K^\dagger$.*

Proof. Let $f \in M^\times$ be such that $f^\dagger \in K$. Then $f^\dagger \in L$, so for $n := [L(f) : L]$,

$$nf^\dagger \;=\; \operatorname{tr}_{L(f)|L}(f^\dagger) \;=\; \mathrm{N}_{L(f)|L}(f)^\dagger \in L^\dagger$$

by an identity in [ADH, 4.4]. Hence $nf^\dagger \in K^\dagger$, and thus $f^\dagger \in K^\dagger$. $\square$

In particular, if $K^\dagger$ is divisible and M is a differential field extension of K and algebraic over K, then $M^\dagger \cap K = K^\dagger$.

In the next two lemmas $a, b \in K$; distinguishing whether or not $a \in K^\dagger$ helps to describe the solutions to the differential equation $y' + ay = b$:

Lemma 1.2.2. *Suppose $\partial K = K$, and let L be differential field extension of K with $C_L = C$. Suppose also $a \in K^\dagger$. Then for some $y_0 \in K^\times$ and $y_1 \in K$,*

$$\{y \in L : y' + ay = b\} \;=\; \{y \in K : y' + ay = b\} \;=\; Cy_0 + y_1.$$

Proof. Take $y_0 \in K^\times$ with $y_0^\dagger = -a$, so $y_0' + ay_0 = 0$. Twisting $\partial + a \in K[\partial]$ by y_0 (see [ADH, p. 243]) transforms the equation $y' + ay = b$ into $z' = y_0^{-1}b$. This gives $y_1 \in K$ with $y_1' + ay_1 = b$. Using $C_L = C$, these y_0, y_1 have the desired properties. $\square$

Lemma 1.2.3. *Let L be a differential field extension of K with $L^\dagger \cap K = K^\dagger$. Assume $a \notin K^\dagger$. Then there is at most one $y \in L$ with $y' + ay = b$.*

Proof. If y_1, y_2 are distinct solutions in L of the equation $y' + ay = b$, then we have $-a = (y_1 - y_2)^\dagger \in L^\dagger \cap K = K^\dagger$, contradicting $a \notin K^\dagger$. $\square$

Logarithmic derivatives under algebraic closure. *In this subsection K is a differential field.* We describe for real closed K how $K^\dagger$ changes if we pass from K to its algebraic closure. More generally, suppose the underlying field of K is euclidean; in particular, -1 is not a square in K. We equip K with the unique ordering making K an ordered field. For $y = a + bi \in K[i]$ $(a, b \in K)$ we let $|y| \in K^\geqslant$ be such that $|y|^2 = a^2 + b^2$. Then $y \mapsto |y| \colon K[i] \to K^\geqslant$ is an absolute value on $K[i]$, i.e., for all $x, y \in K[i]$,

$$|x| \;=\; 0 \iff x = 0, \qquad |xy| \;=\; |x||y|, \qquad |x + y| \;\leqslant\; |x| + |y|.$$

For $a \in K$ we have $|a| = \max\{a, -a\}$. We have the subgroup

$$S \;:=\; \big\{y \in K[\mathrm{i}] : |y| = 1\big\} \;=\; \big\{a + b\mathrm{i} : a, b \in K,\ a^2 + b^2 = 1\big\}$$

of the multiplicative group $K[\mathrm{i}]^\times$. By an easy computation all elements of $K[\mathrm{i}]$ are squares in $K[\mathrm{i}]$; hence $K[\mathrm{i}]^\dagger$ is 2-divisible. The next lemma describes $K[\mathrm{i}]^\dagger$; it partly generalizes [ADH, 10.7.8]. For $a, b \in K$, put $\operatorname{wr}(a, b) := ab' - a'b$ [ADH, 4.1].

Lemma 1.2.4. *We have $K[\mathrm{i}]^\times = K^> \cdot S$ with $K^> \cap S = \{1\}$, and*

$$K[\mathrm{i}]^\dagger \;=\; K^\dagger \oplus S^\dagger \qquad (\textit{internal direct sum of subgroups of } K[\mathrm{i}]^\dagger).$$

For $a, b \in K$ with $a + b\mathrm{i} \in S$ we have $(a + b\mathrm{i})^\dagger = \operatorname{wr}(a, b)\mathrm{i}$. Thus $K[\mathrm{i}]^\dagger \cap K = K^\dagger$.

Proof. Let $y = a + b\mathrm{i} \in K[\mathrm{i}]^\times$ $(a, b \in K)$, and take $r \in K^>$ with $r^2 = a^2 + b^2$; then $y = r \cdot (y/r)$ with $y/r \in S$. Thus $K[\mathrm{i}]^\times = K^> \cdot S$, and clearly $K^> \cap S = \{1\}$. Hence $K[\mathrm{i}]^\dagger = K^\dagger + S^\dagger$. Suppose $a \in K^\times$, $s \in S$, and $a^\dagger = s^\dagger$; then $a = cs$ with $c \in C_{K[\mathrm{i}]}$, and $C_{K[\mathrm{i}]} = C[\mathrm{i}]$ by [ADH, 4.6.20] and hence $\max\{a, -a\} = |a| = |c| \in C$, so $a \in C$ and thus $a^\dagger = s^\dagger = 0$; therefore the sum is direct. Now if $a, b \in K$ and $|a + b\mathrm{i}| = 1$, then

$$\begin{aligned} (a + b\mathrm{i})^\dagger \;&=\; (a' + b'\mathrm{i})(a - b\mathrm{i}) \\ &=\; (aa' + bb') + (ab' - a'b)\mathrm{i} \\ &=\; \tfrac{1}{2}\big(a^2 + b^2\big)' + (ab' - a'b)\mathrm{i} \;=\; (ab' - a'b)\mathrm{i} \;=\; \operatorname{wr}(a, b)\mathrm{i}. \end{aligned}$$

□

Corollary 1.2.5. *For $y \in K[\mathrm{i}]^\times$ we have $\operatorname{Re}(y^\dagger) = |y|^\dagger$, and the group morphism $y \mapsto \operatorname{Re} y^\dagger \colon K[\mathrm{i}]^\times \to K$ has kernel $C^> S$.*

If K is real closed and $\mathcal{O}$ a convex valuation ring of K, then $\mathcal{O}[\mathrm{i}] = \mathcal{O} + \mathcal{O}\mathrm{i}$ is the unique valuation ring of $K[\mathrm{i}]$ that lies over $\mathcal{O}$, and so $S \subseteq \mathcal{O}[\mathrm{i}]^\times$, hence $y \asymp |y|$ for all $y \in K[\mathrm{i}]^\times$. Thus by [ADH, 10.5.2(i)] and Corollary 1.2.5:

Corollary 1.2.6. *If K is a real closed pre-H-field, then for all $y, z \in K[\mathrm{i}]^\times$,*

$$y \prec z \quad \Longrightarrow \quad \operatorname{Re} y^\dagger < \operatorname{Re} z^\dagger.$$

We also have a useful decomposition for S:

Corollary 1.2.7. *Suppose K is a real closed H-field. Then*

$$S \;=\; S_C \cdot \big(S \cap (1 + \mathcal{o}_{K[\mathrm{i}]})\big)$$

where $S_C := S \cap C[\mathrm{i}]^\times$ and $S \cap (1 + \mathcal{o}_{K[\mathrm{i}]})$ are subgroups of $\mathcal{O}[\mathrm{i}]^\times$.

Proof. The inclusion $\supseteq$ is clear. For the reverse inclusion, let $a, b \in K$, $a^2 + b^2 = 1$ and take the unique $c, d \in C$ with $a - c \prec 1$ and $b - d \prec 1$. Then $c^2 + d^2 = 1$ and $a + b\mathrm{i} \sim c + d\mathrm{i}$, and so $(a + b\mathrm{i})/(c + d\mathrm{i}) \in S \cap (1 + \mathcal{o}_{K[\mathrm{i}]})$. □

Logarithmic derivatives in asymptotic fields. *Let K be an asymptotic field.* If K is henselian and $\boldsymbol{k} := \operatorname{res} K$, then by [ADH, remark before 3.3.33], $K^\times$ is divisible iff the groups $\boldsymbol{k}^\times$ and Γ are both divisible. Recall that in [ADH, 14.2] we defined the $\mathcal{O}$-submodule

$$\mathrm{I}(K) \;=\; \{y \in K : y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\}$$

of K. We have $\partial\mathcal{O} \subseteq \mathrm{I}(K)$, hence $(1 + \mathcal{o})^\dagger \subseteq (\mathcal{O}^\times)^\dagger \subseteq \mathrm{I}(K)$. One easily verifies:

Lemma 1.2.8. *Suppose K is pre-d-valued. If $\mathrm{I}(K) \subseteq \partial K$, then $\mathrm{I}(K) = \partial\mathcal{O}$. If $\mathrm{I}(K) \subseteq K^\dagger$, then $\mathrm{I}(K) = (\mathcal{O}^\times)^\dagger$, with $\mathrm{I}(K) = (1 + \mathcal{o})^\dagger$ if K is d-valued.*

If K is d-valued or K is pre-d-valued without a gap, then

$$\mathrm{I}(K) \ = \ \{y \in K : y \preccurlyeq f' \text{ for some } f \in \smallo\}.$$

For $\phi \in K^\times$ we have $\phi\,\mathrm{I}(K^\phi) = \mathrm{I}(K)$. If K has asymptotic integration and L is an asymptotic extension of K, then $\mathrm{I}(K) = \mathrm{I}(L) \cap K$. The following is [ADH, 14.2.5]:

Lemma 1.2.9. *If K is H-asymptotic, has asymptotic integration, and is 1-linearly newtonian, then it is* d*-valued and* $\partial\mathcal{O} = \mathrm{I}(K) = (1+\smallo)^\dagger$.

We now turn our attention to the condition $\mathrm{I}(K) \subseteq K^\dagger$. If $\mathrm{I}(K) \subseteq K^\dagger$, then also $\mathrm{I}(K^\phi) \subseteq (K^\phi)^\dagger$ for $\phi \in K^\times$, where

$$(K^\phi)^\dagger \ := \ \{\phi^{-1} f'/f : \, f \in K^\times\} \ = \ \phi^{-1} K^\dagger.$$

By [ADH, Section 9.5 and 10.4.3]:

Lemma 1.2.10. *Let K be of H-type. If K is* d*-valued, or pre-*d*-valued without a gap, then K has an immediate henselian asymptotic extension L with* $\mathrm{I}(L) \subseteq L^\dagger$.

Corollary 1.2.11. *Suppose K has asymptotic integration. Let L be an asymptotic field extension of K such that $L^\times = K^\times C_L^\times(1+\smallo_L)$. Then $L^\dagger = K^\dagger + (1+\smallo_L)^\dagger$, and if $\mathrm{I}(K) \subseteq K^\dagger$, then $L^\dagger \cap K = K^\dagger$.*

Proof. Let $f \in L^\times$, and take $b \in K^\times$, $c \in C_L^\times$, $g \in \smallo_L$ with $f = bc(1+g)$; then $f^\dagger = b^\dagger + (1+g)^\dagger$, showing $L^\dagger = K^\dagger + (1+\smallo_L)^\dagger$. Next, suppose $\mathrm{I}(K) \subseteq K^\dagger$, let b, c, f, g be as before, and assume $a := f^\dagger \in K$; then

$$a - b^\dagger \in (1+\smallo_L)^\dagger \cap K \ \subseteq \ \mathrm{I}(L) \cap K \ = \ \mathrm{I}(K) \ \subseteq \ K^\dagger$$

and hence $a \in K^\dagger$. This shows $L^\dagger \cap K = K^\dagger$. $\square$

Two cases where the assumption on L in Corollary 1.2.11 is satisfied: (1) L is an immediate asymptotic field extension of K, because then $L^\times = K^\times(1+\smallo_L)$; and (2) L is a d-valued field extension of K with $\Gamma = \Gamma_L$.

If F is a henselian valued field of residue characteristic 0, then clearly the subgroup $1+\smallo_F$ of $F^\times$ is divisible. Hence, if K and L are as in Corollary 1.2.11 and in addition $K^\dagger$ is divisible and L is henselian, then $L^\dagger$ is divisible.

Example 1.2.12. Let C be a field of characteristic 0 and Q be a subgroup of $\mathbb{Q}$ with $1 \in Q$. The Hahn field $C(\!(t^Q)\!) = C[[x^Q]]$, with $x = t^{-1}$, is given the natural derivation with $c' = 0$ for all $c \in C$ and $x' = 1$: this derivation is defined by

$$\left(\sum_{q\in Q} c_q x^q\right)' \ := \ \sum_{q \in Q} q c_q x^{q-1} \qquad (\text{all } c_q \in C).$$

Then $C(\!(t^Q)\!)$ has constant field C, and is d-valued of H-type. Thus $K := C(\!(t^Q)\!)$ satisfies $\mathrm{I}(K) \subseteq K^\dagger$ by Lemma 1.2.10. Hence by Lemma 1.2.8,

$$\mathrm{I}(K) \ = \ (1+\smallo)^\dagger \ = \ \big\{f \in K : \, f \prec x^\dagger = t\big\} \ = \ \smallo\, t.$$

It follows easily that $K^\dagger = Qt \oplus \mathrm{I}(K)$ (internal direct sum of subgroups of $K^\dagger$) and thus $(K^t)^\dagger = Q \oplus \smallo \subseteq \mathcal{O}$. In particular, if $Q = \mathbb{Z}$ (so $K = C(\!(t)\!)$), then $(K^t)^\dagger = \mathbb{Z} \oplus tC[[t]]$. Moreover, if $L := \mathrm{P}(C) \subseteq C(\!(t^{\mathbb{Q}})\!)$ is the differential field of Puiseux series over C, then $(L^t)^\dagger = \mathbb{Q} \oplus \smallo_L$.

The real closed case. *In this subsection H is a real closed asymptotic field whose valuation ring $\mathcal{O}$ is convex with respect to the ordering of H.* (In later use H is often a Hardy field, which is why we use the letter H here.) The valuation ring of the asymptotic field extension $K = H[\mathrm{i}]$ of H is then $\mathcal{O}_K = \mathcal{O} + \mathcal{O}\mathrm{i}$, from which we obtain $\mathrm{I}(K) = \mathrm{I}(H) \oplus \mathrm{I}(H)\mathrm{i}$. Let

$$S \;:=\; \big\{y \in K : |y| = 1\big\}, \qquad W \;:=\; \big\{\mathrm{wr}(a,b) : a,b \in H,\ a^2+b^2 = 1\big\},$$

so S is a subgroup of $\mathcal{O}_K^\times$ with $S^\dagger = W i$ and $K^\dagger = H^\dagger \oplus W i$ by Lemma 1.2.4. Since $\partial\mathcal{O} \subseteq \mathrm{I}(H)$, we have $W \subseteq \mathrm{I}(H)$, and thus: $W = \mathrm{I}(H) \iff \mathrm{I}(H)\mathrm{i} \subseteq K^\dagger$.

Lemma 1.2.13. *The following are equivalent:*

(i) $\mathrm{I}(K) \subseteq K^\dagger$*;*
(ii) $W = \mathrm{I}(H) \subseteq H^\dagger$*.*

Proof. Assume (i). Then $\mathrm{I}(H)\,\mathrm{i} \subseteq \mathrm{I}(K) \subseteq K^\dagger$, so $W = \mathrm{I}(H)$ by the equivalence preceding the lemma. Also $\mathrm{I}(H) \subseteq \mathrm{I}(K)$ and $K^\dagger \cap H = H^\dagger$ (by Lemma 1.2.4), hence $\mathrm{I}(H) \subseteq H^\dagger$, so (ii) holds. For the converse, assume (ii). Then

$$\mathrm{I}(K) \;=\; \mathrm{I}(H) \oplus \mathrm{I}(H)\mathrm{i} \;\subseteq\; H^\dagger \oplus W\mathrm{i} \;=\; K^\dagger. \qquad \square$$

Applying now Lemma 1.2.9 we obtain:

Corollary 1.2.14. *If H is H-asymptotic and has asymptotic integration, and K is* 1*-linearly newtonian, then K is* d*-valued and $\mathrm{I}(K) \subseteq K^\dagger$; in particular, $W = \mathrm{I}(H)$.*

Corollary 1.2.15. *Suppose H has asymptotic integration and $W = \mathrm{I}(H)$. Let F be a real closed asymptotic extension of H whose valuation ring is convex. Then*

$$F[\mathrm{i}]^\dagger \cap K \;=\; (F^\dagger \cap H) \oplus \mathrm{I}(H)\mathrm{i}.$$

If in addition $H^\dagger = H$, then $F[\mathrm{i}]^\dagger \cap K = H \oplus \mathrm{I}(H)\mathrm{i} = K^\dagger$.

Proof. We have

$$F^\dagger \cap H \subseteq F[\mathrm{i}]^\dagger \cap K \quad \text{and} \quad \mathrm{I}(H)\mathrm{i} = W\mathrm{i} \subseteq K^\dagger \cap H\mathrm{i} \subseteq F[\mathrm{i}]^\dagger \cap K,$$

so $(F^\dagger \cap H) \oplus \mathrm{I}(H)\mathrm{i} \subseteq F[\mathrm{i}]^\dagger \cap K$. For the reverse inclusion, $F[\mathrm{i}]^\dagger = F^\dagger \oplus W_F\mathrm{i}$, with

$$W_F \;:=\; \big\{\mathrm{wr}(a,b) : a,b \in F,\ a^2+b^2 = 1\big\} \;\subseteq\; \mathrm{I}(F),$$

hence

$$\begin{aligned} F[\mathrm{i}]^\dagger \cap K \;&=\; (F^\dagger \cap H) \oplus (W_F \cap H)\mathrm{i} \\ &\subseteq\; (F^\dagger \cap H) \oplus \big(\mathrm{I}(F) \cap H\big)\mathrm{i} \;=\; (F^\dagger \cap H) \oplus \mathrm{I}(H)\mathrm{i}, \end{aligned}$$

using $\mathrm{I}(F) \cap H = \mathrm{I}(H)$, a consequence of H having asymptotic integration. If $H^\dagger = H$ then clearly $F^\dagger \cap H = H$, hence $F[\mathrm{i}]^\dagger \cap K = K^\dagger$. $\square$

Trigonometric closure. In this subsection H is a real closed H-field. Let $\mathcal{O}$ be its valuation ring and $\smallo$ the maximal ideal of $\mathcal{O}$. The algebraic closure $K = H[\mathrm{i}]$ of H is a d-valued H-asymptotic extension with valuation ring $\mathcal{O}_K = \mathcal{O} + \mathcal{O}\mathrm{i}$. We have the "complex conjugation" automorphism $z = a + b\mathrm{i} \mapsto \overline{z} = a - b\mathrm{i}$ $(a, b \in H)$ of the valued differential field K. For such z, a, b we have

$$|z| \;=\; \sqrt{z\overline{z}} \;=\; \sqrt{a^2+b^2} \;\in\; H^{\geqslant}.$$

Lemma 1.2.16. *Suppose $\theta \in H$ and $\theta'\mathrm{i} \in K^\dagger$. Then $\theta' \in \partial\smallo$, and there is a unique $y \sim 1$ in K such that $y^\dagger = \theta'\mathrm{i}$. For this y we have $|y| = 1$, so $y^{-1} = \overline{y}$.*

Proof. From $\theta'\mathrm{i} \in K^\dagger$ we get $\theta' \in W \subseteq \mathrm{I}(H)$, so $\theta \preccurlyeq 1$, hence $\theta' \in \partial\mathcal{O} = \partial\mathcal{o}$. Let $z \in K^\times$ and $z^\dagger = \theta'\mathrm{i}$. Then $\operatorname{Re} z^\dagger = 0$, so by Corollaries 1.2.5 and 1.2.7 we have $z = cy$ with $c \in C_K^\times$ and $y \in S \cap (1 + \mathcal{o}_K)$ where $S = \{a \in K : |a| = 1\}$. Hence $y \sim 1$, $|y| = 1$, and $y^\dagger = \theta'\mathrm{i}$. If also $y_1 \in K$ and $y_1 \sim 1$, $y_1^\dagger = \theta'\mathrm{i}$, then $y_1 = c_1 y$ with $c_1 \in C_K^\times$, so $c_1 = 1$ in view of $y \sim y_1$. $\square$

By [ADH, 10.4.3], if y in an H-asymptotic extension L of K satisfies $y \sim 1$ and $y^\dagger \in \partial\mathcal{o}_K$, then the asymptotic field $K(y) \subseteq L$ is an immediate extension of K, and so is any algebraic asymptotic extension of $K(y)$.

Call H **trigonometrically closed** if for all $\theta \prec 1$ in H there is a (necessarily unique) $y \in K$ such that $y \sim 1$ and $y^\dagger = \theta'\mathrm{i}$. (By convention "trigonometrically closed" includes "real closed".) For such θ and y we think of y as $\mathrm{e}^{\mathrm{i}\theta}$ and accordingly of the elements $\frac{y+\overline{y}}{2} = \frac{y+y^{-1}}{2}$ and $\frac{y-\overline{y}}{2\mathrm{i}} = \frac{y-y^{-1}}{2\mathrm{i}}$ of H as $\cos\theta$ and $\sin\theta$; this explains the terminology. By Lemma 1.2.16 the restrictions $\theta \prec 1$ and $y \sim 1$ are harmless. Our aim in this subsection is to construct a canonical trigonometric closure of H.

Our interest in this notion comes from the condition $\mathrm{I}(K) \subseteq K^\dagger$, which appears as a natural hypothesis at many points in Part 4, especially in Section 4.4). Note that if $\mathrm{I}(K) \subseteq K^\dagger$, then H is trigonometrically closed. As a partial converse, if $\mathrm{I}(H) \subseteq H^\dagger \cap \partial H$ and H is trigonometrically closed, then $\mathrm{I}(K) \subseteq K^\dagger$; this is an easy consequence of $\mathrm{I}(K) = \mathrm{I}(H) + \mathrm{I}(H)\mathrm{i}$. Thus for Liouville closed H we have:

$$H \text{ is trigonometrically closed} \iff \mathrm{I}(K) \subseteq K^\dagger.$$

Note also that for trigonometrically closed H there is no y in any H-asymptotic extension of K such that $y \notin K$, $y \sim 1$, and $y^\dagger \in (\partial\mathcal{o})\mathrm{i}$. If H is Schwarz closed, then H is trigonometrically closed by the next lemma:

Lemma 1.2.17. *Suppose H is Liouville closed and $\omega(H)$ is downward closed. Then H is trigonometrically closed.*

Proof. Let $0 \neq \theta \prec 1$ in H. By Lemma 1.2.16 it suffices to show that then $\theta'\mathrm{i} \in K^\dagger$. Note that $h := \theta' \in \mathrm{I}(H)^{\neq}$; we arrange $h > 0$. Now

$$f \;:=\; \omega(-h^\dagger) + 4h^2 \;=\; \sigma(2h), \qquad 2h \in H^> \cap \mathrm{I}(H),$$

hence $2h \in H^> \setminus \Gamma(H)$ by [ADH, 11.8.19]. So $f \in \omega(H)^\downarrow = \omega(H)$ by [ADH, 11.8.31], and thus $\dim_{C_H} \ker 4\partial^2 + f \geqslant 1$ by [ADH, p. 258]. Put $A := \partial^2 - h^\dagger\partial + h^2 \in H[\partial]$. The isomorphism $y \mapsto y\sqrt{h}\colon \ker(4\partial^2 + f) \to \ker A$ of C_H-linear spaces [ADH, 5.1.13] then yields an element of $\ker^{\neq} A$ that for suggestiveness we denote by $\cos\theta$. Put $\sin\theta := -(\cos\theta)'/h$. Then

$$\begin{aligned}(\sin\theta)' \;&=\; -(\cos\theta)''/h + (\cos\theta)'h^\dagger/h \\ &=\; \big(-h^\dagger(\cos\theta)' + h^2\cos\theta\big)/h + (\cos\theta)'h^\dagger/h \;=\; h\cos\theta\end{aligned}$$

and thus $y^\dagger = \theta'\mathrm{i}$ for $y := \cos\theta + \mathrm{i}\sin\theta \in K^\times$. $\square$

If H is H-closed, then H is Schwarz closed by [ADH, 14.2.20], and thus trigonometrically closed. Using also Lemma 1.2.13 and remarks preceding it this yields:

Corollary 1.2.18. *If H is H-closed, then* $\mathrm{I}(K) \subseteq K^\dagger = H \oplus \mathrm{I}(H)\mathrm{i}$.

Suppose now that H is *not* trigonometrically closed; so we have $\theta \prec 1$ in H with $\theta' i \notin K^\dagger$. Then [ADH, 10.4.3] provides an immediate asymptotic extension $K(y)$ of K with $y \sim 1$ and $y^\dagger = \theta' i$. To simplify notation and for suggestiveness we set

$$\cos\theta \ := \ \frac{y + y^{-1}}{2}, \qquad \sin\theta \ := \ \frac{y - y^{-1}}{2i},$$

so $y = \cos\theta + i\sin\theta$ and $(\cos\theta)^2 + (\sin\theta)^2 = 1$. Moreover $(\cos\theta)' = -\theta'\sin\theta$ and $(\sin\theta)' = \theta'\cos\theta$. It follows that $H^+ := H(\cos\theta, \sin\theta)$ is a differential subfield of $K(y)$ with $K(y) = H^+[i]$, and thus H^+, as a valued differential subfield of $H(y)$, is an asymptotic extension of H.

Lemma 1.2.19. *H^+ is an immediate extension of H.*

Proof. Since $(y^{-1})^\dagger = -\theta' i$, the uniqueness property stated in [ADH, 10.4.3] allows us to extend the complex conjugation automorphism of K (which is the identity on H and sends i to $-i$) to an automorphism σ of the valued differential field $K(y)$ such that $\sigma(y) = y^{-1}$. Then $\sigma(\cos\theta) = \cos\theta$ and $\sigma(\sin\theta) = \sin\theta$, so $H^+ = \operatorname{Fix}(\sigma)$. Let $\boldsymbol{k}$ be the residue field of H; so $\boldsymbol{k}[\operatorname{res} i]$ is the residue field of K and of its immediate extension $K(y)$. Now $\sigma(\mathcal{O}_{K(y)}) = \mathcal{O}_{K(y)}$, so σ induces an automorphism of this residue field $\boldsymbol{k}[\operatorname{res} i]$ which is the identity on $\boldsymbol{k}$ and sends $\operatorname{res} i$ to $-\operatorname{res} i$. Hence $\operatorname{res} i$ does not lie in the residue field of H^+, so this residue field is just $\boldsymbol{k}$. $\square$

Equip H^+ with the unique field ordering making it an ordered field extension of H in which $\mathcal{O}_{H^+}$ is convex; see [ADH, 10.5.8]. Then H^+ is an H-field, and its real closure is an immediate real closed H-field extension of H.

Lemma 1.2.20. *The H-field H^+ embeds uniquely over H into any trigonometrically closed H-field extension of H.*

Proof. Let H^* be a trigonometrically closed H-field extension of H. Take the unique $z \sim 1$ in H^* such that $z^\dagger = \theta' i$. Then any H-field embedding $H^+ \to H^*$ over H extends to a valued differential field embedding $H^+[i] = K(y) \to H^*[i]$ sending $i \in K$ to $i \in H^*[i]$, and this extension must send y to z. Hence there is at most one H-field embedding $H^+ \to H^*$ over H. For the existence of such an embedding, the uniqueness properties from [ADH, 10.4.3] yield a valued differential field embedding $K(y) \to H^*[i]$ over H sending $i \in K$ to $i \in H^*[i]$ and y to z. This embedding maps H^+ into H^*. The uniqueness property of the ordering on H^+ shows that this embedding restricts to an H-field embedding $H^+ \to H^*$. $\square$

By iterating the extension step that leads from H to H^+, alternating it with taking real closures, and taking unions at limit stages we obtain:

Proposition 1.2.21. *H has a trigonometrically closed H-field extension H^{trig} that embeds uniquely over H into any trigonometrically closed H-field extension of H.*

This is an easy consequence of Lemma 1.2.20. Note that the universal property stated in Proposition 1.2.21 determines H^{trig} up-to-unique-isomorphism of H-fields over H. We refer to such H^{trig} as the **trigonometric closure** of H. Note that H^{trig} is an immediate extension of H, by Lemma 1.2.19, and that $H^{\text{trig}}[i]$ is a Liouville extension of K and thus of H.

A **trigonometric extension** of H is a real closed H-field extension E of H such that for all $a \in E$ there are real closed H-subfields $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n$ of E such that

(1) $H_0 = H$ and $a \in H_n$;
(2) for $j = 0,\dots,n-1$ there are $\theta_j \in H_j$ and $y_j \in H_{j+1}[\mathrm{i}] \subseteq E[\mathrm{i}]$ such that $y_j \sim 1$, $\theta_j'\mathrm{i} = y_j^\dagger$, and $H_{j+1}[\mathrm{i}]$ is algebraic over $H_j[\mathrm{i}](y_j)$.

If E is a trigonometric extension of H, then E is an immediate extension of H and $E[\mathrm{i}]$ is an immediate Liouville extension of K and thus of H. The next lemma states some further easy consequences of the definition above:

Lemma 1.2.22. *If E is a trigonometric extension of H, then E is a trigonometric extension of any real closed H-subfield $F \supseteq H$ of E. If H is trigonometrically closed, then H has no proper trigonometric extension.*

Induction on m shows that if E is a trigonometric extension of H, then for any $a_1,\dots,a_m \in E$ there are real closed H-subfields $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n$ of E such that $H_0 = H$, $a_1,\dots,a_m \in H_n$ and (2) above holds. This helps in proving:

Corollary 1.2.23. *A trigonometric extension of a trigonometric extension of H is a trigonometric extension of H, and H^{trig} is a trigonometric extension of H.*

Asymptotic fields of Hardy type. Let (Γ,ψ) be an asymptotic couple, $\Psi := \psi(\Gamma^{\neq})$, and let γ, δ range over Γ. Recall that $[\gamma]$ denotes the archimedean class of γ [ADH, 2.4]. Following [24, Section 3] we say that (Γ,ψ) is of **Hardy type** if for all $\gamma,\delta \neq 0$ we have $[\gamma] \leqslant [\delta] \iff \psi(\gamma) \geqslant \psi(\delta)$. Note that then (Γ,ψ) is of H-type, and ψ induces an order-reversing bijection $[\Gamma^{\neq}] \to \Psi$. If Γ is archimedean, then (Γ,ψ) is of Hardy type. If (Γ,ψ) is of Hardy type, then so is $(\Gamma,\psi+\delta)$ for each δ. We also say that an asymptotic field is of Hardy type if its asymptotic couple is. Every asymptotic subfield and every compositional conjugate of an asymptotic field of Hardy type is also of Hardy type. Moreover, every Hardy field is of Hardy type [ADH, 9.1.11]. Let now Δ be a convex subgroup of Γ. Note that Δ contains the archimedean class $[\delta]$ of each $\delta \in \Delta$. Hence, if $\delta \in \Delta^{\neq}$ and $\gamma \notin \Delta$, then $[\delta] < [\gamma]$ and thus:

Lemma 1.2.24. *If (Γ,ψ) is of Hardy type and $\gamma \notin \Delta$, $\delta \in \Delta^{\neq}$, then $\psi(\gamma) < \psi(\delta)$.*

Corollary 1.2.25. *Suppose (Γ,ψ) is of Hardy type with small derivation, $\gamma,\delta \neq 0$, $\psi(\delta) \leqslant 0$, and $[\gamma'] > [\delta]$. Then $\psi(\gamma) < \psi(\delta)$.*

Proof. Let Δ be the smallest convex subgroup of Γ with $\delta \in \Delta$; then $\gamma' \notin \Delta$, and $\psi(\delta) \in \Delta$ by [ADH, 9.2.10(iv)]. Thus $\gamma \notin \Delta$ by [ADH, 9.2.25]. □

In [1, Section 7] we say that an H-field H is *closed under powers* if for all $c \in C$ and $f \in H^\times$ there is a $y \in H^\times$ with $y^\dagger = cf^\dagger$. (Think of y as f^c.) Thus if H is Liouville closed, then H is closed under powers. *In the rest of this subsection we let H be an H-field closed under powers, with asymptotic couple (Γ,ψ) and constant field C.* We recall some basic facts from [1, Section 7]. First, we can make the value group Γ into an ordered vector space over the constant field C:

Lemma 1.2.26. *For all $c \in C$ and $\gamma = vf$ with $f \in H^\times$ and each $y \in H^\times$ with $y^\dagger = cf^\dagger$, the element $vy \in \Gamma$ only depends on (c,γ) (not on the choice of f and y), and is denoted by $c\cdot\gamma$. The scalar multiplication $(c,\gamma) \mapsto c\cdot\gamma\colon C\times\Gamma \to \Gamma$ makes Γ into an ordered vector space over the ordered field C.*

Let G be an ordered vector space over the ordered field C. From [ADH, 2.4] recall that the C-archimedean class of $a \in G$ is defined as

$$[a]_C := \left\{ b \in G : \tfrac{1}{c}|a| \leqslant |b| \leqslant c|a| \text{ for some } c \in C^{>} \right\}.$$

Thus if $C = \mathbb{Q}$, then $[a]_{\mathbb{Q}}$ is just the archimedean class $[a]$ of $a \in G$. Moreover, if C^* is an ordered subfield of C, then $[a]_{C^*} \subseteq [a]_C$ for each $a \in G$, with equality if C^* is cofinal in C. Hence if C is archimedean, then $[a] = [a]_C$ for all $a \in G$. Put $[G]_C := \big\{[a]_C : a \in G\big\}$ and linearly order $[G]_C$ by

$$[a]_C < [b]_C \quad :\Longleftrightarrow \quad [a]_C \neq [b]_C \text{ and } |a| < |b|.$$

Thus $[G]_C$ has smallest element $[0]_C = \{0\}$. We also set $[G^{\neq}]_C := [G]_C \setminus \big\{[0]_C\big\}$. From [1, Proposition 7.5] we have:

Proposition 1.2.27. *For all $\gamma, \delta \neq 0$ we have*

$$[\gamma]_C \leqslant [\delta]_C \quad \Longleftrightarrow \quad \psi(\gamma) \geqslant \psi(\delta).$$

Hence ψ induces an order-reversing bijection $[\Gamma^{\neq}]_C \to \Psi = \psi(\Gamma^{\neq})$.

Proposition 1.2.27 yields:

Corollary 1.2.28. *H is of Hardy type $\Longleftrightarrow$ $[\gamma] = [\gamma]_C$ for all γ. Hence if C is archimedean, then H is of Hardy type; if $\Gamma \neq \{0\}$, then the converse also holds.*

1.3. λ-freeness and ω-freeness

This section contains preservation results for the important properties of λ-*freeness* and ω-*freeness* from [ADH]. Let K be an ungrounded H-asymptotic field such that $\Gamma \neq \{0\}$, and as in [ADH, 11.5], fix a logarithmic sequence (ℓ_ρ) for K and define the pc-sequences $(\lambda_\rho) = (-\ell_\rho^{\dagger\dagger})$ and $(\omega_\rho) = \big(\omega(\lambda_\rho)\big)$ in K, where $\omega(z) := -2z' - z^2$. Recall that K is λ-*free* iff (λ_ρ) does not have a pseudolimit in K, and K is ω-*free* iff (ω_ρ) does not have a pseudolimit in K. If K is ω-free, then K is λ-free. We refer to [ADH, 11.6, 11.7] for this and other basic facts about λ-freeness and ω-freeness used below. As in [ADH], L being λ-free or ω-free includes L being an ungrounded H-asymptotic field with $\Gamma_L \neq \{0\}$.

Preserving λ-freeness and ω-freeness. *In this subsection K is an ungrounded H-asymptotic field with $\Gamma \neq \{0\}$, and (ℓ_ρ), (λ_ρ), (ω_ρ) are as above.* If K has a λ-free H-asymptotic field extension L such that $\Gamma^<$ is cofinal in $\Gamma_L^<$, then K is λ-free, and similarly with "ω-free" in place of "λ-free" [ADH, remarks after 11.6.4, 11.7.19]. The property of ω-freeness is very robust; indeed, by [ADH, 13.6.1]:

Theorem 1.3.1. *If K is ω-free and L is a pre-*d*-valued* d*-algebraic H-asymptotic extension of K, then L is ω-free and $\Gamma^<$ is cofinal in $\Gamma_L^<$.*

In contrast, λ-freeness is more delicate: Theorem 1.3.1 fails with "λ-free" in place of "ω-free", as the next example shows.

Example 1.3.2. The H-field $K = \mathbb{R}\langle\omega\rangle$ from [ADH, 13.9.1] is λ-free, but its H-field extension $L = \mathbb{R}\langle\lambda\rangle$ is not, and this extension is d-algebraic: $2\lambda' + \lambda^2 + \omega = 0$.

In the rest of this subsection we consider cases where parts of Theorem 1.3.1 do hold. Recall from [ADH, 11.6.8] that if K is λ-free, then K has (rational) asymptotic integration, and K is λ-free iff its algebraic closure is λ-free. Moreover, λ-freeness is preserved under adjunction of constants:

Proposition 1.3.3. *Suppose K is λ-free and $L = K(D)$ is an H-asymptotic extension of K with $D \supseteq C$ a subfield of C_L. Then L is λ-free with $\Gamma_L = \Gamma$.*

We are going to deduce this from the next three lemmas. Recall that K is pre-d-valued, by [ADH, 10.1.3]. Let $\mathrm{dv}(K)$ be the d-valued hull of K (see [ADH, 10.3]).

Lemma 1.3.4. *Suppose K is λ-free. Then $L := \mathrm{dv}(K)$ is λ-free and $\Gamma_L = \Gamma$.*

Proof. The first statement is [13, Theorem 10.2], and the second statement follows from [ADH, 10.3.2(i)]. ☐

If $L = K(D)$ is a differential field extension of K with $D \supseteq C$ a subfield of C_L, then $D = C_L$, and K and D are linearly disjoint over C [ADH, 4.6.20]. If K is d-valued and $L = K(D)$ is an H-asymptotic extension of K with $D \supseteq C$ a subfield of C_L, then L is d-valued and $\Gamma_L = \Gamma$ [ADH, 10.5.15].

Lemma 1.3.5. *Suppose K is* d*-valued and λ-free, and $L = K(D)$ is an H-asymptotic extension of K with $D \supseteq C$ a subfield of C_L. Then L is λ-free.*

Proof. First, (λ_ρ) is of transcendental type over K: otherwise, [ADH, 3.2.7] would give an algebraic extension of K that is not λ-free. Next, our logarithmic sequence (ℓ_ρ) for K remains a logarithmic sequence for L.

Zorn and the $\forall\exists$-form of the λ-freeness axiom [ADH, 1.6.1(ii)] reduce us to the case $D = C(d)$, $d \notin C$, d transcendental over K, so $L = K(d)$. Suppose L is not λ-free. Then $\lambda_\rho \rightsquigarrow \lambda \in L$, and such λ is transcendental over K and gives an immediate extension $K(\lambda)$ of K by [ADH, 3.2.6]. Hence L is algebraic over $K(\lambda)$, so $\operatorname{res} L$ is algebraic over $\operatorname{res} K(\lambda) = \operatorname{res} K \cong C$ and thus d is algebraic over C, a contradiction. ☐

Lemma 1.3.6. *Suppose K is λ-free and L is an H-asymptotic extension of K, where $L = K(d)$ with $d \in C_L$. Then L is pre*-d*-valued.*

Proof. Let L^{a} be an algebraic closure of the H-asymptotic field L, and let K^{a} be the algebraic closure of K inside L^{a}. Then K^{a} is pre-d-valued by [ADH, 10.1.22]. Replacing K, L by K^{a}, $K^{\mathrm{a}}(d)$ we arrange that K is algebraically closed. We may assume $d \notin C$, so d is transcendental over K by [ADH, 4.1.1, 4.1.2].

Suppose first that $\operatorname{res}(d) \in \operatorname{res}(K) \subseteq \operatorname{res}(L)$, and take $b \in \mathcal{O}$ such that $y := b - d \prec 1$. Then $b' \notin \partial\smallo$: otherwise $y' = b' = \delta'$ with $\delta \in \smallo$, so $y = \delta \in K$ and hence $d \in K$, a contradiction. Also $vb' \in (\Gamma^{>})'$: otherwise $vb' < (\Gamma^{>})'$, by [ADH, 9.2.14], and vb' would be a gap in K, contradicting λ-freeness of K. Hence $L = K(y)$ is pre-d-valued by [ADH, 10.2.4, 10.2.5(iii)] applied to $s := b'$.

If $\operatorname{res}(d) \notin \operatorname{res}(K)$, then $\operatorname{res}(d)$ is transcendental over $\operatorname{res}(K)$ by [ADH, 3.1.17], hence $\Gamma_L = \Gamma$ by [ADH, 3.1.11], and so L has asymptotic integration and thus is pre-d-valued by [ADH, 10.1.3]. ☐

Proof of Proposition 1.3.3. By Zorn we reduce to the case $L = K(d)$ with $d \in C_L$. Then L is pre-d-valued by Lemma 1.3.6. By Lemma 1.3.4, the d-valued hull $K_1 := \mathrm{dv}(K)$ of K is λ-free with $\Gamma_{K_1} = \Gamma$, and by the universal property of d-valued hulls we may arrange that K_1 is a d-valued subfield of $L_1 := \mathrm{dv}(L)$ [ADH, 10.3.1]. The proof of [ADH, 10.3.1] gives $L_1 = L(E)$ where $E = C_{L_1}$, and so $L_1 = K_1(E)$. Hence by Lemma 1.3.5 and the remarks preceding it, L_1 is λ-free with $\Gamma_{L_1} = \Gamma_{K_1} = \Gamma$. Thus L is λ-free with $\Gamma_L = \Gamma$. ☐

Lemma 1.3.7. *Let H be a λ-free real closed H-field. Then the trigonometric closure H^{trig} of H is λ-free.*

Proof. We show that H^+ as in Lemma 1.2.19 is λ-free. There $H^+[i] = K(y)$ where K is the H-asymptotic extension $H[i]$ of H and $y \sim 1$, $y^\dagger \notin K^\dagger$, $y^\dagger \in i\partial\mathcal{O}_H$. Then K is λ-free, so $K(y)$ is λ-free by [13, Proposition 7.2], hence H^+ is λ-free. $\square$

In Example 1.3.2 we have a λ-free K and an H-asymptotic extension L of K that is not λ-free, with $\operatorname{trdeg}(L|K) = 1$. The next proposition shows that the second part of the conclusion of Theorem 1.3.1 nevertheless holds for such K, L.

Proposition 1.3.8. *The following are equivalent:*

(i) *K has rational asymptotic integration;*

(ii) *for every H-asymptotic extension L of K with $\operatorname{trdeg}(L|K) \leqslant 1$ we have that $\Gamma^<$ is cofinal in $\Gamma_L^<$.*

Proof. For (i) $\Rightarrow$ (ii), assume (i), and let L be an H-asymptotic extension of K with $\operatorname{trdeg}(L|K) \leqslant 1$. Towards showing that $\Gamma^<$ is cofinal in $\Gamma_L^<$ we can arrange that K and L are algebraically closed. Suppose towards a contradiction that $\gamma \in \Gamma_L$ and $\Gamma^< < \gamma < 0$. Then $\Psi < \gamma' < (\Gamma^>)'$, and so Γ is dense in $\Gamma + \mathbb{Q}\gamma'$ by [ADH, 2.4.16, 2.4.17], in particular, $\gamma \notin \Gamma + \mathbb{Q}\gamma'$. Thus γ, γ' are $\mathbb{Q}$-linearly independent over Γ, which contradicts $\operatorname{trdeg}(L|K) \leqslant 1$ by [ADH, 3.1.11].

As to (ii) $\Rightarrow$ (i), we prove the contrapositive, so assume K does not have rational asymptotic integration. We arrange again that K is algebraically closed. Then K has a gap vs with $s \in K^\times$, and so [ADH, 10.2.1 and its proof] gives an H-asymptotic extension $K(y)$ of K with $y' = s$ and $0 < vy < \Gamma^>$. $\square$

Recall from [ADH, 11.6] that Liouville closed H-fields are λ-free. To prove the next result we also use Gehret's theorem [13, Theorem 12.1(1)] that an H-field H has up to isomorphism over H exactly one Liouville closure iff H is grounded or λ-free. Here *isomorphism* means of course *isomorphism of H-fields*, and likewise with the embeddings referred to in the next result:

Proposition 1.3.9. *Let H be a grounded or λ-free H-field. Then H has a trigonometrically closed and Liouville closed H-field extension H^{tl} that embeds over H into any trigonometrically closed Liouville closed H-field extension of H.*

Proof. We build real closed H-fields $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ as follows: H_0 is a real closure of H, and, recursively, H_{2n+1} is a Liouville closure of H_{2n}, and $H_{2n+2} := H_{2n+1}^{\text{trig}}$ is the trigonometric closure of H_{2n+1}. Then $H^* := \bigcup_n H_n$ is a trigonometrically closed Liouville closed H-field extension of H. Induction using Lemma 1.3.7 shows that all H_n with $n \geqslant 1$ are λ-free, and that H_{2n} has for all n up to isomorphism over H a unique Liouville closure. Given any trigonometrically closed Liouville closed H-field extension E of H we then use the embedding properties of *Liouville closure* and *trigonometric closure* to construct by a similar recursion embeddings $H_n \to E$ that extend to an embedding $H^* \to E$ over H. $\square$

For H as in Proposition 1.3.9, the H^* constructed in its proof is minimal: Let $E \supseteq H$ be any trigonometrically closed Liouville closed H-subfield of H^*. Then induction on n yields $H_n \subseteq E$ for all n, so $E = H^*$. It follows that any H^{tl} as in Proposition 1.3.9 is isomorphic over H to H^*, and we refer to such H^{tl} as a **trigonometric-Liouville closure** of H. Here are some useful facts about H^{tl}:

Corollary 1.3.10. *Let H be a λ-free H-field. Then $C_{H^{\text{tl}}}$ is a real closure of C_H, the H-asymptotic extension $K^{\text{tl}} := H^{\text{tl}}[i]$ of H^{tl} is a Liouville extension of H*

with $\mathrm{I}(K^{\mathrm{tl}}) \subseteq (K^{\mathrm{tl}})^\dagger$*, and* $\Gamma_H^<$ *is cofinal in* $\Gamma_{H^{\mathrm{tl}}}^<$*. Moreover,*

$$H \text{ is } \omega\text{-free} \iff H^{\mathrm{tl}} \text{ is } \omega\text{-free}.$$

Proof. The construction of H^* in the proof of Proposition 1.3.9 gives that C_{H^*} is a real closure of C_H, and that the H-asymptotic extension $K^* := H^*[i]$ of H^* is a Liouville extension of H with $\mathrm{I}(K^*) \subseteq (K^*)^\dagger$. Induction using Lemma 1.3.7 and Proposition 1.3.8 shows that H_n is λ-free and $\Gamma_H^<$ is cofinal in $\Gamma_{H_n}^<$, for all n, so $\Gamma_H^<$ is cofinal in $\Gamma_{H^*}^<$.

The final equivalence follows from Theorem 1.3.1 and a remark preceding it. □

Proposition 1.3.8 and [ADH, remarks after 11.6.4 and after 11.7.19] yield:

Corollary 1.3.11. *Suppose K has rational asymptotic integration, and let L be an H-asymptotic extension of K with* $\operatorname{trdeg}(L|K) \leqslant 1$*. If L is λ-free, then so is K, and if L is ω-free, then so is K.*

We also have a similar characterization of λ-freeness:

Proposition 1.3.12. *The following are equivalent:*

(i) *K is λ-free;*
(ii) *every H-asymptotic extension L of K with* $\operatorname{trdeg}(L|K) \leqslant 1$ *has asymptotic integration.*

Proof. Assume K is λ-free, and let L be an H-asymptotic extension of K such that $\operatorname{trdeg}(L|K) \leqslant 1$. By Proposition 1.3.8, $\Gamma^<$ is cofinal in $\Gamma_L^<$, so L is ungrounded. Towards a contradiction, suppose vf ($f \in L^\times$) is a gap in L. Passing to algebraic closures we arrange that K and L are algebraically closed. Set $\lambda := -f^\dagger$. Then for all active a in L we have $\lambda + a^\dagger \prec a$ by [ADH, 11.5.9] and hence $\lambda_\rho \rightsquigarrow \lambda$ by [ADH, 11.5.6]. By λ-freeness of K and [ADH, 3.2.6, 3.2.7], the valued field extension $K(\lambda) \supseteq K$ is immediate of transcendence degree 1, so $L \supseteq K(\lambda)$ is algebraic and $\Gamma = \Gamma_L$. Hence vf is a gap in K, a contradiction. This shows (i) ⇒ (ii).

To show the contrapositive of (ii) ⇒ (i), suppose $\lambda \in K$ is a pseudolimit of (λ_ρ). If the algebraic closure K^{a} of K does not have asymptotic integration, then clearly (ii) fails. If K^{a} has asymptotic integration, then $-\lambda$ creates a gap over K by [ADH, 11.5.14] applied to K^{a} in place of K, hence (ii) also fails. □

The next two lemmas include converses to Lemmas 1.3.4 and 1.3.5.

Lemma 1.3.13. *Let E be a pre-*d*-valued H-asymptotic field. Then:*

(i) *if E is not λ-free, then* $\mathrm{dv}(E)$ *is not λ-free;*
(ii) *if E is not ω-free, then* $\mathrm{dv}(E)$ *is not ω-free.*

Proof. This is clear if E has no rational asymptotic integration, because then $\mathrm{dv}(E)$ has no rational asymptotic integration either, by [ADH, 10.3.2]. Assume E has rational asymptotic integration. Then $\mathrm{dv}(E)$ is an immediate extension of E by [ADH, 10.3.2], and then (i) and (ii) follow from the characterizations of λ-freeness and ω-freeness in terms of nonexistence of certain pseudolimits. □

Lemma 1.3.14. *Let E be a* d*-valued H-asymptotic field and F an H-asymptotic extension of E such that $F = E(C_F)$. Then:*

(i) *if E is not λ-free, then F is not λ-free;*
(ii) *if E is not ω-free, then F is not ω-free.*

Proof. By [ADH, 10.5.15] E and F have the same value group. The rest of the proof is like that for the previous lemma, with F instead of $\operatorname{dv}(E)$. $\square$

In the rest of this subsection K is in addition a pre-H-field and L a pre-H-field extension of K. The following is shown in the proof of [13, Lemma 12.5]:

Proposition 1.3.15 (Gehret)**.** *Suppose K is a λ-free H-field and L is a Liouville H-field extension of K. Then L is λ-free and $\Gamma^{<}$ is cofinal in $\Gamma_L^{<}$.*

Example 1.3.16. Let $K = \mathbb{R}\langle\omega\rangle$ be the λ-free but non-ω-free H-field from [ADH, 13.9.1]. Then K has a unique Liouville closure L, up to isomorphism over K, by [13, Theorem 12.1(1)]. By Proposition 1.3.15, L is not ω-free; another proof of this fact is in [2]. By [ADH, 13.9.5] we can take here K to be a Hardy field, and then L is isomorphic over K to a Hardy field extension of K [ADH, 10.6.11].

Applying Corollary 1.3.10 to $H := \mathbb{R}\langle\omega\rangle$ yields a Liouville closed H-field H^{tl} that is not ω-free but does satisfy $\mathrm{I}(K^{\text{tl}}) \subseteq (K^{\text{tl}})^{\dagger}$ for $K^{\text{tl}} := H^{\text{tl}}[i]$.

Lemma 1.3.17. *Suppose K is λ-free, $\lambda \in \Lambda(L)^{\downarrow}$, $\omega := \omega(\lambda) \in K$, and suppose $\omega\big(\Lambda(K)\big) < \omega < \sigma\big(\Gamma(K)\big)$. Then $\lambda_\rho \rightsquigarrow \lambda$, and the pre-$H$-subfield $K\langle\lambda\rangle = K(\lambda)$ of L is an immediate extension of K (and so $K\langle\lambda\rangle$ is not λ-free).*

Proof. From $\Lambda(L) < \Delta(L)$ [ADH, p. 522] and $\Delta(K) \subseteq \Delta(L)$ we obtain $\lambda < \Delta(K)$. The restriction of ω to $\Lambda(L)^{\downarrow}$ is strictly increasing [ADH, p. 526] and $\Lambda(K) \subseteq \Lambda(L)$, so $\omega\big(\Lambda(K)\big) < \omega = \omega(\lambda)$ gives $\Lambda(K) < \lambda$. Hence $\lambda_\rho \rightsquigarrow \lambda$ by [ADH, 11.8.16]. Also $\omega_\rho \rightsquigarrow \omega$ by [ADH, 11.8.30]. Thus $K\langle\lambda\rangle$ is an immediate extension of K by [ADH, 11.7.13]. $\square$

Achieving ω-freeness for pre-H-fields. *In the rest of this section H is a pre-H-field and L is a Liouville closed* d*-algebraic H-field extension of H.* Thus if H is ω-free, then so is L, by Theorem 1.3.1. The lemmas below give conditions guaranteeing that L is ω-free, while H is not.

Lemma 1.3.18. *Suppose H is grounded or has a gap. Then L is ω-free.*

Proof. Suppose H is grounded. Let H_ω be the ω-free pre-H-field extension of H introduced in connection with [ADH, 11.7.17] (where we use the letter F instead of H). Identifying H_ω with its image in L under an embedding $H_\omega \to L$ over H of pre-H-fields, we apply Theorem 1.3.1 to $K := H_\omega$ to conclude that L is ω-free.

Next, suppose H has a gap $\beta = vb$, $b \in H^{\times}$. Take $a \in L$ with $a' = b$ and $a \not\asymp 1$. Then $\alpha := va$ satisfies $\alpha' = \beta$, and so the pre-H-field $H(a) \subseteq L$ is grounded, by [ADH, 9.8.2 and remarks following its proof]. Now apply the previous case to $H(a)$ in place of H. $\square$

Lemma 1.3.19. *Suppose H has asymptotic integration and divisible value group, and $s \in H$ creates a gap over H. Then L is ω-free.*

Proof. Take $f \in L^{\times}$ with $f^{\dagger} = s$. Then by [ADH, remark after 11.5.14], vf is a gap in $H\langle f\rangle = H(f)$, so L is ω-free by Lemma 1.3.18 applied to $H\langle f\rangle$ in place of H. $\square$

Lemma 1.3.20. *Suppose H is not λ-free. Then L is ω-free.*

Proof. By [ADH, 11.6.8], the real closure H^{rc} of H inside L is not λ-free, hence replacing H by H^{rc} we arrange that H is real closed. If H does not have asymptotic integration, then we are done by Lemma 1.3.18. So suppose H has asymptotic integration. Then some $s \in H$ creates a gap over H, by [ADH, 11.6.1], so L is ω-free by Lemma 1.3.19. $\square$

Corollary 1.3.21. *Suppose H is λ-free and $\lambda \in \Lambda(L)^{\downarrow}$ is such that $\omega := \omega(\lambda) \in H$ and $\omega\big(\Lambda(H)\big) < \omega < \sigma\big(\Gamma(H)\big)$. Then L is ω-free.*

Proof. By Lemma 1.3.17, the pre-H-subfield $H\langle\lambda\rangle = H(\lambda)$ of L is an immediate non-λ-free extension of H. Now apply Lemma 1.3.20 to $H\langle\lambda\rangle$ in place of H. $\square$

1.4. Complements on Linear Differential Operators

In this section we tie up loose ends from the material on linear differential operators in [ADH, 14.2] and [3, Section 8]. *Throughout K is an ungrounded asymptotic field with $\Gamma = v(K^{\times}) \neq \{0\}$, a, b, f, g, h range over arbitrary elements of K, and ϕ over those active in K, in particular, $\phi \neq 0$.* Recall our use of the term "eventually": a property $S(\phi)$ of ϕ is said to hold *eventually* if for some active ϕ_0 in K, $S(\phi)$ holds for all $\phi \preccurlyeq \phi_0$.

We shall consider linear differential operators $A \in K[\partial]^{\neq}$ and set $r := \operatorname{order}(A)$. In [ADH, 11.1] we showed that for each $\gamma \in \Gamma$ the quantity $\operatorname{dwt}_{A^\phi}(\gamma)$ is eventually constant; its eventual value is denoted by $\operatorname{nwt}_A(\gamma)$. We also introduced the set

$$\mathscr{E}^{\mathrm{e}}(A) \ = \ \mathscr{E}^{\mathrm{e}}_K(A) \ := \ \big\{\gamma \in \Gamma : \operatorname{nwt}_A(\gamma) \geqslant 1\big\} \ = \ \bigcap_{\phi} \mathscr{E}(A^{\phi})$$

of *eventual exceptional values of A*. For $a \neq 0$ we have $\mathscr{E}^{\mathrm{e}}(aA) = \mathscr{E}^{\mathrm{e}}(A)$ and $\mathscr{E}^{\mathrm{e}}(Aa) = \mathscr{E}^{\mathrm{e}}(A) - va$. An easy consequence of the definitions: $\mathscr{E}^{\mathrm{e}}(A^f) = \mathscr{E}^{\mathrm{e}}(A)$ for $f \neq 0$. A key fact about $\mathscr{E}^{\mathrm{e}}(A)$ is that if $y \in K^{\times}$, $vy \notin \mathscr{E}^{\mathrm{e}}(A)$, then $A(y) \asymp A^{\phi}y$, eventually. Since $A^{\phi}y \neq 0$ for $y \in K^{\times}$, this gives $v(\ker^{\neq} A) \subseteq \mathscr{E}^{\mathrm{e}}(A)$.

Lemma 1.4.1. *If L is an ungrounded asymptotic extension of K, then we have $\mathscr{E}^{\mathrm{e}}_L(A) \cap \Gamma \subseteq \mathscr{E}^{\mathrm{e}}(A)$, with equality if Ψ is cofinal in Ψ_L.*

Proof. For the inclusion, use that $\operatorname{dwt}(A^{\phi})$ decreases as $v\phi$ strictly increases [ADH, 11.1.12]. Thus its eventual value $\operatorname{nwt}(A)$, evaluated in K, cannot strictly increase when evaluated in an ungrounded asymptotic extension of K. $\square$

In the rest of this section we assume in addition that K is H-asymptotic with asymptotic integration. Then by [ADH, 14.2.8]:

Proposition 1.4.2. *If K is r-linearly newtonian, then $v(\ker^{\neq} A) = \mathscr{E}^{\mathrm{e}}(A)$.*

Remark 1.4.3. If K is d-valued, then $|v(\ker^{\neq} A)| = \dim_C \ker A \leqslant r$ by [ADH, 5.6.6], using a reduction to the case of "small derivation" by compositional conjugation.

Corollary 1.4.4. *Suppose K is* d*-valued, $\mathscr{E}^{\mathrm{e}}(A) = v(\ker^{\neq} A)$, and $0 \neq f \in A(K)$. Then $A(y) = f$ for some $y \in K$ with $vy \notin \mathscr{E}^{\mathrm{e}}(A)$.*

Proof. Let $y \in K$, $A(y) = f$, with vy maximal. Then $vy \notin \mathscr{E}^{\mathrm{e}}(A)$: otherwise we have $z \in \ker A$ with $z \sim y$, so $A(y - z) = f$ and $v(y - z) > vy$. $\square$

Corollary 1.4.5. *Suppose K is ω-free. Then $\sum_{\gamma \in \Gamma} \operatorname{nwt}_A(\gamma) = |\mathscr{E}^{\mathrm{e}}(A)| \leqslant r$.*

Proof. By (0.7) we have an immediate newtonian asymptotic extension L of K. Then L is d-valued by Lemma 1.2.9, hence $|\mathscr{E}^{\mathrm{e}}(A)| = |\mathscr{E}^{\mathrm{e}}_L(A)| \leqslant r$ by Proposition 1.4.2 and Remark 1.4.3. By [ADH, 13.7.10] we have $\operatorname{nwt}_A(\gamma) \leqslant 1$ for all $\gamma \in \Gamma$, thus $\sum_{\gamma \in \Gamma} \operatorname{nwt}_A(\gamma) = |\mathscr{E}^{\mathrm{e}}(A)|$. $\square$

In [ADH, 11.1] we defined $v_A^{\mathrm{e}}\colon \Gamma \to \Gamma$ by requiring that for all $\gamma \in \Gamma$:

$$v_{A^\phi}(\gamma) \ = \ v_A^{\mathrm{e}}(\gamma) + \operatorname{nwt}_A(\gamma) v\phi, \qquad \text{eventually.} \tag{1.4.1}$$

We recall from that reference that for $a \neq 0$ and $\gamma \in \Gamma$ we have

$$v_{aA}^{\mathrm{e}}(\gamma) \ = \ va + v_A^{\mathrm{e}}(\gamma), \qquad v_{Aa}^{\mathrm{e}}(\gamma) = v_A^{\mathrm{e}}(va + \gamma).$$

As an example from [ADH, p. 481], $v_\partial^{\mathrm{e}}(\gamma) = \gamma + \psi(\gamma)$ for $\gamma \in \Gamma \setminus \{0\}$ and $v_\partial^{\mathrm{e}}(0) = 0$. By [ADH, 14.2.7 and the remark preceding it] we have:

Lemma 1.4.6. *The restriction of v_A^{e} to a function $\Gamma \setminus \mathscr{E}^{\mathrm{e}}(A) \to \Gamma$ is strictly increasing, and $v\big(A(y)\big) = v_A^{\mathrm{e}}(vy)$ for all $y \in K$ with $vy \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A)$. Moreover, if K is ω-free, then $v_A^{\mathrm{e}}\big(\Gamma \setminus \mathscr{E}^{\mathrm{e}}(A)\big) = \Gamma$.*

A differential field F is said to be *r-linearly surjective* ($r \in \mathbb{N}$) if $A(F) = F$ for every $A \in F[\partial]^{\neq}$ of order at most r. The following is [ADH, 14.2.10] without the hypothesis of ω-freeness:

Corollary 1.4.7. *Suppose K is r-linearly newtonian. Then for each $f \neq 0$ there exists $y \in K^\times$ such that $A(y) = f$, $vy \notin \mathscr{E}^{\mathrm{e}}(A)$, and $v_A^{\mathrm{e}}(vy) = vf$.*

Proof. If $r = 0$, then $\mathscr{E}^{\mathrm{e}}(A) = \emptyset$ and our claim is obviously valid. Suppose $r \geqslant 1$. Then K is d-valued by Lemma 1.2.9, and $v(\ker^{\neq} A) = \mathscr{E}^{\mathrm{e}}(A)$ by Proposition 1.4.2, Moreover, by [ADH, 14.2.2], K is r-linearly surjective, hence $f \in A(K)$. Now Corollary 1.4.4 yields $y \in K^\times$ with $A(y) = f$ and $vy \notin \mathscr{E}^{\mathrm{e}}(A)$. By Lemma 1.4.6 we have $v_A^{\mathrm{e}}(vy) = v\big(A(y)\big) = vf$. □

From the proof of [ADH, 14.2.10] we extract the following:

Corollary 1.4.8. *Suppose K is r-linearly newtonian with small derivation, and $A \in \mathcal{O}[\partial]$ with $a_0 := A(1) \asymp 1$, and $f \asymp^\flat 1$. Then there is $y \in K^\times$ such that $A(y) = f$ and $y \sim f/a_0$. For any such y we have $vy \notin \mathscr{E}^{\mathrm{e}}(A)$ and $v_A^{\mathrm{e}}(vy) = vf$.*

Proof. The case $r = 0$ is trivial. Assume $r \geqslant 1$, so K is d-valued by Lemma 1.2.9. Hence $f^\dagger \prec 1$, that is, $f' \prec f$, so $f^{(n)} \prec f$ for all $n \geqslant 1$ by [ADH, 4.4.2]. Then $Af \preccurlyeq f$ by [ADH, (5.1.3), (5.1.2)], and $A(f) \sim a_0 f$, so $A_{\ltimes f} \in \mathcal{O}[\partial]$ and $A_{\ltimes f}(1) \sim a_0$. Thus we may replace A, f by $A_{\ltimes f}$, 1 to arrange $f = 1$. Now $a_0 \asymp 1$ gives $\operatorname{dwm}(A) = 0$, so $\operatorname{dwt}(A^\phi) = 0$ eventually, by [ADH, 11.1.11(ii)], that is, $\operatorname{nwt}(A) = 0$. Also $A^\phi(1) = A(1) = a_0 \asymp 1$, so $v^{\mathrm{e}}(A) = 0$. Arguing as in the proof of [ADH, 14.2.10] we obtain $y \in K^\times$ with $A(y) = 1$ and $y \sim 1/a_0$. It is clear that $vy = 0 \notin \mathscr{E}^{\mathrm{e}}(A)$ and $v_A^{\mathrm{e}}(vy) = v^{\mathrm{e}}(A) = 0 = vf$ for any such y. □

In the next few subsections below we consider more closely the case of order $r = 1$, and in the last subsection the case of arbitrary order.

First-order operators. *In this subsection $A = \partial - g$.* By [ADH, p. 481],

$$\mathscr{E}^{\mathrm{e}}(A) \ = \ \mathscr{E}_K^{\mathrm{e}}(A) \ = \ \big\{vy : y \in K^\times,\ v(g - y^\dagger) > \Psi\big\}$$

has at most one element. We also have $|v(\ker^{\neq} A)| = \dim_C \ker A \leqslant 1$ in view of $C^\times \subseteq \mathcal{O}^\times$. Proposition 1.4.2 holds under a weaker assumption on K for $r = 1$:

Lemma 1.4.9. *Suppose $\mathrm{I}(K) \subseteq K^\dagger$. Then $v(\ker^{\neq} A) = \mathscr{E}^{\mathrm{e}}(A)$.*

Proof. It remains to show "$\supseteq$". Suppose $\mathscr{E}^{\mathrm{e}}(A) = \{0\}$. Then $g - y^\dagger \in \mathrm{I}(K)$ with $y \asymp 1$ in K, hence $g \in \mathrm{I}(K) \subseteq K^\dagger$, so $g = h^\dagger$ with $h \asymp 1$, and thus $0 = vh \in v(\ker^{\neq} A)$. The general case reduces to the case $\mathscr{E}^{\mathrm{e}}(A) = \{0\}$ by twisting. □

Lemma 1.4.10. *Suppose L is an ungrounded H-asymptotic extension of K. Then $\mathscr{E}^{\mathrm{e}}_L(A) \cap \Gamma = \mathscr{E}^{\mathrm{e}}(A)$.*

Proof. Lemma 1.4.1 gives $\mathscr{E}^{\mathrm{e}}_L(A) \cap \Gamma \subseteq \mathscr{E}^{\mathrm{e}}(A)$. Next, let $vy \in \mathscr{E}^{\mathrm{e}}(A)$, $y \in K^\times$. Then $v(g - y^\dagger) > \Psi$ and so $v(g - y^\dagger) \in (\Gamma^>)'$ since K has asymptotic integration. Hence $v(g - y^\dagger) > \Psi_L$ and thus $vy \in \mathscr{E}^{\mathrm{e}}_L(A)$, by [ADH, p. 481]. $\square$

Recall also from [ADH, 9.7] that for an ordered abelian group G and $U \subseteq G$, a function $\eta\colon U \to G$ is said to be *slowly varying* if $\eta(\alpha) - \eta(\beta) = o(\alpha - \beta)$ for all $\alpha \neq \beta$ in U; then the function $\gamma \mapsto \gamma + \eta(\gamma)\colon U \to G$ is strictly increasing. The quintessential example of a slowly varying function is $\psi\colon \Gamma^{\neq} \to \Gamma$ [ADH, 6.5.4(ii)].

Proposition 1.4.11. *There is a unique slowly varying function $\psi_A\colon \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A) \to \Gamma$ such that for all $y \in K^\times$ with $vy \notin \mathscr{E}^{\mathrm{e}}(A)$ we have $v\big(A(y)\big) = vy + \psi_A(vy)$.*

Proof. For d-valued K, use [3, 8.4]. In general, pass to the d-valued hull $L := \mathrm{dv}(K)$ of K from [ADH, 10.3] and use $\Gamma_L = \Gamma$ [ADH, 10.3.2]. $\square$

If $b \neq 0$, then $\mathscr{E}^{\mathrm{e}}(A_{\ltimes b}) = \mathscr{E}^{\mathrm{e}}(A) - vb$ and $\psi_{A_{\ltimes b}}(\gamma) = \psi_A(\gamma + vb)$ for $\gamma \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A_{\ltimes b})$.

Example. We have $\mathscr{E}^{\mathrm{e}}(\partial) = \{0\}$ and $\psi_\partial = \psi$. More generally, if $g = b^\dagger$, $b \neq 0$, then $A_{\ltimes b} = \partial$ and so $\mathscr{E}^{\mathrm{e}}(A) = \{vb\}$ and $\psi_A(\gamma) = \psi(\gamma - vb)$ for $\gamma \in \Gamma \setminus \{vb\}$.

If Γ is divisible, then $\Gamma \setminus v\big(A(K)\big)$ has at most one element by [ADH, 11.6.16]. Also, K is $\uplambda$-free iff $v\big(A(K)\big) = \Gamma_\infty$ for all $A = \partial - g$ by [ADH, 11.6.17].

Lemma 1.4.12. *Suppose K is $\uplambda$-free and $f \neq 0$. Then for some $y \in K^\times$ we have $A(y) \asymp f$ and $vy \notin \mathscr{E}^{\mathrm{e}}(A)$. (Hence $\gamma \mapsto \gamma + \psi_A(\gamma)\colon \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A) \to \Gamma$ is surjective.)*

Proof. [ADH, 11.6.17] gives $y \in K^\times$ with $A^\phi y \asymp f$ eventually. Now

$$A^\phi y \;=\; \phi y \updelta - (g - y^\dagger)y \;\text{ in } K^\phi[\updelta], \qquad \updelta := \phi^{-1}\partial.$$

Since $v(A^\phi y) = vf$ eventually, this forces $g - y^\dagger \succ \phi$ eventually, so $vy \notin \mathscr{E}^{\mathrm{e}}(A)$. $\square$

Call A **steep** if $g \succ^\flat 1$, that is, $g \succ 1$ and $g^\dagger \succcurlyeq 1$. If K has small derivation and A is steep, then $g^\dagger \prec g$ by [ADH, 9.2.10].

Lemma 1.4.13. *Suppose K has small derivation, A is steep, and $y \in K^\times$ such that $A(y) = f \neq 0$, $g \succ f^\dagger$, and $vy \notin \mathscr{E}^{\mathrm{e}}(A)$. Then $y \sim -f/g$.*

Proof. We have

$$(f/g)^\dagger - g = f^\dagger - g^\dagger - g \sim -g \succ g^\dagger,$$

hence $v(f/g) \notin \mathscr{E}^{\mathrm{e}}(A)$, and

$$A(f/g) \;=\; (f/g)' - (f/g)g \;=\; (f/g)\cdot\big(f^\dagger - g^\dagger - g\big) \;\sim\; (f/g)\cdot(-g) \;=\; -f.$$

Since $A(y) = f \sim A(-f/g)$ and $vy, v(f/g) \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A)$, this gives $y = u \cdot f/g$ where $u \asymp 1$, by Proposition 1.4.11. Now $u^\dagger \prec 1 \prec g$ and $(f/g)^\dagger = f^\dagger - g^\dagger \prec g$, hence $y^\dagger \prec g$ and so

$$f = A(y) = y \cdot (y^\dagger - g) \sim -yg.$$

Therefore $y \sim -f/g$. $\square$

Lemma 1.4.14. *Suppose K has small derivation and $y \in K^\times$ is such that $A(y) = f \neq 0$, $g - f^\dagger \succ^\flat 1$ and $vy \notin \mathscr{E}^{\mathrm{e}}(A)$. Then $y \sim f/(f^\dagger - g)$.*

Proof. From $g - f^\dagger \succ 1$ we get $vf \notin \mathscr{E}^{\mathrm{e}}(A)$. Now $A(y) = f \prec f(f^\dagger - g) = A(f)$, so $y \prec f$ by [ADH, 5.6.8], and $v(y/f) \notin \mathscr{E}^{\mathrm{e}}(A_{\ltimes f}) = \mathscr{E}^{\mathrm{e}}(A) - vf$. Since $A_{\ltimes f} = \partial - (g - f^\dagger)$ is steep, Lemma 1.4.13 applies to $A_{\ltimes f}$, y/f, 1 in the role of A, y, f. $\square$

Suppose K is $\uplambda$-free and $f \neq 0$. Then [ADH, 11.6.1] gives an active ϕ_0 in K with $f^\dagger - g - \phi^\dagger \succcurlyeq \phi_0$ for all $\phi \prec \phi_0$. The convex subgroups $\Gamma^\flat_\phi$ of Γ become arbitrarily small as we let $v\phi$ increase cofinally in $\Psi^\downarrow$, so $\phi \prec^\flat_\phi \phi_0$ eventually, and hence $f^\dagger - g - \phi^\dagger \succ^\flat_\phi \phi$ eventually, that is, $\phi^{-1}(f/\phi)^\dagger - g/\phi \succ^\flat_\phi 1$ eventually. So replacing K by K^ϕ, A by $\phi^{-1}A^\phi = \updelta - (g/\phi)$ in $K^\phi[\updelta]$, and f and g by f/ϕ and g/ϕ, for suitable ϕ, we arrange $f^\dagger - g \succ^\flat 1$. Thus by Lemma 1.4.14:

Corollary 1.4.15. *If K is $\uplambda$-free, $y \in K^\times$, $A(y) = f \neq 0$, and $vy \notin \mathscr{E}^{\mathrm{e}}(A)$, then $y \sim f/\big((f/\phi)^\dagger - g\big)$, eventually.*

Example. If K is $\uplambda$-free and $y \in K$, $y' = f \neq 0$ with $y \not\asymp 1$, then $y \sim f/(f/\phi)^\dagger$, eventually.

From K to $K[\mathrm{i}]$. *In this subsection K is a real closed H-field.* Then $K[\mathrm{i}]$ ($\mathrm{i}^2 = -1$) is an H-asymptotic extension of K, with $\Gamma_{K[\mathrm{i}]} = \Gamma$. Consider a linear differential operator $B = \partial - (g + h\mathrm{i})$ over $K[\mathrm{i}]$. Note that $g + h\mathrm{i} \in K[\mathrm{i}]^\dagger$ iff $g \in K^\dagger$ and $h\mathrm{i} \in K[\mathrm{i}]^\dagger$, by Lemma 1.2.4. Under further assumptions on K, the next two results give explicit descriptions of ψ_B when $g \in K^\dagger$.

Proposition 1.4.16. *Suppose $K[\mathrm{i}]$ is 1-linearly newtonian and $g \in K^\dagger$. Then:*

(i) *if $h\mathrm{i} \in K[\mathrm{i}]^\dagger$, then for some $\beta \in \Gamma$ we have*

$$\mathscr{E}^{\mathrm{e}}(B) = \{\beta\}, \qquad \psi_B(\gamma) = \psi(\gamma - \beta) \ \text{ for all } \gamma \in \Gamma \setminus \{\beta\};$$

(ii) *if $h\mathrm{i} \notin K[\mathrm{i}]^\dagger$ and $g = b^\dagger$, $b \neq 0$, then*

$$\mathscr{E}^{\mathrm{e}}(B) = \emptyset, \qquad \psi_B(\gamma) = \min\big(\psi(\gamma - vb), vh\big) \ \text{ for all } \gamma \in \Gamma.$$

Proof. As to (i), apply the example following Proposition 1.4.11 to $K[\mathrm{i}]$, B, $g + h\mathrm{i}$ in the roles of K, A, g. For (ii), assume $h\mathrm{i} \notin K[\mathrm{i}]^\dagger$, $g = b^\dagger$, $b \neq 0$. Replacing B by $B_{\ltimes b}$ we arrange $g = 0$, $b = 1$, $B = \partial - h\mathrm{i}$. Corollary 1.2.14 gives $K[\mathrm{i}]^\dagger = K^\dagger \oplus \mathrm{I}(K)\mathrm{i}$, so $h \notin \mathrm{I}(K)$, and thus $vh \in \Psi^\downarrow$. Let $y \in K[\mathrm{i}]^\times$, and take $z \in K^\times$ and $s \in \mathrm{I}(K)$ with $y^\dagger = z^\dagger + s\mathrm{i}$. Then $vh < vs$, hence

$$v(y^\dagger - h\mathrm{i}) = \min\big(v(z^\dagger), v(s - h)\big) = \min\big(v(z^\dagger), vs, vh\big) = \min\big(v(y^\dagger), vh\big),$$

where the last equality uses $v(y^\dagger) = \min\big(v(z^\dagger), vs\big)$. Thus $v(y^\dagger - h\mathrm{i}) \in \Psi^\downarrow$ and

$$v\big(B(y)\big) - vy = v(y^\dagger - h\mathrm{i}) = \min\big(v(y^\dagger), vh\big) = \min\big(\psi(vy), vh\big),$$

which gives the desired result. $\square$

Corollary 1.4.17. *Suppose K is $\upomega$-free, $g \in K^\dagger$, $g = b^\dagger$, $b \neq 0$. Then either for some $\beta \in \Gamma$ we have $\mathscr{E}^{\mathrm{e}}(B) = \{\beta\}$ and $\psi_B(\gamma) = \psi(\gamma - \beta)$ for all $\gamma \in \Gamma \setminus \{\beta\}$, or $\mathscr{E}^{\mathrm{e}}(B) = \emptyset$ and $\psi_B(\gamma) = \min\big(\psi(\gamma - vb), vh\big)$ for all $\gamma \in \Gamma$.*

Proof. By (0.7) we have an immediate newtonian extension L of K. Then L is still a real closed H-field [ADH, 10.5.8, 3.5.19], and $L[\mathrm{i}]$ is newtonian by (0.10), so Proposition 1.4.16 applies to L in place of K. $\square$

Higher-order operators. We begin with the following observation:

Lemma 1.4.18. *Let $B \in K[\partial]^{\neq}$ and $\gamma \in \Gamma$. Then* $\operatorname{nwt}_{AB}(\gamma) \geqslant \operatorname{nwt}_B(\gamma)$*, and*

$$\gamma \notin \mathscr{E}^{\mathrm{e}}(B) \implies \operatorname{nwt}_{AB}(\gamma) \ = \ \operatorname{nwt}_A\big(v^{\mathrm{e}}_B(\gamma)\big) \text{ and } v^{\mathrm{e}}_{AB}(\gamma) \ = \ v^{\mathrm{e}}_A\big(v^{\mathrm{e}}_B(\gamma)\big).$$

Proof. We have $\operatorname{nwt}_{AB}(\gamma) = \operatorname{dwt}_{(AB)^\phi}(\gamma)$ eventually, and $(AB)^\phi = A^\phi B^\phi$. Hence by [ADH, 5.6] and the definition of $v^{\mathrm{e}}_B(\gamma)$ in (1.4.1):

$$\begin{aligned} \operatorname{nwt}_{AB}(\gamma) \ &= \ \operatorname{dwt}_{A^\phi}\big(v_{B^\phi}(\gamma)\big) + \operatorname{dwt}_{B^\phi}(\gamma) \\ &= \ \operatorname{dwt}_{A^\phi}\big(v^{\mathrm{e}}_B(\gamma) + \operatorname{nwt}_B(\gamma)v\phi\big) + \operatorname{nwt}_B(\gamma), \text{ eventually,} \end{aligned}$$

so $\operatorname{nwt}_{AB}(\gamma) \geqslant \operatorname{nwt}_B(\gamma)$. Now suppose $\gamma \notin \mathscr{E}^{\mathrm{e}}(B)$. Then $\operatorname{nwt}_B(\gamma) = 0$, so

$$\operatorname{nwt}_{AB}(\gamma) = \operatorname{dwt}_{A^\phi}\big(v^{\mathrm{e}}_B(\gamma)\big) = \operatorname{nwt}_A\big(v^{\mathrm{e}}_B(\gamma)\big), \qquad \text{eventually.}$$

Moreover, $v_{(AB)^\phi} = v_{A^\phi B^\phi} = v_{A^\phi} \circ v_{B^\phi}$, hence using (1.4.1):

$$v_{(AB)^\phi}(\gamma) \ = \ v_{A^\phi}\big(v_{B^\phi}(\gamma)\big) \ = \ v_{A^\phi}\big(v^{\mathrm{e}}_B(\gamma)\big), \text{ eventually,}$$

and thus eventually

$$\begin{aligned} v^{\mathrm{e}}_{AB}(\gamma) \ &= \ v_{(AB)^\phi}(\gamma) - \operatorname{nwt}_{AB}(\gamma)v\phi \\ &= \ v_{A^\phi}\big(v^{\mathrm{e}}_B(\gamma)\big) - \operatorname{nwt}_A\big(v^{\mathrm{e}}_B(\gamma)\big)v\phi \ = \ v^{\mathrm{e}}_A\big(v^{\mathrm{e}}_B(\gamma)\big). \end{aligned}$$ □

Lemmas 1.4.6 and 1.4.18 yield:

Corollary 1.4.19. *Let $B \in K[\partial]^{\neq}$. Then*

$$\mathscr{E}^{\mathrm{e}}(AB) \ = \ (v^{\mathrm{e}}_B)^{-1}\big(\mathscr{E}^{\mathrm{e}}(A)\big) \cup \mathscr{E}^{\mathrm{e}}(B)$$

and hence $|\mathscr{E}^{\mathrm{e}}(AB)| \leqslant |\mathscr{E}^{\mathrm{e}}(A)| + |\mathscr{E}^{\mathrm{e}}(B)|$, with equality if $v^{\mathrm{e}}_B\big(\Gamma \setminus \mathscr{E}^{\mathrm{e}}(B)\big) = \Gamma$.

As an easy consequence we have a variant of Corollary 1.4.5:

Corollary 1.4.20. *If A splits over K, then $|\mathscr{E}^{\mathrm{e}}(A)| \leqslant r$.*

To study v^{e}_A in more detail we introduce the function

$$\psi_A \ : \ \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A) \to \Gamma, \qquad \gamma \mapsto v^{\mathrm{e}}_A(\gamma) - \gamma.$$

For monic A of order 1 this agrees with ψ_A as defined in Proposition 1.4.11. For $A = a$ ($a \neq 0$) we have $\mathscr{E}^{\mathrm{e}}(A) = \emptyset$ and $\psi_A(\gamma) = va$ for all $\gamma \in \Gamma$.

Lemma 1.4.21. *Let $B \in K[\partial]^{\neq}$ and $\gamma \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(AB)$. Then*

$$\psi_{AB}(\gamma) \ = \ \psi_A\big(v^{\mathrm{e}}_B(\gamma)\big) + \psi_B(\gamma).$$

Proof. We have $\gamma \notin \mathscr{E}^{\mathrm{e}}(B)$ and $v^{\mathrm{e}}_B(\gamma) \notin \mathscr{E}^{\mathrm{e}}(A)$ by Corollary 1.4.19, hence

$$\psi_{AB}(\gamma) \ = \ v^{\mathrm{e}}_A\big(v^{\mathrm{e}}_B(\gamma)\big) - \gamma \ = \ v^{\mathrm{e}}_B(\gamma) + \psi_A\big(v^{\mathrm{e}}_B(\gamma)\big) - \gamma \ = \ \psi_A\big(v^{\mathrm{e}}_B(\gamma)\big) + \psi_B(\gamma)$$

by Lemma 1.4.18. □

Thus for $a \neq 0$ and $\gamma \in \Gamma$ we have

$\psi_{aA}(\gamma) = va + \psi_A(\gamma)$ if $\gamma \notin \mathscr{E}^{\mathrm{e}}(A)$, $\quad \psi_{Aa}(\gamma) = \psi_A(va+\gamma) + va$ if $\gamma \notin \mathscr{E}^{\mathrm{e}}(A) - va$.

Example. Suppose K has small derivation and $x \in K$, $x' \asymp 1$. Then $vx < 0$ and $\mathscr{E}^{\mathrm{e}}(\partial^2) = \{vx, 0\}$, and $\psi_{\partial^2}(\gamma) = \psi\big(\gamma + \psi(\gamma)\big) + \psi(\gamma)$ for $\gamma \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(\partial^2)$.

Lemma 1.4.22. *Suppose ψ_A is slowly varying. Let Δ be a convex subgroup of Γ and let $y, z \in K^\times$ be such that $vy, vz \notin \mathscr{E}^{\mathrm{e}}(A)$. Then*

$$v_\Delta(y) < v_\Delta(z) \iff v_\Delta\big(A(y)\big) < v_\Delta\big(A(z)\big).$$

Proof. By Lemma 1.4.6 we have

$$v\big(A(y)\big) - v\big(A(z)\big) \;=\; v_A^{\mathrm{e}}(vy) - v_A^{\mathrm{e}}(vz) \;=\; vy - vz + \psi_A(vy) - \psi_A(vz)$$

and $\psi_A(vy) - \psi_A(vz) = o(vy - vz)$ if $vy \neq vz$. $\square$

Call A **asymptotically surjective** if $v_A^{\mathrm{e}}\big(\Gamma \setminus \mathscr{E}^{\mathrm{e}}(A)\big) = \Gamma$ and ψ_A is slowly varying. If A is asymptotically surjective, then so are aA and Aa for $a \neq 0$, and if A has order 0, then A is asymptotically surjective. If K is $\uplambda$-free and A has order 1, then A is asymptotically surjective, thanks to Proposition 1.4.11 and Lemma 1.4.12.
The next lemma has an obvious proof.

Lemma 1.4.23. *Let G be an ordered abelian group and $U, V \subseteq G$. If $\eta_1, \eta_2 \colon U \to G$ are slowly varying, then so is $\eta_1 + \eta_2$. If $\eta \colon U \to G$ and $\zeta \colon V \to G$ are slowly varying and $\gamma + \zeta(\gamma) \in U$ for all $\gamma \in V$, then the function $\gamma \mapsto \eta\big(\gamma + \zeta(\gamma)\big) \colon V \to G$ is also slowly varying.*

Lemma 1.4.24. *If A and $B \in K[\partial]^{\neq}$ are asymptotically surjective, then so is AB.*

Proof. Let A, B be asymptotically surjective and $\gamma \in \Gamma$. This gives $\alpha \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A)$ with $v_A^{\mathrm{e}}(\alpha) = \gamma$ and $\beta \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(B)$ with $v_B^{\mathrm{e}}(\beta) = \alpha$. Then $\beta \notin \mathscr{E}^{\mathrm{e}}(AB)$ by Corollary 1.4.19, and $v_{AB}^{\mathrm{e}}(\beta) = \gamma$ by Lemma 1.4.18. Moreover, ψ_{AB} is slowly varying by Lemmas 1.4.21 and 1.4.23. $\square$

A straightforward induction on r using this lemma yields:

Corollary 1.4.25. *If K is $\uplambda$-free and A splits over K, then A is asymptotically surjective.*

We can now add to Lemma 1.4.6:

Corollary 1.4.26. *Suppose K is $\upomega$-free. Then A is asymptotically surjective.*

Proof. By the second part of Lemma 1.4.6 it is enough to show that ψ_A is slowly varying. For this we may replace K by any $\upomega$-free extension L of K with Ψ cofinal in Ψ_L. Thus we can arrange by (0.7) and (0.10) that K is newtonian, and by passing to the algebraic closure, algebraically closed. Then A splits over K by (0.8) and Lemma 1.2.9, so A is asymptotically surjective by Corollary 1.4.25. $\square$

1.5. Special Elements

Let K be a valued field and let $\widehat{a}$ be an element of an immediate extension of K with $\widehat{a} \notin K$. Recall that

$$v(\widehat{a} - K) \;=\; \big\{v(\widehat{a} - a) : a \in K\big\}$$

is a nonempty downward closed subset of $\Gamma := v(K^\times)$ without a largest element. Call $\widehat{a}$ *special* over K if some nontrivial convex subgroup of Γ is cofinal in $v(\widehat{a} - K)$ [ADH, p. 167]. In this case $v(\widehat{a} - K) \cap \Gamma^{>} \neq \emptyset$, and there is a unique such nontrivial convex subgroup Δ of Γ, namely

$$\Delta \;=\; \big\{\delta \in \Gamma : |\delta| \in v(\widehat{a} - K)\big\}.$$

We also call $\widehat{a}$ *almost special* over K if $\widehat{a}/\mathfrak{m}$ is special over K for some $\mathfrak{m} \in K^\times$. If $\Gamma \neq \{0\}$ is archimedean, then $\widehat{a}$ is special over K iff $v(\widehat{a} - K) = \Gamma$, iff $\widehat{a}$ is the limit of a divergent c-sequence in K. (Recall that “c-sequence” abbreviates “cauchy sequence” [ADH, p. 82].) In the next lemma a ranges over K and $\mathfrak{m}$, $\mathfrak{n}$ over $K^\times$.

Lemma 1.5.1. *Suppose $\widehat{a} \prec \mathfrak{m}$ and $\widehat{a}/\mathfrak{m}$ is special over K. Then for all a, $\mathfrak{n}$, if $\widehat{a} - a \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$, then $(\widehat{a} - a)/\mathfrak{n}$ is special over K.*

Proof. Replacing $\widehat{a}$, a, $\mathfrak{m}$, $\mathfrak{n}$ by $\widehat{a}/\mathfrak{m}$, $a/\mathfrak{m}$, 1, $\mathfrak{n}/\mathfrak{m}$, respectively, we arrange $\mathfrak{m} = 1$. So let $\widehat{a}$ be special over K with $\widehat{a} \prec 1$. It is enough to show: (1) $\widehat{a} - a$ is special over K, for all a; (2) for all $\mathfrak{n}$, if $\widehat{a} \prec \mathfrak{n} \preccurlyeq 1$, then $\widehat{a}/\mathfrak{n}$ is special over K. Here (1) follows from $v(\widehat{a} - a - K) = v(\widehat{a} - K)$. For (2), note that if $\widehat{a} \prec \mathfrak{n} \preccurlyeq 1$, then $v\mathfrak{n} \in \Delta$ with Δ as above, and so $v(\widehat{a}/\mathfrak{n} - K) = v(\widehat{a} - K) - v\mathfrak{n} = v(\widehat{a} - K)$. □

The remainder of this section is devoted to showing that (almost) special elements arise naturally in the analysis of certain immediate d-algebraic extensions of valued differential fields. We first treat the case of asymptotic fields with small derivation, and then focus on the linearly newtonian H-asymptotic case.

We recall some notation: for an ordered abelian group Γ and $\alpha \in \Gamma_\infty$, $\beta \in \Gamma$, $\gamma \in \Gamma^>$ we mean by "$\alpha \geqslant \beta + o(\gamma)$" that $\alpha \geqslant \beta - (1/n)\gamma$ for all $n \geqslant 1$, while "$\alpha < \beta + o(\gamma)$" is its negation, that is, $\alpha < \beta - (1/n)\gamma$ for some $n \geqslant 1$; see [ADH, p. 312]. Here and later inequalities are in the sense of the ordered divisible hull $\mathbb{Q}\Gamma$ of the relevant Γ.

A source of special elements. *In this subsection K is an asymptotic field with small derivation, value group $\Gamma = v(K^\times) \neq \{0\}$, and differential residue field $\boldsymbol{k}$; we also let $r \in \mathbb{N}^{\geqslant 1}$.* Below we use the notion *neatly surjective* from [ADH, 5.6]: $A \in K[\partial]^{\neq}$ is neatly surjective iff for all $b \in K^\times$ there exists $a \in K^\times$ with $A(a) = b$ and $v_A(va) = vb$. For use in the next proof, recall from [ADH, 7.1] the notion of a valued differential field being r-*differential-henselian* (or r-d-*henselian,* for short). We often let $\widehat{f}$ be an element in an immediate asymptotic extension $\widehat{K}$ of K, but in the statement of the next lemma we take $\widehat{f} \in K$:

Lemma 1.5.2. *Assume $\boldsymbol{k}$ is r-linearly surjective, $A \in K[\partial]^{\neq}$ of order $\leqslant r$ is neatly surjective, $\gamma \in \mathbb{Q}\Gamma$, $\gamma > 0$, $\widehat{f} \in K^\times$, and $v\big(A(\widehat{f})\big) \geqslant v(A\widehat{f}) + \gamma$. Then $A(f) = 0$ and $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \gamma + o(\gamma)$ for some $f \in K$.*

Proof. Set $B := g^{-1}A\widehat{f}$, where we take $g \in K^\times$ such that $vg = v(A\widehat{f})$. Then $B \asymp 1$, B is still neatly surjective, and $B(1) = g^{-1}A(\widehat{f})$, $v\big(B(1)\big) \geqslant \gamma$. It suffices to find $y \in K$ such that $B(y) = 0$ and $v(y-1) \geqslant \gamma + o(\gamma)$, because then $f := \widehat{f}y$ has the desired property. If $B(1) = 0$, then $y = 1$ works, so assume $B(1) \neq 0$. By [ADH, 7.2.7] we have an immediate extension $\widehat{K}$ of K that is r-differential henselian. Then $\widehat{K}$ is asymptotic by [ADH, 9.4.2 and 9.4.5]. Set $R(Z) := \operatorname{Ri}(B) \in K\{Z\}$. Then the proof of [ADH, 7.5.1] applied to $\widehat{K}$ and B in the roles of K and A yields $z \prec 1$ in $\widehat{K}$ with $R(z) = 0$. Now $R(0) = B(1)$, hence by [ADH, 7.2.2] we can take such z with $v(z) \geqslant \beta + o(\beta)$ where $\beta := v\big(B(1)\big) \geqslant \gamma$. As in the proof of [ADH, 7.5.1] we next take $y \in \widehat{K}$ with $v(y-1) > 0$ and $y^\dagger = z$ to get $B(y) = 0$, and observe that then $v(y-1) \geqslant \beta + o(\beta)$, by [ADH, 9.2.10(iv)], hence $v(y-1) \geqslant \gamma + o(\gamma)$. It remains to note that $y \in K$ by [ADH, 7.5.7]. □

By a remark following the proof of [ADH, 7.5.1] the assumption that $\boldsymbol{k}$ is r-linearly surjective in the lemma above can be replaced for $r \geqslant 2$ by the assumption that $\boldsymbol{k}$ is $(r-1)$-linearly surjective.

Next we establish a version of the above with $\widehat{f}$ in an immediate asymptotic extension of K. Recall that an asymptotic extension of K with the same value group as K has small derivation, by [ADH, 9.4.1].

Lemma 1.5.3. *Assume $\boldsymbol{k}$ is r-linearly surjective, $A \in K[\partial]^{\neq}$ of order $\leqslant r$ is neatly surjective, $\gamma \in \mathbb{Q}\Gamma$, $\gamma > 0$, $\widehat{K}$ is an immediate asymptotic extension of K, $\widehat{f} \in \widehat{K}^{\times}$, and $v\big(A(\widehat{f})\big) \geqslant v(A\widehat{f}) + \gamma$. Then for some $f \in K$ we have*

$$A(f) \ = \ 0, \qquad v(\widehat{f} - f) \ \geqslant \ v(\widehat{f}) + \gamma + o(\gamma).$$

Proof. By extending $\widehat{K}$ we can arrange that $\widehat{K}$ is r-differential henselian, so A remains neatly surjective as an element of $\widehat{K}[\partial]$, by [ADH, 7.1.8]. Then by Lemma 1.5.2 with $\widehat{K}$ in the role of K we get $f \in \widehat{K}$ such that $A(f) = 0$ and $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \gamma + o(\gamma)$. It remains to note that $f \in K$ by [ADH, 7.5.7]. $\square$

We actually need an inhomogeneous variant of the above:

Lemma 1.5.4. *Assume $\boldsymbol{k}$ is r-linearly surjective, $A \in K[\partial]^{\neq}$ of order $\leqslant r$ is neatly surjective, $b \in K$, $\gamma \in \mathbb{Q}\Gamma$, $\gamma > 0$, $v(A) = o(\gamma)$, $v(b) \geqslant o(\gamma)$, $\widehat{K}$ is an immediate asymptotic extension of K, $\widehat{f} \in \widehat{K}$, $\widehat{f} \preccurlyeq 1$, and $v\big(A(\widehat{f}) - b\big) \geqslant \gamma + o(\gamma)$. Then*

$$A(f) \ = \ b, \qquad v(\widehat{f} - f) \ \geqslant \ (1/2)\gamma + o(\gamma)$$

for some $f \in K$.

Proof. Take $y \in K$ with $A(y) = b$ and $v(y) \geqslant o(\gamma)$. Then $A(\widehat{g}) = A(\widehat{f}) - b$ for $\widehat{g} := \widehat{f} - y$, so $v\big(A(\widehat{g})\big) \geqslant \gamma + o(\gamma)$ and $v(\widehat{g}) \geqslant o(\gamma)$. We distinguish two cases:

(1) $v(\widehat{g}) \geqslant (1/2)\gamma + o(\gamma)$. Then $v(\widehat{f} - y) \geqslant (1/2)\gamma + o(\gamma)$, so $f := y$ works.

(2) $v(\widehat{g}) < (1/2)\gamma + o(\gamma)$. Then by [ADH, 6.1.3],

$$v(A\widehat{g}) \ < \ (1/2)\gamma + o(\gamma), \qquad v\big(A(\widehat{g})\big) \ \geqslant \ \gamma + o(\gamma),$$

so $v\big(A(\widehat{g})\big) \geqslant v(A\widehat{g}) + (1/2)\gamma$. Then Lemma 1.5.3 gives an element $g \in K$ such that $A(g) = 0$ and $v(\widehat{g} - g) \geqslant (1/2)\gamma + o(\gamma)$. Hence $f := y + g$ works. $\square$

Recall from [ADH, 7.2] that $\mathcal{O}$ is said to be *r-linearly surjective* if for every A in $K[\partial]^{\neq}$ of order r with $v(A) = 0$ there exists $y \in \mathcal{O}$ with $A(y) = 1$.

Proposition 1.5.5. *Assume $\mathcal{O}$ is r-linearly surjective, $P \in K\{Y\}$,* $\operatorname{order}(P) \leqslant r$, $\operatorname{ddeg} P = 1$*, and $P(\widehat{a}) = 0$, where $\widehat{a} \preccurlyeq 1$ lies in an immediate asymptotic extension of K and $\widehat{a} \notin K$. Then $\widehat{a}$ is special over K.*

Proof. The hypothesis on $\mathcal{O}$ yields: $\boldsymbol{k}$ is r-linearly surjective and all $A \in K[\partial]^{\neq}$ of order $\leqslant r$ are neatly surjective. Let $0 < \gamma \in v(\widehat{a} - K)$; we claim that $v(\widehat{a} - K)$ has an element $\geqslant (4/3)\gamma$. We arrange $P \asymp 1$. Take $a \in K$ with $v(\widehat{a} - a) = \gamma$. Then $P_{+a} \asymp 1$, $\operatorname{ddeg} P_{+a} = 1$, so

$$P_{+a,1} \ \asymp \ 1, \quad P_{+a,>1} \ \prec \ 1, \quad P_{+a} \ = \ P(a) + P_{+a,1} + P_{+a,>1}$$

and

$$0 \ = \ P(\widehat{a}) \ = \ P_{+a}(\widehat{a} - a) \ = \ P(a) + P_{+a,1}(\widehat{a} - a) + P_{+a,>1}(\widehat{a} - a),$$

with

$$v\big(P_{+a,1}(\widehat{a} - a) + P_{+a,>1}(\widehat{a} - a)\big) \geqslant \gamma + o(\gamma),$$

and thus $v(P(a)) \geqslant \gamma + o(\gamma)$. Take $g \in K^\times$ with $vg = \gamma$ and set $Q := g^{-1}P_{+a,\times g}$, so $Q = Q_0 + Q_1 + Q_{>1}$ with

$$Q_0 \;=\; Q(0) \;=\; g^{-1}P(a), \quad Q_1 \;=\; g^{-1}(P_{+a,1})_{\times g}, \quad Q_{>1} \;=\; g^{-1}(P_{+a,>1})_{\times g},$$

hence

$$v(Q_0) \geqslant o(\gamma), \quad v(Q_1) = o(\gamma), \quad v(Q_{>1}) \geqslant \gamma + o(\gamma).$$

We set $\widehat{f} := g^{-1}(\widehat{a} - a)$, so $Q(\widehat{f}) = 0$ and $\widehat{f} \asymp 1$, and $A := L_Q \in K[\partial]$. Then $Q(\widehat{f}) = 0$ gives

$$Q_0 + A(\widehat{f}) \;=\; Q_0 + Q_1(\widehat{f}) \;=\; -Q_{>1}(\widehat{f}), \quad \text{with } v\big(Q_{>1}(\widehat{f})\big) \;\geqslant\; \gamma + o(\gamma),$$

so $v\big(Q_0 + A(\widehat{f})\big) \geqslant \gamma + o(\gamma)$. Since $v(A) = v(Q_1) = o(\gamma)$, Lemma 1.5.4 then gives $f \in K$ with $v(\widehat{f} - f) \geqslant (1/3)\gamma$. In view of $\widehat{a} - a = g\widehat{f}$, this yields

$$v\big(\widehat{a} - (a + gf)\big) \;=\; \gamma + v(\widehat{f} - f) \;\geqslant\; (4/3)\gamma,$$

which proves our claim. It gives the desired result. □

A source of almost special elements. *In this subsection K, Γ, $\boldsymbol{k}$, and r are as in the previous subsection, and we assume that $\mathcal{O}$ is r-linearly surjective.* (So $\boldsymbol{k}$ is r-linearly surjective, and $\sup \Psi = 0$ by [ADH, 9.4.2].) Let $\widehat{a}$ be an element in an immediate asymptotic extension of K such that $\widehat{a} \notin K$ and $K\langle\widehat{a}\rangle$ has transcendence degree $\leqslant r$ over K. We shall use Proposition 1.5.5 to show:

Proposition 1.5.6. *If Γ is divisible, then $\widehat{a}$ is almost special over K.*

Towards the proof we first note that $\widehat{a}$ has a minimal annihilator $P(Y)$ over K of order $\leqslant r$. We also fix a divergent pc-sequence (a_ρ) in K such that $a_\rho \rightsquigarrow \widehat{a}$. (See [ADH, 4.1] for "minimal annihilator", and [ADH, 4.4] for "minimal differential polynomial of (a_ρ) over K".) We next show how to improve $\widehat{a}$ and P (without assuming divisibility of Γ):

Lemma 1.5.7. *For some $\widehat{b}$ in an immediate asymptotic extension of K we have:*

- (i) *$v(\widehat{a} - K) = v(\widehat{b} - K)$;*
- (ii) *(a_ρ) has a minimal differential polynomial Q over K of order $\leqslant r$ such that Q is also a minimal annihilator of $\widehat{b}$ over K.*

Proof. By [ADH, 6.8.1, 6.9.2], (a_ρ) is of d-algebraic type over K with a minimal differential polynomial Q over K such that $\operatorname{order} Q \leqslant \operatorname{order} P \leqslant r$. By [ADH, 6.9.3, 9.4.5] this gives an element $\widehat{b}$ in an immediate asymptotic extension of K such that Q is a minimal annihilator of $\widehat{b}$ over K and $a_\rho \rightsquigarrow \widehat{b}$. Then Q and $\widehat{b}$ have the desired properties. □

Proof of Proposition 1.5.6. Replace $\widehat{a}$ and P by $\widehat{b}$ and Q from Lemma 1.5.7 (and rename) to arrange that P is a minimal differential polynomial of (a_ρ) over K. Now assuming Γ is divisible, [22, Proposition 3.1] gives $a \in K$ and $g \in K^\times$ such that $\widehat{a} - a \asymp g$ and $\operatorname{ddeg} P_{+a,\times g} = 1$.

Set $F := P_{+a,\times g}$ and $\widehat{f} := (\widehat{a} - a)/g$. Then $\operatorname{ddeg} F = 1$, $F(\widehat{f}) = 0$, and $\widehat{f} \preccurlyeq 1$. Applying Proposition 1.5.5 to F and $\widehat{f}$ in the role of P and $\widehat{a}$ yields a nontrivial convex subgroup Δ of Γ that is cofinal in $v(\widehat{f} - K)$. Setting $\alpha := vg$, it follows that $\alpha + \Delta$ is cofinal in $v\big((\widehat{a} - a) - K\big) = v(\widehat{a} - K)$. □

We can trade the divisibility assumption in Proposition 1.5.6 against a stronger hypothesis on K, the proof using [22, 3.3] instead of [22, 3.1]:

Corollary 1.5.8. *If K is henselian and $\boldsymbol{k}$ is linearly surjective, then $\widehat{a}$ is almost special over K.*

The linearly newtonian setting. *In this subsection K is an $\upomega$-free r-linearly newtonian H-asymptotic field, $r \geqslant 1$.* Thus K is d-valued by Lemma 1.2.9. We let ϕ range over the elements active in K. We now mimick the material in the previous two subsections. Note that for $A \in K[\partial]^{\neq}$ and any element $\widehat{f}$ in an asymptotic extension of K we have $A(\widehat{f}) \preccurlyeq A^{\phi}\widehat{f}$, since $A(\widehat{f}) = A^{\phi}(\widehat{f})$.

Lemma 1.5.9. *Assume that $A \in K[\partial]^{\neq}$ has order $\leqslant r$, $\gamma \in \mathbb{Q}\Gamma$, $\gamma > 0$, $\widehat{f} \in K^{\times}$, and $v\big(A(\widehat{f})\big) \geqslant v(A^{\phi}\widehat{f}) + \gamma$, eventually. Then there exists an $f \in K$ with $A(f) = 0$ and $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \gamma + o(\gamma)$.*

Proof. Take ϕ such that $v\phi \geqslant \gamma^{\dagger}$ and $v\big(A(\widehat{f})\big) \geqslant v(A^{\phi}\widehat{f}) + \gamma$. Next, take $\beta \in \Gamma$ such that $\beta \geqslant \gamma$ and $v\big(A(\widehat{f})\big) \geqslant v(A^{\phi}\widehat{f}) + \beta$. Then $v\phi \geqslant \beta^{\dagger}$, so $\beta > \Gamma^{\flat}_{\phi}$, hence the valuation ring of the flattening $(K^{\phi}, v^{\flat}_{\phi})$ is r-linearly surjective, by [ADH, 14.2.1]. We now apply Lemma 1.5.2 to

$$(K^{\phi}, v^{\flat}_{\phi}), \quad A^{\phi}, \quad \dot{\beta} := \beta + \Gamma^{\flat}_{\phi}$$

in the role of K, A, γ to give $f \in K$ with $A(f) = 0$ and $v^{\flat}_{\phi}(\widehat{f} - f) \geqslant v^{\flat}_{\phi}(\widehat{f}) + \dot{\beta} + o(\dot{\beta})$. Then also $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \beta + o(\beta)$, and thus $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \gamma + o(\gamma)$. □

Lemma 1.5.10. *Assume $A \in K[\partial]^{\neq}$ has order $\leqslant r$, $\widehat{K}$ is an immediate* d*-algebraic asymptotic extension of K, $\gamma \in \mathbb{Q}\Gamma$, $\gamma > 0$, $\widehat{f} \in \widehat{K}^{\times}$, and $v\big(A(\widehat{f})\big) \geqslant v(A^{\phi}\widehat{f}) + \gamma$ eventually. Then $A(f) = 0$ and $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \gamma + o(\gamma)$ for some $f \in K$.*

Proof. Since K is $\upomega$-free, so is $\widehat{K}$ by Theorem 1.3.1. By (0.7) we can extend $\widehat{K}$ to arrange that $\widehat{K}$ is also newtonian. Then by Lemma 1.5.9 with $\widehat{K}$ in the role of K we get $f \in \widehat{K}$ with $A(f) = 0$ and $v(\widehat{f} - f) \geqslant v(\widehat{f}) + \gamma + o(\gamma)$. Now use that $f \in K$ by [ADH, line before 14.2.10]. □

Lemma 1.5.11. *Assume $A \in K[\partial]^{\neq}$ has order $\leqslant r$, $b \in K$, $\gamma \in \mathbb{Q}\Gamma$, $\gamma > 0$, $\widehat{K}$ is an immediate* d*-algebraic asymptotic extension of K, and $\widehat{f} \in \widehat{K}$, $v(\widehat{f}) \geqslant o(\gamma)$. Assume also that eventually $v(b) \geqslant v(A^{\phi}) + o(\gamma)$ and $v\big(A(\widehat{f}) - b\big) \geqslant v(A^{\phi}) + \gamma + o(\gamma)$. Then for some $f \in K$ we have $A(f) = b$ and $v(\widehat{f} - f) \geqslant (1/2)\gamma + o(\gamma)$.*

Proof. We take $y \in K$ with $A(y) = b$ as follows: If $b = 0$, then $y := 0$. If $b \neq 0$, then Corollary 1.4.7 yields $y \in K^{\times}$ such that $A(y) = b$, $vy \notin \mathscr{E}^{\mathrm{e}}(A)$, and $v^{\mathrm{e}}_{A}(vy) = vb$. In any case, $vy \geqslant o(\gamma)$: when $b \neq 0$, the sentence preceding [ADH, 14.2.7] gives $v_{A^{\phi}}(vy) = vb$, eventually, to which we apply [ADH, 6.1.3].

Now $A(\widehat{g}) = A(\widehat{f}) - b$ for $\widehat{g} := \widehat{f} - y$, so $v(\widehat{g}) \geqslant o(\gamma)$, and eventually $v\big(A(\widehat{g})\big) \geqslant v(A^{\phi}) + \gamma + o(\gamma)$. We distinguish two cases:

(1) $v(\widehat{g}) \geqslant (1/2)\gamma + o(\gamma)$. Then $v(\widehat{f} - y) \geqslant (1/2)\gamma + o(\gamma)$, so $f := y$ works.

(2) $v(\widehat{g}) < (1/2)\gamma + o(\gamma)$. Then by [ADH, 6.1.3] we have eventually

$$v(A^{\phi}\widehat{g}) \;<\; v(A^{\phi}) + (1/2)\gamma + o(\gamma), \qquad v\big(A(\widehat{g})\big) \;\geqslant\; v(A^{\phi}) + \gamma + o(\gamma),$$

so $v(A(\widehat{g})) \geqslant v(A^{\phi}\widehat{g}) + (1/2)\gamma$, eventually. Lemma 1.5.10 gives an element $g \in K$ with $A(g) = 0$ and $v(\widehat{g} - g) \geqslant (1/2)\gamma + o(\gamma)$. Hence $f := y + g$ works. $\square$

Proposition 1.5.12. *Suppose that $P \in K\{Y\}$, $\operatorname{order} P \leqslant r$, $\operatorname{ndeg} P = 1$, and $P(\widehat{a}) = 0$, where $\widehat{a} \preccurlyeq 1$ lies in an immediate asymptotic extension of K and $\widehat{a} \notin K$. Then $\widehat{a}$ is special over K.*

The proof is like that of Proposition 1.5.5, but there are some differences that call for further details.

Proof. Given $0 < \gamma \in v(\widehat{a} - K)$, we claim that $v(\widehat{a} - K)$ has an element $\geqslant (4/3)\gamma$. Take $a \in K$ with $v(\widehat{a} - a) = \gamma$. Then $\operatorname{ndeg} P_{+a} = 1$ by [ADH, 11.2.3(i)], so eventually we have

$$P(a) \preccurlyeq P^{\phi}_{+a,1} \succ P^{\phi}_{+a,>1}, \quad P^{\phi}_{+a} = P(a) + P^{\phi}_{+a,1} + P^{\phi}_{+a,>1}$$

and

$$\begin{aligned} 0 = P(\widehat{a}) &= P^{\phi}_{+a}(\widehat{a} - a) \\ &= P(a) + P^{\phi}_{+a,1}(\widehat{a} - a) + P^{\phi}_{+a,>1}(\widehat{a} - a), \\ &\qquad v\big(P^{\phi}_{+a,1}(\widehat{a} - a) + P^{\phi}_{+a,>1}(\widehat{a} - a)\big) \geqslant v(P^{\phi}_{+a,1}) + \gamma + o(\gamma), \end{aligned}$$

and thus eventually $v\big(P(a)\big) \geqslant v(P^{\phi}_{+a,1}) + \gamma + o(\gamma)$. Take $g \in K^{\times}$ with $vg = \gamma$ and set $Q := g^{-1}P_{+a,\times g}$, so $Q = Q_0 + Q_1 + Q_{>1}$ with

$$Q_0 = Q(0) = g^{-1}P(a), \quad Q_1 = g^{-1}(P_{+a,1})_{\times g}, \quad Q_{>1} = g^{-1}(P_{+a,>1})_{\times g}.$$

Then $v(Q_0) = v\big(P(a)\big) - \gamma \geqslant v(P^{\phi}_{+a,1}) + o(\gamma)$, eventually. By [ADH, 6.1.3],

$$v(Q^{\phi}_1) = v(P^{\phi}_{+a,1}) + o(\gamma), \qquad v(Q^{\phi}_{>1}) \geqslant v(P^{\phi}_{+a,>1}) + \gamma + o(\gamma)$$

for all ϕ. Since $P^{\phi}_{+a,>1} \preccurlyeq P^{\phi}_{+a,1}$, eventually, the last two displayed inequalities give $v(Q^{\phi}_{>1}) \geqslant v(Q^{\phi}_1) + \gamma + o(\gamma)$, eventually. We set $\widehat{f} := g^{-1}(\widehat{a} - a)$, so $Q(\widehat{f}) = 0$ and $\widehat{f} \asymp 1$. Set $A := L_Q \in K[\partial]$. Then $Q(\widehat{f}) = 0$ gives

$$Q_0 + A(\widehat{f}) = Q_0 + Q_1(\widehat{f}) = -Q^{\phi}_{>1}(\widehat{f}),$$

with $v\big(Q^{\phi}_{>1}(\widehat{f})\big) \geqslant v(Q^{\phi}_1) + \gamma + o(\gamma)$, eventually, so

$$v\big(Q_0 + A(\widehat{f})\big) \geqslant v(A^{\phi}) + \gamma + o(\gamma), \quad \text{eventually.}$$

Moreover, $v(Q_0) \geqslant v(A^{\phi}) + o(\gamma)$, eventually. Lemma 1.5.11 then gives $f \in K$ with $v(\widehat{f} - f) \geqslant (1/3)\gamma$. In view of $\widehat{a} - a = g\widehat{f}$, this yields

$$v\big(\widehat{a} - (a + gf)\big) = \gamma + v(\widehat{f} - f) \geqslant (4/3)\gamma,$$

which proves our claim. $\square$

In the rest of this subsection we assume that $\widehat{a} \notin K$ lies in an immediate asymptotic extension of K and $K\langle\widehat{a}\rangle$ has transcendence degree $\leqslant r$ over K.

Proposition 1.5.13. *If Γ is divisible, then $\widehat{a}$ is almost special over K.*

Towards the proof, we fix a minimal annihilator $P(Y)$ of $\widehat{a}$ over K, so $\operatorname{order} P \leqslant r$. We also fix a divergent pc-sequence (a_ρ) in K such that $a_\rho \rightsquigarrow \widehat{a}$. We next show how to improve $\widehat{a}$ and P if necessary:

Lemma 1.5.14. *For some $\widehat{b}$ in an immediate asymptotic extension of K we have:*

(i) $v(\widehat{a}-a)=v(\widehat{b}-a)$ *for all* $a\in K$;

(ii) (a_ρ) *has a minimal differential polynomial* Q *over* K *of order* $\leqslant r$ *such that* Q *is also a minimal annihilator of* $\widehat{b}$ *over* K.

Proof. By the remarks following the proof of [ADH, 11.4.3] we have $P\in Z(K,\widehat{a})$. Take $Q\in Z(K,\widehat{a})$ of minimal complexity. Then $\operatorname{order} Q\leqslant \operatorname{order} P\leqslant r$, and Q is a minimal differential polynomial of (a_ρ) over K by [ADH, 11.4.13]. By [ADH, 11.4.8 and its proof] this gives an element $\widehat{b}$ in an immediate asymptotic extension of K such that (i) holds and Q is a minimal annihilator of $\widehat{b}$ over K. Then Q and $\widehat{b}$ have the desired properties. □

Proof of Proposition 1.5.13. Assume Γ is divisible. Replace $\widehat{a}$, P by $\widehat{b}$, Q from Lemma 1.5.14 and rename to arrange that P is a minimal differential polynomial of (a_ρ) over K. By [ADH, 14.5.1] we have $a\in K$ and $g\in K^\times$ such that $\widehat{a}-a\asymp g$ and $\operatorname{ndeg} P_{+a,\times g}=1$. Set $F:=P_{+a,\times g}$ and $\widehat{f}:=(\widehat{a}-a)/g$. Then $\operatorname{ndeg} F=1$, $F(\widehat{f})=0$, and $\widehat{f}\preccurlyeq 1$. Applying Proposition 1.5.12 to F and $\widehat{f}$ in the role of P and $\widehat{a}$ yields a nontrivial convex subgroup Δ of Γ that is cofinal in $v(\widehat{f}-K)$. Setting $\alpha:=vg$, it follows that $\alpha+\Delta$ is cofinal in $v\big((\widehat{a}-a)-K\big)=v(\widehat{a}-K)$. □

Corollary 1.5.15. *If* K *is henselian, then* $\widehat{a}$ *is almost special over* K.

The proof is like that of Proposition 1.5.13, using [21, 3.3] instead of [ADH, 14.5.1].

The case of order 1. We show here that Proposition 1.5.12 goes through in the case of order 1 under weaker assumptions: *in this subsection K is a 1-linearly newtonian H-asymptotic field with asymptotic integration.* Then K is d-valued with $\mathrm{I}(K)\subseteq K^\dagger$, by Lemma 1.2.9, and λ-free, by [ADH, 14.2.3]. We let ϕ range over elements active in K. In the next two lemmas $A\in K[\partial]^{\neq}$ has order $\leqslant 1$, $\gamma\in\mathbb{Q}\Gamma$, $\gamma>0$, and $\widehat{K}$ is an immediate asymptotic extension of K.

Lemma 1.5.16. *Let* $\widehat{f}\in\widehat{K}^\times$ *be such that* $v\big(A(\widehat{f})\big)\geqslant v(A^\phi\widehat{f})+\gamma$ *eventually. Then there exists* $f\in K$ *such that* $A(f)=0$ *and* $v(\widehat{f}-f)\geqslant v(\widehat{f})+\gamma$.

Proof. Note that $\operatorname{order}(A)=1$; we arrange $A=\partial-g$ $(g\in K)$. If $A(\widehat{f})=0$, then $\widehat{f}$ is in K [ADH, line before 14.2.10], and $f:=\widehat{f}$ works. Assume $A(\widehat{f})\neq 0$. Then

$$v\big(A^\phi(\widehat{f})\big)=v\big(A(\widehat{f})\big)\geqslant v(A^\phi\widehat{f})+\gamma>v(A^\phi\widehat{f}),\qquad \text{eventually,}$$

so $v(\widehat{f})\in\mathscr{E}^{\mathrm{e}}(A)$, and Lemma 1.4.9 yields an $f\in K$ with $f\sim\widehat{f}$ and $A(f)=0$. We claim that this f has the desired property. Set $b:=A(\widehat{f})$. By the remarks preceding Corollary 1.4.15 we can replace K, $\widehat{K}$, A, b by K^ϕ, $\widehat{K}^\phi$, $\phi^{-1}A^\phi$, $\phi^{-1}b$, respectively, for suitable ϕ, to arrange that K has small derivation and $b^\dagger-g\succ^\flat 1$. Using the hypothesis of the lemma we also arrange $vb\geqslant v(A\widehat{f})+\gamma$. It remains to show that for $\widehat{g}:=\widehat{f}-f\neq 0$ we have $v(\widehat{g})\geqslant v(\widehat{f})+\gamma$. Now $A(\widehat{g})=b$ with $v(\widehat{g})\notin\mathscr{E}^{\mathrm{e}}(A)$, hence $\widehat{g}\sim b/(b^\dagger-g)\prec^\flat b$ by Lemma 1.4.14, and thus $v(\widehat{g})>vb\geqslant v(A\widehat{f})+\gamma$, so it is enough to show $v(A\widehat{f})\geqslant v(\widehat{f})$. Now $b=A(\widehat{f})=\widehat{f}(\widehat{f}^\dagger-g)$ and $A\widehat{f}=\widehat{f}\big(\partial+\widehat{f}^\dagger-g\big)$. As $vb\geqslant v(A\widehat{f})+\gamma>v(A\widehat{f})$, this yields $v(\widehat{f}^\dagger-g)>0$, so $v(A\widehat{f})=v(\widehat{f})$. □

Lemma 1.5.17. *Let* $b\in K$ *and* $\widehat{f}\in\widehat{K}$ *with* $v(\widehat{f})\geqslant o(\gamma)$. *Assume also that eventually* $v(b)\geqslant v(A^\phi)+o(\gamma)$ *and* $v\big(A(\widehat{f})-b\big)\geqslant v(A^\phi)+\gamma+o(\gamma)$. *Then for some* $f\in K$ *we have* $A(f)=b$ *and* $v(\widehat{f}-f)\geqslant(1/2)\gamma+o(\gamma)$.

The proof is like that of Lemma 1.5.11, using Lemma 1.5.16 instead of Lemma 1.5.10. In the same way Lemma 1.5.11 gave Proposition 1.5.12, Lemma 1.5.17 now yields:

Proposition 1.5.18. *If $P \in K\{Y\}$, $\operatorname{order} P \leqslant 1$, $\operatorname{ndeg} P = 1$, and $P(\widehat{a}) = 0$, where $\widehat{a} \preccurlyeq 1$ lies in an immediate asymptotic extension of K and $\widehat{a} \notin K$, then $\widehat{a}$ is special over K.*

Remark. Proposition 1.5.13 does not hold for $r = 1$ under present assumptions. To see this, let K be a Liouville closed H-field which is not $\upomega$-free, as in Example 1.3.16 or [2]. Then K is 1-linearly newtonian by Corollary 1.7.29 below. Consider the pc-sequences $(\uplambda_\rho)$ and $(\upomega_\rho)$ in K as in [ADH, 11.7], let $\upomega \in K$ with $\upomega_\rho \rightsquigarrow \upomega$, and $P = 2Y' + Y^2 + \upomega$. Then [ADH, 11.7.13] gives an element $\uplambda$ in an immediate asymptotic extension of K but not in K with $\uplambda_\rho \rightsquigarrow \uplambda$ and $P(\uplambda) = 0$. However, $\uplambda$ is not almost special over K [ADH, 3.4.13, 11.5.2].

Relating $Z(K,\widehat{a})$ and $v(\widehat{a} - K)$ for special $\widehat{a}$. *In this subsection K is a valued differential field with small derivation $\partial \neq 0$ such that $\Gamma \neq \{0\}$ and $\Gamma^{>}$ has no least element.* We recall from [3] that a valued differential field extension L of K is said to be *strict* if for all $\phi \in K^\times$,

$$\partial\smallo \subseteq \phi\smallo \ \Rightarrow\ \partial\smallo_L \subseteq \phi\smallo_L, \qquad \partial\mathcal{O} \subseteq \phi\smallo \ \Rightarrow\ \partial\mathcal{O}_L \subseteq \phi\smallo_L.$$

(If K is asymptotic, then any immediate asymptotic extension of K is automatically strict, by [3, 1.11].) Let $\widehat{a}$ lie in an immediate strict extension of K such that $\widehat{a} \preccurlyeq 1$, $\widehat{a} \notin K$, and $\widehat{a}$ is special over K. We adopt from [3, Sections 2, 4] the definitions of $\operatorname{ndeg} P$ for $P \in K\{Y\}^{\neq}$ and of the set $Z(K,\widehat{a}) \subseteq K\{Y\}^{\neq}$. Also recall that $\Gamma(\partial) := \{v\phi :\ \phi \in K^\times,\ \partial\smallo \subseteq \phi\smallo\}$.

Lemma 1.5.19. *Let $P \in Z(K,\widehat{a})$ and $P \asymp 1$. Then $v\big(P(\widehat{a})\big) > v(\widehat{a} - K)$.*

Proof. Take a divergent pc-sequence (a_ρ) in $\mathcal{O}$ with $a_\rho \rightsquigarrow \widehat{a}$, and as in [ADH, 11.2] let $\boldsymbol{a} := c_K(a_\rho)$. Then $\operatorname{ndeg}_{\boldsymbol{a}} P \geqslant 1$ by [3, 4.7]. We arrange $\gamma_\rho := v(\widehat{a} - a_\rho)$ to be strictly increasing as a function of ρ, with $0 < 2\gamma_\rho < \gamma_{s(\rho)}$ for all ρ. Take $g_\rho \in \smallo$ with $g_\rho \asymp \widehat{a} - a_\rho$; then $1 \leqslant d := \operatorname{ndeg}_{\boldsymbol{a}} P = \operatorname{ndeg} P_{+a_\rho,\times g_\rho}$ for all sufficiently large ρ, and we arrange that this holds for all ρ. We have $\widehat{a} = a_\rho + g_\rho y_\rho$ with $y_\rho \asymp 1$, and

$$P(\widehat{a}) \ =\ P_{+a_\rho,\times g_\rho}(y_\rho) \ =\ \sum_i (P_{+a_\rho,\times g_\rho})_i(y_\rho).$$

Pick for every ρ an element $\phi_\rho \in K^\times$ such that $0 \leqslant v(\phi_\rho) \in \Gamma(\partial)$ and $(P^{\phi_\rho}_{+a_\rho,\times g_\rho})_i \preccurlyeq (P^{\phi_\rho}_{+a_\rho,\times g_\rho})_d$ for all i. Then for all ρ and i,

$$\begin{aligned} &(P_{+a_\rho,\times g_\rho})_i(y_\rho) \ = (P^{\phi_\rho}_{+a_\rho,\times g_\rho})_i(y_\rho) \ \preccurlyeq\ (P^{\phi_\rho}_{+a_\rho,\times g_\rho})_i \ \preccurlyeq\ (P^{\phi_\rho}_{+a_\rho,\times g_\rho})_d \ \text{ with} \\ &v\big((P^{\phi_\rho}_{+a_\rho,\times g_\rho})_d\big) \ \geqslant\ d\gamma_\rho + o(\gamma_\rho) \ \geqslant\ \gamma_\rho + o(\gamma_\rho), \end{aligned}$$

where for the next to last inequality we use [ADH, 11.1.1, 5.7.1, 5.7.5, 6.1.3]. Hence $v\big(P(\widehat{a})\big) \geqslant \gamma_\rho + o(\gamma_\rho)$ for all ρ, and thus $v\big(P(\widehat{a})\big) > v(\widehat{a} - K)$. □

We also have a converse under extra assumptions:

Lemma 1.5.20. *Assume K is asymptotic and $\Psi \subseteq v(\widehat{a} - K)$. Let $P \in K\{Y\}$ be such that $P \asymp 1$ and $v\big(P(\widehat{a})\big) > v(\widehat{a} - K)$. Then $P \in Z(K,\widehat{a})$.*

Proof. Let Δ be the nontrivial convex subgroup of Γ that is cofinal in $v(\widehat{a}-K)$. Let $\kappa := \operatorname{cf}(\Delta)$. Take a divergent pc-sequence $(a_\rho)_{\rho<\kappa}$ in K such that $a_\rho \rightsquigarrow \widehat{a}$. We arrange $\gamma_\rho := v(\widehat{a}-a_\rho)$ is strictly increasing as a function of ρ, with $\gamma_\rho > 0$ for all ρ; thus $a_\rho \preccurlyeq 1$ for all ρ. Consider the Δ-coarsening $\dot{v} = v_\Delta$ of the valuation v of K; it has valuation ring $\dot{\mathcal{O}}$ with differential residue field $\dot{K}$. Consider likewise the Δ-coarsening of the valuation of the immediate extension $L = K\langle\widehat{a}\rangle$ of K. Let a^* be the image of $\widehat{a}$ in the differential residue field $\dot{L}$ of $(L,\dot{v})$. Note that $\dot{L}$ is an immediate extension of $\dot{K}$. The pc-sequence (a_ρ) then yields a sequence $(\dot{a_\rho})$ in $\dot{K}$ with $v(a^*-\dot{a_\rho}) = \gamma_\rho$ for all ρ. Thus $(\dot{a}_\rho)$ is a c-sequence in $\dot{K}$ with $\dot{a}_\rho \to a^*$, so $\dot{P}(\dot{a}_\rho) \to \dot{P}(a^*)$ by [ADH, 4.4.5]. From $v\big(P(\widehat{a})\big) > \Delta$ we obtain $\dot{P}(a^*) = 0$, and so $\dot{P}(\dot{a}_\rho) \to 0$. So far we have not used our assumption that K is asymptotic and $\Psi \subseteq v(\widehat{a}-K)$. Using this now, we note that for $\alpha \in \Delta^>$ we have $0 < \alpha' = \alpha + \alpha^\dagger$, so $\alpha' \in \Delta$, hence the derivation of $\dot{K}$ is nontrivial. Thus we can apply [ADH, 4.4.10] to $\dot{K}$ and modify the a_ρ without changing $\gamma_\rho = v(a^*-\dot{a}_\rho)$ to arrange that in addition $\dot{P}(\dot{a}_\rho) \neq 0$ for all ρ. Since $\kappa = \operatorname{cf}(\Delta)$, we can replace (a_ρ) by a cofinal subsequence so that $P(a_\rho) \rightsquigarrow 0$, hence $P \in Z(K,\widehat{a})$ by [3, 4.6]. □

To elaborate on this, let Δ be a convex subgroup of Γ and $\dot{K}$ the valued differential residue field of the Δ-coarsening v_Δ of the valuation v of K. We view $\dot{K}$ in the usual way as a valued differential subfield of the valued differential residue field $\dot{\widehat{K}}$ of the Δ-coarsening of the valuation of $\widehat{K}$ by Δ; see [ADH, pp. 159–160 and 4.4.4].

Corollary 1.5.21. *Suppose K is asymptotic, $\Psi \subseteq v(\widehat{a}-K)$, and Δ is cofinal in $v(\widehat{a}-K)$. Let $P \in K\{Y\}$ with $P \asymp 1$. Then $P \in Z(K,\widehat{a})$ if and only if $\dot{P}(\dot{\widehat{a}}) = 0$ in $\dot{\widehat{K}}$. Also, P is an element of $Z(K,\widehat{a})$ of minimal complexity if and only if $\dot{P}$ is a minimal annihilator of $\dot{\widehat{a}}$ over $\dot{K}$ and $\dot{P}$ has the same complexity as P.*

Proof. The first statement is immediate from Lemmas 1.5.19 and 1.5.20. For the rest use that for $R \in \dot{\mathcal{O}}\{Y\}$ we have $\mathrm{c}(\dot{R}) \leqslant \mathrm{c}(R)$ and that for all $Q \in \dot{K}\{Y\}$ there is an $R \in \dot{\mathcal{O}}\{Y\}$ with $Q = \dot{R}$ and $Q_{\boldsymbol{i}} \neq 0$ iff $R_{\boldsymbol{i}} \neq 0$ for all $\boldsymbol{i}$, so $\mathrm{c}(\dot{R}) = \mathrm{c}(R)$. □

1.6. Differential Henselianity of the Completion

Let K be a valued differential field with small derivation. We let $\Gamma := v(K^\times)$ be the value group of K and $\boldsymbol{k} := \operatorname{res}(K)$ be the differential residue field of K, and we let $r \in \mathbb{N}$. The following summarizes [ADH, 7.1.1, 7.2.1]:

Lemma 1.6.1. *The valued differential field K is r-*d*-henselian iff for each P in $K\{Y\}$ of order $\leqslant r$ with* $\operatorname{ddeg} P = 1$ *there is a $y \in \mathcal{O}$ with $P(y) = 0$.*

Recall that the derivation of K being small, it is continuous [ADH, 4.4.6], and hence extends uniquely to a continuous derivation on the completion K^{c} of the valued field K [ADH, 4.4.11]. We equip K^{c} with this derivation, which remains small [ADH, 4.4.12], so K^{c} is an immediate valued differential field extension of K with small derivation, in particular, $\boldsymbol{k} = \operatorname{res}(K^{\mathrm{c}})$.

Below we characterize in a first-order way when K^{c} is r-d-henselian. We shall use tacitly that for $P \in K\{Y\}$ we have $P(g) \preccurlyeq P_{\times g}$ for all $g \in K$; to see this, replace P by $P_{\times g}$ to reduce to $g = 1$, and observe that $P(1) = \sum_{\|\boldsymbol{\sigma}\|=0} P_{[\boldsymbol{\sigma}]} \preccurlyeq P$.

Lemma 1.6.2. *Let $P \in K^{\mathrm{c}}\{Y\}$, $b \in K^{\mathrm{c}}$ with $b \preccurlyeq 1$ and $P(b) = 0$, and $\gamma \in \Gamma^>$. Then there is an $a \in \mathcal{O}$ with $v\big(P(a)\big) > \gamma$.*

Proof. To find an a as claimed we take $f \in K$ satisfying $f \asymp P$ and replace P, γ by $f^{-1}P$, $\gamma - vf$, respectively, to arrange $P \asymp 1$ and thus $P_{+b} \asymp 1$. We also assume $b \neq 0$. Since K is dense in K^{c} we can take $a \in K$ such that $a \sim b$ (so $a \in \mathcal{O}$) and $v(a-b) > 2\gamma$. Then with $g := a-b$, using [ADH, 4.5.1(i) and 6.1.4] we conclude

$$v\big(P(a)\big) = v\big(P_{+b}(g)\big) \geqslant v\big((P_{+b})_{\times g}\big) \geqslant v(P_{+b}) + vg + o(vg) = vg + o(vg) > \gamma$$

as required. □

Recall that if K is asymptotic, then so is K^{c} by [ADH, 9.1.6].

Lemma 1.6.3. *Suppose K is asymptotic, $\Gamma \neq \{0\}$, and for every $P \in K\{Y\}$ of order at most r with $\operatorname{ddeg} P = 1$ and every $\gamma \in \Gamma^{>}$ there is an $a \in \mathcal{O}$ with $v\big(P(a)\big) > \gamma$. Then K^{c} is r-*d*-henselian.*

Proof. The hypothesis applied to $P \in \mathcal{O}\{Y\}$ of order $\leqslant r$ with $\operatorname{ddeg} P = \deg P = 1$ yields that $\boldsymbol{k}$ is r-linearly surjective. Let now $P \in K^{\mathrm{c}}\{Y\}$ be of order $\leqslant r$ with $\operatorname{ddeg} P = 1$. We need to show that there exists $b \in K^{\mathrm{c}}$ with $b \preccurlyeq 1$ and $P(b) = 0$. First we arrange $P \asymp 1$. By [ADH, remarks after 9.4.11] we can take $b \preccurlyeq 1$ in an immediate d-henselian asymptotic field extension L of K^{c} with $P(b) = 0$. We prove below that $b \in K^{\mathrm{c}}$. We may assume $b \notin K$, so $v(b-K)$ has no largest element, since $L \supseteq K$ is immediate. Note also that $\operatorname{ddeg} P_{+b} = 1$ by [ADH, 6.6.5(i)]; since $P(b) = 0$ we thus have $\operatorname{ddeg} P_{+b,\times g} = 1$ for all $g \preccurlyeq 1$ in $L^{\times}$ by [ADH, 6.6.7].

Claim: *Let $\gamma \in \Gamma^{>}$ and $a \in K$ with $v(b-a) \geqslant 0$. There is a $y \in \mathcal{O}$ such that $v\big(P(y)\big) > \gamma$ and $v(b-y) \geqslant v(b-a)$.*

To prove this claim, take $g \in K^{\times}$ with $vg = v(b-a)$. Then by [ADH, 6.6.6] and the observation preceding the claim we have $\operatorname{ddeg} P_{+a,\times g} = \operatorname{ddeg} P_{+b,\times g} = 1$. Thanks to density of K in K^{c} we may take $Q \in K\{Y\}$ of order $\leqslant r$ with $P_{+a,\times g} \sim Q$ and $v(P_{+a,\times g} - Q) > \gamma$. Then $\operatorname{ddeg} Q = 1$, so by the hypothesis of the lemma we have $z \in \mathcal{O}$ with $v\big(Q(z)\big) > \gamma$. Set $y := a + gz \in \mathcal{O}$; then we have $v\big(P(y)\big) = v\big(P_{+a,\times g}(z)\big) > \gamma$ and $v(b-y) = v(b-a-gz) \geqslant v(b-a) = vg$ as claimed.

Let now $\gamma \in \Gamma^{>}$; to show that $b \in K^{\mathrm{c}}$, it is enough by [ADH, 3.2.15, 3.2.16] to show that then $v(a-b) > \gamma$ for some $a \in K$. Let $A := L_{P_{+b}} \in L[\partial]$; then $A \asymp 1$. Since $|\mathscr{E}_L(A)| \leqslant r$ by [ADH, 7.5.3], the claim gives an $a \in \mathcal{O}$ with $v\big(P(a)\big) > 2\gamma$ and $0 < v(b-a) \notin \mathscr{E}_L(A)$. Put $g := a-b$ and $R := (P_{+b})_{>1}$. Then $R \prec 1$ and

$$P(a) \;=\; P_{+b}(g) \;=\; A(g) + R(g)$$

where by the definition of $\mathscr{E}_L(A)$ and [ADH, 6.4.1(iii), 6.4.3] we have in $\mathbb{Q}\Gamma$:

$$v\big(A(g)\big) \;=\; v_A(vg) \;=\; vg + o(vg) \;<\; vR + (3/2)vg \;\leqslant\; v_R(vg) \;\leqslant\; v\big(R(g)\big)$$

and so $v\big(P(a)\big) = vg + o(vg) > 2\gamma$. Therefore $v(a-b) = vg > \gamma$ as required. □

The last two lemmas yield an analogue of [ADH, 3.3.7] for r-d-henselianity and a partial generalization of [ADH, 7.2.15]:

Corollary 1.6.4. *Suppose K is asymptotic and $\Gamma \neq \{0\}$. Then the following are equivalent:*

(i) *K^{c} is r-*d*-henselian;*
(ii) *for every $P \in K\{Y\}$ of order at most r with $\operatorname{ddeg} P = 1$ and every $\gamma \in \Gamma^{>}$ there exists $a \in \mathcal{O}$ with $v\big(P(a)\big) > \gamma$.*

*In particular, if K is r-*d*-henselian, then so is K^{c}.*

In this section K is an ungrounded H-asymptotic field with $\Gamma = v(K^\times) \neq \{0\}$. Note that then K^{c} is also H-asymptotic. We let r range over $\mathbb{N}$ and ϕ over the active elements of K. Our first aim is a newtonian analogue of Corollary 1.6.4:

Proposition 1.7.1. *For* d*-valued and* ω*-free* K*, the following are equivalent:*

(i) K^{c} *is* r*-newtonian;*
(ii) *for every* $P \in K\{Y\}$ *of order at most* r *with* $\operatorname{ndeg} P = 1$ *and every* $\gamma \in \Gamma^>$ *there is an* $a \in \mathcal{O}$ *with* $v\big(P(a)\big) > \gamma$.

If K *is* d*-valued,* ω*-free, and* r*-newtonian, then so is* K^{c}.

The final statement in this proposition extends [ADH, 14.1.5]. Towards the proof we first state a variant of [ADH, 13.2.2] which follows easily from [ADH, 11.1.4]:

Lemma 1.7.2. *Assume* K *has small derivation and let* $P, Q \in K\{Y\}^{\neq}$ *and* $\phi \preccurlyeq 1$. *Then* $P^\phi \asymp^\flat P$*, and so we have the three implications*

$$P \preccurlyeq^\flat Q \implies P^\phi \preccurlyeq^\flat Q^\phi, \quad P \prec^\flat Q \implies P^\phi \prec^\flat Q^\phi, \quad P \sim^\flat Q \implies P^\phi \sim^\flat Q^\phi.$$

The last implication gives: $P \sim^\flat Q \implies \operatorname{ndeg} P = \operatorname{ndeg} Q$ *and* $\operatorname{nmul} P = \operatorname{nmul} Q$.

For $P^\phi \asymp^\flat P$ and the subsequent three implications in the lemma above we can drop the assumption that K is ungrounded.

Lemma 1.7.3. *Suppose* K *is* d*-valued,* ω*-free, and for every* $P \in K\{Y\}$ *of order at most* r *with* $\operatorname{ndeg} P = 1$ *and every* $\gamma \in \Gamma^>$ *there is an* $a \in \mathcal{O}$ *with* $v\big(P(a)\big) > \gamma$. *Then* K^{c} *is* d*-valued,* ω*-free, and* r*-newtonian.*

Proof. By [ADH, 9.1.6 and 11.7.20], K^{c} is d-valued and ω-free. Let $P \in K^{\mathrm{c}}\{Y\}$ be of order $\leqslant r$ with $\operatorname{ndeg} P = 1$. We need to show that $P(b) = 0$ for some $b \preccurlyeq 1$ in K^{c}. To find b we may replace K, P by K^ϕ, P^ϕ; in particular we may assume that K has small derivation and $\Gamma^\flat \neq \Gamma$. By (0.7) we can take $b \preccurlyeq 1$ in an immediate newtonian extension L of K^{c} such that $P(b) = 0$. We claim that $b \in K^{\mathrm{c}}$. To show this we may assume $b \notin K$, so $v(b - K)$ does not have a largest element. By [ADH, 11.2.3(i)] we have $\operatorname{ndeg} P_{+b} = 1$ and so $\operatorname{ndeg} P_{+b,\times g} = 1$ for all $g \preccurlyeq 1$ in $L^\times$ by [ADH, 11.2.5], in view of $P(b) = 0$.

Claim: *Let* $\gamma \in \Gamma^>$ *and* $a \in K$ *with* $v(b - a) \geqslant 0$. *There is a* $y \in \mathcal{O}$ *such that* $v\big(P(y)\big) > \gamma$ *and* $v(b - y) \geqslant v(b - a)$.

The proof is similar to that of the claim in the proof of Lemma 1.6.3: Take $g \in K^\times$ with $vg = v(b-a)$. Then $\operatorname{ndeg} P_{+a,\times g} = \operatorname{ndeg} P_{+b,\times g} = 1$ by [ADH, 11.2.4] and the observation preceding the claim. Density of K in K^{c} yields $Q \in K\{Y\}$ of order $\leqslant r$ with $v(P_{+a,\times g} - Q) > \gamma$ and $P_{+a,\times g} \sim^\flat Q$, the latter using $\Gamma^\flat \neq \Gamma$. Then $\operatorname{ndeg} Q = \operatorname{ndeg} P_{+a,\times g} = 1$ by Lemma 1.7.2, so the hypothesis of the lemma gives $z \in \mathcal{O}$ with $v\big(Q(z)\big) > \gamma$. Setting $y := a + gz \in \mathcal{O}$ we have $v\big(P(y)\big) = v\big(P_{+a,\times g}(z)\big) > \gamma$ and $v(b - y) = v(b - a - gz) \geqslant vg = v(b - a)$.

Let $\gamma \in \Gamma^>$; to get $b \in K^{\mathrm{c}}$, it is enough to show that then $v(a - b) > \gamma$ for some $a \in K$. Let $A := L_{P_{+b}} \in L[\partial]$. Since $|\mathscr{E}^{\mathrm{e}}_L(A)| \leqslant r$ by [ADH, 14.2.9], by the claim we can take $a \in \mathcal{O}$ with $v\big(P(a)\big) > 2\gamma$ and $0 < v(b - a) \notin \mathscr{E}^{\mathrm{e}}_L(A)$. Now put $g := a - b$ and take ϕ with $vg \notin \mathscr{E}_{L^\phi}(A^\phi)$; note that then $A^\phi = L_{P^\phi_{+b}}$. Replacing K, L, P by K^ϕ, L^ϕ, P^ϕ we arrange $vg \notin \mathscr{E}_L(A)$, and (changing ϕ if necessary) $\operatorname{ddeg} P_{+b} = 1$. We also arrange $P_{+b} \asymp 1$, and then $(P_{+b})_{>1} \prec 1$. As in the proof of Lemma 1.6.3 above we now derive $v(a-b) = vg > \gamma$. $\square$

Combining Lemmas 1.6.2 and 1.7.3 now yields Proposition 1.7.1. $\square$

To show that newtonianity is preserved under specialization, we assume below that $\Psi \cap \Gamma^{>} \neq \emptyset$, so K has small derivation. Let $\Delta \neq \{0\}$ be a convex subgroup of Γ with $\psi(\Delta^{\neq}) \subseteq \Delta$. Then $1 \in \Delta$ where 1 denotes the unique positive element of Γ fixed by the function ψ: use that $\psi(\gamma) \geqslant 1$ for $0 < \gamma < 1$. (Conversely, any convex subgroup G of Γ with $1 \in G$ satisfies $\psi(G^{\neq}) \subseteq G$.) Let $\dot{v}$ be the coarsening of the valuation v of K by Δ, with valuation ring $\dot{\mathcal{O}}$, maximal ideal $\dot{o}$ of $\dot{\mathcal{O}}$, and residue field $\dot{K} = \dot{\mathcal{O}}/\dot{o}$. The derivation of K is small with respect to $\dot{v}$, and $\dot{K}$ with the induced valuation $v\colon \dot{K}^{\times} \to \Delta$ and induced derivation as in [ADH, p. 405] is an asymptotic field with asymptotic couple $(\Delta, \psi|\Delta^{\neq})$, and so is of H-type with small derivation. If K is d-valued, then so is $\dot{K}$ by [ADH, 10.1.8], and if K is ω-free, then so is $\dot{K}$ by [ADH, 11.7.24]. The residue map $a \mapsto \dot{a} := a + \dot{o}\colon \dot{\mathcal{O}} \to \dot{K}$ is a differential ring morphism, extends to a differential ring morphism $P \mapsto \dot{P}\colon \dot{\mathcal{O}}\{Y\} \to \dot{K}\{Y\}$ of differential rings sending Y to Y, and $\operatorname{ddeg} P = \operatorname{ddeg} \dot{P}$ for $P \in \dot{\mathcal{O}}\{Y\}$ with $\dot{P} \neq 0$.

We now restrict ϕ to range over active elements of $\mathcal{O}$. Then $v\phi \leqslant 1{+}1$, so $v\phi \in \Delta$, and hence ϕ is a unit of $\dot{\mathcal{O}}$. It follows that $\dot{\phi}$ is active in $\dot{K}$, and every active element of $\dot{K}$ lying in its valuation ring is of this form. Moreover, the differential subrings $\dot{\mathcal{O}}$ of K and $\dot{\mathcal{O}}^{\phi} := (\dot{\mathcal{O}})^{\phi}$ of K^{ϕ} have the same underlying ring, and the derivation of K^{ϕ} is small with respect to $\dot{v}$. Thus the differential residue fields $\dot{K} = \dot{\mathcal{O}}/\dot{o}$ and $\dot{K^{\phi}} := \dot{\mathcal{O}}^{\phi}/\dot{o}$ have the same underlying field and are related as follows:

$$\dot{K^{\phi}} \;=\; (\dot{K})^{\dot{\phi}}.$$

For $P \in \dot{\mathcal{O}}\{Y\}$ we have $P^{\phi} \in \dot{\mathcal{O}}^{\phi}\{Y\}$, and the image of P^{ϕ} under the residue map $\dot{\mathcal{O}}^{\phi}\{Y\} \to \dot{K^{\phi}}\{Y\}$ equals $\dot{P}^{\dot{\phi}}$; hence $\operatorname{ndeg} P = \operatorname{ndeg} \dot{P}$ for $P \in \dot{\mathcal{O}}\{Y\}$ satisfying $\dot{P} \neq 0$. These remarks imply:

Lemma 1.7.4. *If K is r-newtonian, then $\dot{K}$ is r-newtonian.*

Combining Proposition 1.7.1 and Lemmas 1.7.3 and 1.7.4 yields:

Corollary 1.7.5. *Suppose K is* d*-valued,* ω*-free, and r-newtonian. Then $\dot{K}$ and its completion are* d*-valued,* ω*-free, and r-newtonian.*

We finish with a newtonian analogue of [ADH, 7.1.7]:

Lemma 1.7.6. *Suppose $(K, \dot{\mathcal{O}})$ is r-*d*-henselian and $\dot{K}$ is r-newtonian. Then K is r-newtonian.*

Proof. Let $P \in K\{Y\}$ be quasilinear of order $\leqslant r$; we need to show the existence of $b \in \mathcal{O}$ with $P(b) = 0$. Replacing K, P by K^{ϕ}, P^{ϕ} for suitable ϕ (and renaming) we arrange $\operatorname{ddeg} P = 1$; this also uses [ADH, 7.3, subsection on compositional conjugation]. We can further assume that $P \asymp 1$. Put $Q := \dot{P} \in \dot{K}\{Y\}$, so $\operatorname{ndeg} Q = 1$, and thus r-newtonianity of $\dot{K}$ yields an $a \in \mathcal{O}$ with $Q(\dot{a}) = 0$. Then $P(a) \mathrel{\dot{\prec}} 1$, $P_{+a,1} \sim P_1 \asymp 1$, and $P_{+a,>1} \prec 1$. Since $(K, \dot{\mathcal{O}})$ is r-d-henselian, this gives $y \in \dot{o}$ with $P_{+a}(y) = 0$, and then $P(b) = 0$ for $b := a + y \in \mathcal{O}$. $\square$

Lemmas 1.7.4, 1.7.6, and [ADH, 14.1.2] yield:

Corollary 1.7.7. *K is r-newtonian iff $(K, \dot{\mathcal{O}})$ is r-*d*-henselian and $\dot{K}$ is r-newtonian.*

Invariance of Newton quantities. *In this subsection* $P \in K\{Y\}^{\neq}$. In [ADH, 11.1] we associated to P its Newton weight $\operatorname{nwt} P$, Newton degree $\operatorname{ndeg} P$, and Newton multiplicity $\operatorname{nmul} P$ at 0, all elements of $\mathbb{N}$, as well as the element $v^{\mathrm{e}}(P)$ of Γ; these quantities do not change when passing to an H-asymptotic extension L of K with Ψ cofinal in Ψ_L, cf. [ADH, p. 480], where the assumptions on K, L are slightly weaker. Thus by Theorem 1.3.1, these quantities do not change for ω-free K in passing to an H-asymptotic pre-d-valued d-algebraic extension of K. Below we improve on this in several ways. First, for order $P \leqslant 1$ we can drop Ψ being cofinal in Ψ_L by a strengthening of [ADH, 11.2.13]:

Lemma 1.7.8. *Suppose K is H-asymptotic with rational asymptotic integration and $P \in K[Y, Y']^{\neq}$. Then there are $w \in \mathbb{N}$, $\alpha \in \Gamma^{>}$, $A \in K[Y]^{\neq}$, and an active ϕ_0 in K such that for every asymptotic extension L of K and active $f \preccurlyeq \phi_0$ in L,*

$$P^f \ = \ f^w A(Y)(Y')^w + R_f, \quad R_f \in L^f[Y, Y'], \quad v(R_f) \ \geqslant \ v(P^f) + \alpha.$$

For such w, A we have for any ungrounded H-asymptotic extension L of K,

$$\operatorname{nwt}_L P \ = \ w, \quad \operatorname{ndeg}_L P \ = \ \deg A + w, \quad \operatorname{nmul}_L P \ = \ \operatorname{mul} A + w, \quad v^{\mathrm{e}}_L(P) = v(A).$$

Proof. Let w be as in the proof of [ADH, 11.2.13]. Using its notations, this proof yields an active ϕ_0 in K such that

$$(1.7.1) \qquad w\gamma + v(A_w) \ < \ j\gamma + v(A_j)$$

for all $\gamma \geqslant v(\phi_0)$ in $\Psi^{\downarrow}$ and $j \in J \setminus \{w\}$. This gives $\beta \in \mathbb{Q}\Gamma$ such that $\beta > \Psi$ and (1.7.1) remains true for all $\gamma \in \Gamma$ with $v(\phi_0) \leqslant \gamma < \beta$. Since $(\mathbb{Q}\Gamma, \psi)$ has asymptotic integration, β is not a gap in $(\mathbb{Q}\Gamma, \psi)$, so $\beta > \beta_0 > \Psi$ with $\beta_0 \in \mathbb{Q}\Gamma$. This yields an element $\alpha \in (\mathbb{Q}\Gamma)^{>}$ such that for all $\gamma \in \mathbb{Q}\Gamma$ with $v(\phi_0) \leqslant \gamma \leqslant \beta_0$ we have

$$(1.7.2) \qquad w\gamma + v(A_w) + \alpha \ \leqslant \ j\gamma + v(A_j)$$

Since Γ has no least positive element, we can decrease α to arrange $\alpha \in \Gamma^{>}$. Now (1.7.2) remains true for all elements γ of any divisible ordered abelian group extending $\mathbb{Q}\Gamma$ with $v(\phi_0) \leqslant \gamma \leqslant \beta_0$. Thus w, α, $A = A_w$, and ϕ_0 are as required.

For any ungrounded H-asymptotic extension L of K we obtain for active $f \preccurlyeq \phi_0$ in L that $v(P^f) = v(A) + wv(f)$, so $v^{\mathrm{e}}_L(P) = v(A)$ in view of the identity in [ADH, 11.1.15] defining $v^{\mathrm{e}}_L(P)$. □

For quasilinear P we have:

Lemma 1.7.9. *Suppose K is λ-free and* $\operatorname{ndeg} P = 1$. *Then there are active ϕ_0 in K and $a, b \in K$ with $a \preccurlyeq b \neq 0$ such that either* (i) *or* (ii) *below holds:*

(i) $P^f \sim^{\flat}_{\phi_0} a + bY$ *for all active $f \preccurlyeq \phi_0$ in all H-asymptotic extensions of K;*

(ii) $P^f \sim^{\flat}_{\phi_0} \frac{f}{\phi_0} bY'$ *for all active $f \preccurlyeq \phi_0$ in all H-asymptotic extensions of K.*

In particular, for each ungrounded H-asymptotic extension L of K,

$$\operatorname{nwt}_L P = \operatorname{nwt} P \leqslant 1, \quad \operatorname{ndeg}_L P = 1, \quad \operatorname{nmul}_L P = \operatorname{nmul} P, \quad v^{\mathrm{e}}_L(P) = v^{\mathrm{e}}(P).$$

Proof. From [ADH, 13.7.10] we obtain an active ϕ_0 in K and $a, b \in K$ with $a \preccurlyeq b$ such that in $K^{\phi_0}\{Y\}$, either $P^{\phi_0} \sim^{\flat}_{\phi_0} a + bY$ or $P^{\phi_0} \sim^{\flat}_{\phi_0} bY'$ (so $b \neq 0$). In the first case, set $A(Y) := a + bY$, $w := 0$; in the second case, set $A(Y) := bY'$, $w := 1$. Then $P^{\phi_0} = A + R$ where $R \prec^{\flat}_{\phi_0} b \asymp P^{\phi_0}$ in $K^{\phi_0}\{Y\}$.

Let L be an H-asymptotic extension of K. Then $R \prec^{\flat}_{\phi_0} P^{\phi_0}$ remains true in $L^{\phi_0}\{Y\}$, and if $f \preccurlyeq \phi_0$ is active in L, then $P^f = (P^{\phi_0})^{f/\phi_0} = (f/\phi_0)^w A + R^{f/\phi_0}$ where $R^{f/\phi_0} \prec^{\flat}_{\phi_0} P^f$ by Lemma 1.7.2 and the remark following its proof. As to $v^{\mathrm{e}}_L(P) = v^{\mathrm{e}}(P)$ for ungrounded L, the identity from [ADH, 11.1.15] defining these quantities shows that both are vb in case (i), and $v(b) - v(\phi_0)$ in case (ii). $\square$

Lemma 1.7.9 has the following consequence, partly generalizing Corollary 1.4.5:

Corollary 1.7.10. *Suppose K is λ-free, $A \in K[\partial]^{\neq}$ and L is an ungrounded H-asymptotic extension of K. Then for $\gamma \in \Gamma$ the quantities $\operatorname{nwt}_A(\gamma) \leqslant 1$ and $v^{\mathrm{e}}_A(\gamma)$ do not change when passing from K to L; in particular,*

$$\mathscr{E}^{\mathrm{e}}(A) \ = \ \big\{\gamma \in \Gamma : \operatorname{nwt}_A(\gamma) = 1\big\} \ = \ \mathscr{E}^{\mathrm{e}}_L(A) \cap \Gamma.$$

This leads to a variant of Corollary 1.4.20:

Corollary 1.7.11. *Suppose K is λ-free. Then $|\mathscr{E}^{\mathrm{e}}(A)| \leqslant \operatorname{order} A$ for all $A \in K[\partial]^{\neq}$.*

Proof. By [ADH, 10.1.3], K is pre-d-valued, hence by [ADH, 11.7.18] it has an ω-free H-asymptotic extension. It remains to appeal to Corollaries 1.4.5 and 1.7.10. $\square$

For completeness we next state a version of Lemma 1.7.9 for $\operatorname{ndeg} P = 0$; the proof using [ADH, 13.7.9] is similar, but simpler, and hence omitted.

Lemma 1.7.12. *Suppose K is λ-free and $\operatorname{ndeg} P = 0$. Then there are an active ϕ_0 in K and $a \in K^{\times}$ such that $P^f \sim^{\flat}_{\phi_0} a$ for all active $f \preccurlyeq \phi_0$ in all H-asymptotic extensions of K.*

In particular, for K, P as in Lemma 1.7.12, no H-asymptotic extension of K contains any $y \preccurlyeq 1$ such that $P(y) = 0$.

For general P and ω-free K we can still do better than stated earlier:

Lemma 1.7.13. *Suppose K is ω-free. Then there are $w \in \mathbb{N}$, $A \in K[Y]^{\neq}$, and an active ϕ_0 in K such that for all active $f \preccurlyeq \phi_0$ in all H-asymptotic extensions of K,*

$$P^f \ \sim^{\flat}_{\phi_0} \ (f/\phi_0)^w A(Y)(Y')^w.$$

For such w, A, ϕ_0 we have for any ungrounded H-asymptotic extension L of K,

$$\operatorname{nwt}_L P \ = \ w, \qquad \operatorname{ndeg}_L P \ = \ \deg A + w,$$
$$\operatorname{nmul}_L P \ = \ \operatorname{mul} A + w, \qquad v^{\mathrm{e}}_L(P) \ = \ v(A) - wv(\phi_0).$$

Proof. By [ADH, 13.6.11] we have active ϕ_0 in K and $A \in K[Y]^{\neq}$ such that

$$P^{\phi_0} \ = \ A \cdot (Y')^w + R, \quad w := \operatorname{nwt} P, \quad R \in K^{\phi_0}\{Y\},\ R \prec^{\flat}_{\phi_0} P^{\phi_0}.$$

(Here ϕ_0 and A are the e and aA in [ADH, 13.6.11].) The rest of the argument is just like in the second part of the proof of Lemma 1.7.9. $\square$

Remarks on newton position. For the next lemma we put ourselves in the setting of [ADH, 14.3]: K is ω-free, $P \in K\{Y\}^{\neq}$, and a ranges over K. Recall that P is said to be in *newton position at a* if $\operatorname{nmul} P_{+a} = 1$.

Suppose P is in newton position at a; then $A := L_{P_{+a}} \in K[\partial]^{\neq}$. Recall the definition of $v^{\mathrm{e}}(P,a) = v^{\mathrm{e}}_K(P,a) \in \Gamma_{\infty}$: if $P(a) = 0$, then $v^{\mathrm{e}}(P,a) = \infty$; if $P(a) \neq 0$, then $v^{\mathrm{e}}(P,a) = vg$ where $g \in K^{\times}$ satisfies $P(a) \asymp (P_{+a})^{\phi}_{1,\times g}$ eventually, that is, $v_{A^{\phi}}(vg) = v\big(P(a)\big)$ eventually. In the latter case $\operatorname{nwt}_A(vg) = 0$, that is, $vg \notin \mathscr{E}^{\mathrm{e}}(A)$, and $v^{\mathrm{e}}_A(vg) = v\big(P(a)\big)$, since $v_{A^{\phi}}(vg) = v^{\mathrm{e}}_A(vg) + \operatorname{nwt}_A(vg)v\phi$ eventually. For any $f \in K^\times$, P^f is also in newton position at a, and $v^{\mathrm{e}}(P^f, a) = v^{\mathrm{e}}(P, a)$. Note also that P_{+a} is in newton position at 0 and $v^{\mathrm{e}}(P_{+a}, 0) = v^{\mathrm{e}}(P, a)$. Moreover, in passing from K to an ω-free extension, P remains in newton position at a and $v^{\mathrm{e}}(P, a)$ does not change, by Lemma 1.7.13.

In the rest of this subsection P is in newton position at a, and $\widehat{a}$ is an element of an H-asymptotic extension $\widehat{K}$ of K such that $P(\widehat{a}) = 0$. (We allow $\widehat{a} \in K$.) We first generalize part of [ADH, 14.3.1], with a similar proof:

Lemma 1.7.14. *$v^{\mathrm{e}}(P, a) > 0$ and $v(\widehat{a} - a) \leqslant v^{\mathrm{e}}(P, a)$.*

Proof. This is clear if $P(a) = 0$. Assume $P(a) \neq 0$. Replace $P, \widehat{a}, a$ by $P_{+a}, \widehat{a}-a, 0$, respectively, to arrange $a = 0$. Recall that K^ϕ has small derivation. Set $\gamma := v^{\mathrm{e}}(P, 0) \in \Gamma$ and take $g \in K$ with $vg = \gamma$. Now $(P_1^\phi)_{\times g} \asymp P_0$, eventually, and $\operatorname{nmul} P = 1$ gives $P(0) \prec P_1^\phi$, eventually, hence $g \prec 1$. Moreover, for $j \geqslant 2$, $P_1^\phi \succcurlyeq P_j^\phi$, eventually, so $(P_1^\phi)_{\times g} \succ (P_j^\phi)_{\times g}$, eventually, by [ADH, 6.1.3]. Thus for $j \geqslant 1$ we have $(P_{\times g}^\phi)_j = (P_j^\phi)_{\times g} \preccurlyeq P(0)$, eventually; in particular, there is no $y \prec 1$ in any H-asymptotic extension of K with $P_{\times g}(y) = 0$. Since $P(\widehat{a}) = 0$, this yields $v(\widehat{a}) \leqslant \gamma = v^{\mathrm{e}}(P, 0)$. $\square$

Here is a situation where $v(\widehat{a} - a) = v^{\mathrm{e}}(P, a)$:

Lemma 1.7.15. *Suppose Ψ is cofinal in $\Psi_{\widehat{K}}$, $\widehat{a} - a \prec 1$, and $v(\widehat{a} - a) \notin \mathscr{E}^{\mathrm{e}}_{\widehat{K}}(A)$ where $A := L_{P_{+a}}$. Then $v(\widehat{a} - a) = v^{\mathrm{e}}(P, a)$.*

Proof. Note that $\widehat{K}$ is ungrounded, so $\mathscr{E}^{\mathrm{e}}_{\widehat{K}}(A)$ is defined, and $\widehat{K}$ is pre-d-valued. As in the proof of Lemma 1.7.14 we arrange $a = 0$. As an asymptotic subfield of $\widehat{K}$, $K\langle\widehat{a}\rangle$ is pre-d-valued. Hence $K\langle\widehat{a}\rangle$ is ω-free by Theorem 1.3.1. The remarks preceding Lemma 1.7.14 then allow us to replace K by $K\langle\widehat{a}\rangle$ to arrange $\widehat{a} \in K$. The case $\widehat{a} = 0$ is trivial, so assume $0 \neq \widehat{a} \prec 1$. Now $\operatorname{nmul} P = 1$ gives for $j \geqslant 2$ that $P_1^\phi \succcurlyeq P_j^\phi$, eventually, hence $(P_1^\phi)_{\times\widehat{a}} \succ (P_j^\phi)_{\times\widehat{a}}$, eventually, by [ADH, 6.1.3]. Moreover, $P_1(\widehat{a}) = A(\widehat{a}) = A^\phi(\widehat{a}) \asymp A^\phi\widehat{a}$, eventually, using $v(\widehat{a}) \notin \mathscr{E}^{\mathrm{e}}_{\widehat{K}}(A)$ in the last step, so for $j \geqslant 2$, eventually

$$P_1(\widehat{a}) \asymp (P_1^\phi)_{\times\widehat{a}} \succ (P_j^\phi)_{\times\widehat{a}} \succcurlyeq P_j^\phi(\widehat{a}) = P_j(\widehat{a}).$$

Also $P_1(\widehat{a}) \neq 0$, since $A^\phi\widehat{a} \neq 0$. Then $P(\widehat{a}) = 0$ gives $P(0) \asymp P_1(\widehat{a})$. Thus $v\big(P(0)\big) = v_{A^\phi}\big(v(\widehat{a})\big)$, eventually, so $v^{\mathrm{e}}(P, 0) = v(\widehat{a})$ by the definition of $v^{\mathrm{e}}(P, 0)$. $\square$

Corollary 1.7.16. *Suppose $\widehat{K}$ is ungrounded and equipped with an ordering making it a pre-H-field, and assume $\widehat{a} - a \prec 1$ and $v(\widehat{a} - a) \notin \mathscr{E}^{\mathrm{e}}_{\widehat{K}}(A)$ where $A := L_{P_{+a}}$. Then $v(\widehat{a} - a) = v^{\mathrm{e}}(P, a)$.*

Proof. In view of Lemma 1.4.1 and using [ADH, 14.5.11] we can extend $\widehat{K}$ to arrange that it is an ω-free newtonian Liouville closed H-field. Next, let H be the real closure of the H-field hull of $K\langle\widehat{a}\rangle$, all inside $\widehat{K}$. Then H is ω-free, by Theorem 1.3.1, and hence has a Newton-Liouville closure L inside $\widehat{K}$ [ADH, 14.5]. Since $L \preccurlyeq \widehat{K}$ by [ADH, 16.2.5], we have $v(\widehat{a} - a) \notin \mathscr{E}^{\mathrm{e}}_L(A)$. Now L is d-algebraic over K by [ADH, 14.5.9], so Ψ is cofinal in Ψ_L by Theorem 1.3.1. It remains to apply Lemma 1.7.15. $\square$

Newton position in the order 1 case. *In this subsection K is λ-free, $P \in K\{Y\}$ has order 1, and $a \in K$.* We basically copy here a definition and two lemmas from [ADH, 14.3] with the ω-free assumption there replaced by the weaker λ-freeness, at the cost of restricting P to have order 1.

Suppose $\operatorname{nmul} P = 1$, $P_0 \neq 0$. Then [ADH, 11.6.17] yields $g \in K^\times$ such that $P_0 \asymp P^\phi_{1,\times g}$, eventually. Since $P_0 \prec P^\phi_1$, eventually, we have $g \prec 1$. Moreover, if $i \geqslant 2$, then $P^\phi_1 \succcurlyeq P^\phi_i$, eventually, hence $P^\phi_{1,\times g} \succ P^\phi_{i,\times g}$, eventually. Therefore $\operatorname{ndeg} P_{\times g} = 1$.

Define P to be in **newton position at** a if $\operatorname{nmul} P_{+a} = 1$. Suppose P is in newton position at a; set $Q := P_{+a}$, so $Q(0) = P(a)$. If $P(a) \neq 0$, then the above yields $g \in K^\times$ with $P(a) = Q(0) \asymp Q^\phi_{1,\times g}$, eventually; as vg does not depend on the choice of such g, we set $v^{\mathrm{e}}(P,a) := vg$. If $P(a) = 0$, then we set $v^{\mathrm{e}}(P,a) := \infty \in \Gamma_\infty$. In passing from K to a λ-free extension, P remains in newton position at a and $v^{\mathrm{e}}(P,a)$ does not change, by Lemma 1.7.8. *In the rest of this subsection we assume P is in newton position at a.*

Lemma 1.7.17. *If $P(a) \neq 0$, then there exists $b \in K$ with the following properties:*

(i) *P is in newton position at b, $v(a-b) = v^{\mathrm{e}}(P,a)$, and $P(b) \prec P(a)$;*

(ii) *for all $b^* \in K$ with $v(a-b^*) \geqslant v^{\mathrm{e}}(P,a)$: $P(b^*) \prec P(a) \Leftrightarrow a-b \sim a-b^*$;*

(iii) *for all $b^* \in K$, if $a-b \sim a-b^*$, then P is in newton position at b^* and $v^{\mathrm{e}}(P,b^*) > v^{\mathrm{e}}(P,a)$.*

This is shown as in [ADH, 14.3.2]. Next an analogue of [ADH, 14.3.3], with the same proof, but using Lemma 1.7.17 in place of [ADH, 14.3.2]:

Lemma 1.7.18. *If there is no b with $P(b) = 0$ and $v(a-b) = v^{\mathrm{e}}(P,a)$, then there is a divergent pc-sequence $(a_\rho)_{\rho<\lambda}$ in K, indexed by all ordinals ρ smaller than some infinite limit ordinal λ, such that $a_0 = a$, $v(a_\rho - a_{\rho'}) = v^{\mathrm{e}}(P,a_\rho)$ for all $\rho < \rho' < \lambda$, and $P(a_\rho) \rightsquigarrow 0$.*

The next result is proved just like Lemma 1.7.14:

Lemma 1.7.19. *If $P(\widehat{a}) = 0$ with $\widehat{a}$ in an H-asymptotic extension of K, then $v^{\mathrm{e}}(P,a) > 0$ and $v(\widehat{a}-a) \leqslant v^{\mathrm{e}}(P,a)$.*

Next an analogue of Lemma 1.7.15 using Propositions 1.3.8 and 1.3.12 in its proof:

Lemma 1.7.20. *Suppose $\widehat{a}$ in an ungrounded H-asymptotic extension $\widehat{K}$ of K satisfies $P(\widehat{a}) = 0$, $\widehat{a} - a \prec 1$, and $v(\widehat{a}-a) \notin \mathscr{E}^{\mathrm{e}}_{\widehat{K}}(A)$, where $A := L_{P_{+a}}$. Then $v(\widehat{a}-a) = v^{\mathrm{e}}(P,a)$.*

Proof. We arrange $a = 0$ and assume $\widehat{a} \neq 0$. Then $L := K\langle\widehat{a}\rangle$ has asymptotic integration, by Proposition 1.3.12, and $v(\widehat{a}) \notin \mathscr{E}^{\mathrm{e}}_L(A)$ by Lemma 1.4.10 (applied with L, $\widehat{K}$ in place of K, L). Moreover, Ψ is cofinal in Ψ_L by Proposition 1.3.8. As in the proof of Lemma 1.7.15 this leads to $P_1(\widehat{a}) = A(\widehat{a}) = A^\phi(\widehat{a}) \asymp A^\phi\widehat{a}$, eventually, and then as in the rest of that proof we derive $v^{\mathrm{e}}(P,0) = v(\widehat{a})$. $\square$

Zeros of differential polynomials of order and degree 1. *In this subsection K has asymptotic integration.* We fix a differential polynomial

$$P(Y) \;=\; a(Y' + gY - u) \in K\{Y\} \qquad (a,g,u \in K,\ a \neq 0),$$

and set $A := L_P = a(\partial + g) \in K[\partial]$. Section 1.2 gives for $y \in K$ the equivalence $y \in \mathrm{I}(K) \Leftrightarrow vy > \Psi$, so by Section 1.4, $\mathscr{E}^{\mathrm{e}}(A) = \emptyset \Leftrightarrow g \notin \mathrm{I}(K) + K^\dagger$, and $v(\ker^{\neq}_{\widehat{K}} A) \subseteq \mathscr{E}^{\mathrm{e}}(A)$ for each immediate H-asymptotic field extension $\widehat{K}$ of K. Thus:

Lemma 1.7.21. *If $g \notin \mathrm{I}(K) + K^{\dagger}$, then each immediate H-asymptotic extension of K contains at most one y such that $P(y) = 0$.*

If $\partial K = K$ and $g \in K^{\dagger}$, then $P(y) = 0$ for some $y \in K$, and if moreover K is d-valued, then any y in any immediate H-asymptotic extension of K with $P(y) = 0$ lies in K. (Lemma 1.2.2.) If $y \prec 1$ in an immediate H-asymptotic extension of K satisfies $P(y) = 0$, then by [ADH, 11.2.3(ii), 11.2.1] we have

$$\operatorname{nmul} P \;=\; \operatorname{nmul} P_{+y} \;=\; \operatorname{mul} P_{+y} \;=\; 1.$$

Lemma 1.7.18 yields the following partial converse (a variant of [3, Lemma 8.5]):

Corollary 1.7.22. *Suppose K is $\uplambda$-free and* $\operatorname{nmul} P = 1$. *Then there is a $y \prec 1$ in an immediate H-asymptotic extension of K with $P(y) = 0$.*

Proof. Replacing K by its henselization and using [ADH, 11.6.7], we arrange that K is henselian. Suppose that P has no zero in $\mathcal{O}$. Then P is in newton position at 0, and so Lemma 1.7.18 yields a divergent pc-sequence $(a_\rho)_{\rho<\lambda}$ in K, indexed by all ordinals ρ smaller than some infinite limit ordinal λ, with $a_0 = 0$, $v(a_\rho - a_{\rho'}) = v^{\mathrm{e}}(P, a_\rho)$ for all $\rho < \rho' < \lambda$, and $P(a_\rho) \rightsquigarrow 0$. Since $\deg P = \operatorname{order} P = 1$ and K is henselian, P is a minimal differential polynomial of (a_ρ) over K, and $v(a_\rho) = v^{\mathrm{e}}(P, 0) > 0$ for all $\rho > 0$. Hence [ADH, 9.7.6] yields a pseudolimit y of (a_ρ) in an immediate asymptotic extension of K with $P(y) = 0$ and $y \prec 1$, as required. $\square$

We say that P is **proper** if $u \neq 0$ and $g + u^{\dagger} \succ^{\flat} 1$. If P is proper, then so is bP for each $b \in K^{\times}$. For $\mathfrak{m} \in K^{\times}$ we have

$$P_{\times\mathfrak{m}} \;=\; a\mathfrak{m}\big(Y' + (g + \mathfrak{m}^{\dagger})Y - u\mathfrak{m}^{-1}\big),$$

hence if P is proper, then so is $P_{\times\mathfrak{m}}$. If $u \neq 0$, then P is proper iff $a^{-1}A_{\ltimes u} = \partial + (g + u^{\dagger})$ is steep, as defined in Section 1.4. Note that

$$P^{\phi} \;=\; a\phi\big(Y' + (g/\phi)Y - (u/\phi)\big).$$

Lemma 1.7.23. *Suppose K has small derivation, and P is proper. Then P^{ϕ} is proper (with respect to K^{ϕ}) for all $\phi \preccurlyeq 1$.*

Proof. Let $\phi \preccurlyeq 1$. Then we have $\phi \asymp^{\flat} 1$ and hence $\phi^{\dagger} \asymp^{\flat} \phi' \preccurlyeq 1 \prec^{\flat} g + u^{\dagger}$. Thus

$$g + (u/\phi)^{\dagger} = (g + u^{\dagger}) - \phi^{\dagger} \sim^{\flat} g + u^{\dagger} \succ^{\flat} 1 \succcurlyeq \phi,$$

hence $(g/\phi) + \phi^{-1}(u/\phi)^{\dagger} \succ^{\flat} 1$ and so $(g/\phi) + \phi^{-1}(u/\phi)^{\dagger} \succ^{\flat}_{\phi} 1$. Therefore P^{ϕ} is proper (with respect to K^{ϕ}). $\square$

Lemma 1.7.24. *Suppose K is $\uplambda$-free and $u \neq 0$. Then there is an active ϕ_0 in K such that for all $\phi \prec \phi_0$, P^{ϕ} is proper with $g + (u/\phi)^{\dagger} \sim g + (u/\phi_0)^{\dagger}$.*

Proof. The argument before Corollary 1.4.15 yields an active ϕ_0 in K such that $u^{\dagger} + g - \phi^{\dagger} \succcurlyeq \phi_0$ for all $\phi \prec \phi_0$. For such ϕ we have $\phi^{\dagger} - \phi_0^{\dagger} \prec \phi_0$ as noted just before [ADH, 11.5.3], and so $(u/\phi)^{\dagger} + g \sim (u/\phi_0)^{\dagger} + g$. The argument before Corollary 1.4.15 also gives $\phi^{-1}(u/\phi)^{\dagger} + g/\phi \succ^{\flat}_{\phi} 1$ eventually, and if $\phi^{-1}(u/\phi)^{\dagger} + g/\phi \succ^{\flat}_{\phi} 1$, then P^{ϕ} is proper. $\square$

Lemma 1.7.25. *We have* $\operatorname{nmul} P = 1$ *iff $u \prec g$ or $u \in \mathrm{I}(K)$. Moreover, if K is $\uplambda$-free,* $\operatorname{nmul} P = 1$, *and $u \neq 0$, then $u \prec^{\flat}_{\phi} g + (u/\phi)^{\dagger}$, eventually.*

Proof. For the equivalence, note that the identity above for P^ϕ yields:

$$\operatorname{nmul} P = 0 \iff u \succcurlyeq g, \text{ and } u/\phi \succcurlyeq 1 \text{ eventually.}$$

Suppose K is $\boldsymbol{\lambda}$-free, $\operatorname{nmul} P = 1$, and $u \neq 0$. If $u \in \mathrm{I}(K)$, then $u \prec \phi \prec^\flat_\phi g + (u/\phi)^\dagger$, eventually, by Lemma 1.7.24. Suppose $u \notin \mathrm{I}(K)$. Then $v(u) \in \Psi^\downarrow$ and $u \prec g$. Hence by [ADH, 9.2.11] we have $(u/\phi)^\dagger \prec u \prec g$, eventually, and thus $u \prec g \sim g + (u/\phi)^\dagger$, eventually. Thus $u \prec^\flat_\phi g + (u/\phi)^\dagger$, eventually. □

Assume now $P(y) = 0$ with y in an immediate H-asymptotic extension of K; so $A(y) = u$. Note: if $vy \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A)$, then $u \neq 0$. From Lemma 1.4.14 we get:

Lemma 1.7.26. *If K has small derivation, P is proper, and $vy \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A)$, then $y \sim u/(g + u^\dagger)$.*

By Lemmas 1.7.24 and 1.7.26, and using Lemma 1.7.25 for the last part:

Corollary 1.7.27. *If K is $\boldsymbol{\lambda}$-free and $vy \in \Gamma \setminus \mathscr{E}^{\mathrm{e}}(A)$, then*

$$y \sim u/\big(g + (u/\phi)^\dagger\big) \qquad \textit{eventually.}$$

If in addition $\operatorname{nmul} P = 1$, then $y \prec 1$.

A characterization of 1-linear newtonianity. *In this subsection K has asymptotic integration.* We first expand [ADH, 14.2.4]:

Proposition 1.7.28. *The following are equivalent:*

(i) *K is 1-linearly newtonian;*
(ii) *every $P \in K\{Y\}$ with $\operatorname{nmul} P = \deg P = 1$ and $\operatorname{order} P \leqslant 1$ has a zero in $\smallo$;*
(iii) *K is* d*-valued, $\boldsymbol{\lambda}$-free, and 1-linearly surjective, with $\mathrm{I}(K) \subseteq K^\dagger$.*

Proof. The equivalence of (i) and (ii) is [ADH, 14.2.4], and the implication (i) ⇒ (iii) follows from [ADH, 14.2.2, 14.2.3, 14.2.5]. To show (iii) ⇒ (ii), suppose (iii) holds, and let $g, u \in K$ and $P = Y' + gY - u$ with $\operatorname{nmul} P = 1$. We need to find $y \in \smallo$ such that $P(y) = 0$. Corollary 1.7.22 gives an element $y \prec 1$ in an immediate H-asymptotic extension L of K with $P(y) = 0$. It suffices to show that then $y \in K$ (and thus $y \in \smallo$). If $g \notin K^\dagger$, then this follows from Lemma 1.7.21, using $\mathrm{I}(K) \subseteq K^\dagger$ and 1-linear surjectivity of K; if $g \in K^\dagger$, then this follows from Lemma 1.2.2 and $\partial K = K$. □

By the next corollary, each Liouville closed H-field is 1-linearly newtonian:

Corollary 1.7.29. *Suppose $K^\dagger = K$. Then the following are equivalent:*

(i) *K is 1-linearly newtonian;*
(ii) *K is* d*-valued and 1-linearly surjective;*
(iii) *K is* d*-valued and $\partial K = K$.*

Proof. Note that K is $\boldsymbol{\lambda}$-free by [ADH, remarks following 11.6.2]. Hence the equivalence of (i) and (ii) follows from Proposition 1.7.28. For the equivalence of (ii) with (iii), see [ADH, example following 5.5.22]. □

Linear newtonianity descends. *In this subsection H is* d*-valued with valuation ring $\mathcal{O}$ and constant field C. Let $r \in \mathbb{N}^{\geqslant 1}$.* If H is $\upomega$-free, Γ is divisible, and H has a newtonian algebraic extension $K = H(C_K)$, then H is also newtonian, by (0.11). Here is an analogue of this for r-linear newtonianity:

Lemma 1.7.30. *Let $K = H(C_K)$ be an algebraic asymptotic extension of H which is r-linearly newtonian. Then H is r-linearly newtonian.*

Proof. Take a basis B of the C-linear space C_K with $1 \in B$, and let b range over B. We have $H(C_K) = H[C_K]$, and H is linearly disjoint from C_K over C [ADH, 4.6.16], so B is a basis of the H-linear space $H[C_K]$. Let $P \in H\{Y\}$ with $\deg P = 1$ and $\operatorname{order}(P) \leqslant r$ be quasilinear; then P as element of $K\{Y\}$ remains quasilinear, since $\Gamma_K = \Gamma$ by [ADH, 10.5.15]. Let $y \in \mathcal{O}_K$ be a zero of P. Take $y_b \in H$ $(b \in B)$ with $y_b = 0$ for all but finitely many b and $y = \sum_b y_b\, b$. Then $y_b \in \mathcal{O}$ for all b, and

$$0 \ = \ P(y) \ = \ P_0 + P_1(y) \ = \ P_0 + \sum_b P_1(y_b) b,$$

so $P(y_1) = P_0 + P_1(y_1) = 0$. $\square$

Thus if $H[\mathrm{i}]$ with $\mathrm{i}^2 = -1$ is r-linearly newtonian, then H is r-linearly newtonian.

Cases of bounded order. *In the rest of this section $r \in \mathbb{N}^{\geqslant 1}$.* Define K to be **strongly r-newtonian** if K is r-newtonian and for each divergent pc-sequence (a_ρ) in K with minimal differential polynomial $G(Y)$ over K of order $\leqslant r$ we have $\operatorname{ndeg}_{\boldsymbol{a}} G = 1$, where $\boldsymbol{a} := c_K(a_\rho)$. Given $P \in K\{Y\}^{\neq}$, a **K-external zero of P** is an element $\widehat{a}$ of some immediate asymptotic extension $\widehat{K}$ of K such that $P(\widehat{a}) = 0$ and $\widehat{a} \notin K$. Now [ADH, 14.1.11] extends as follows with the same proof:

Lemma 1.7.31. *Suppose K has rational asymptotic integration and K is strongly r-newtonian. Then no $P \in K\{Y\}^{\neq}$ of order $\leqslant r$ can have a K-external zero.*

The following is important in certain inductions on the order. A differential field F is *r-linearly closed* ($r \in \mathbb{N}$) if every $A \in F[\partial]^{\neq}$ of order $\leqslant r$ splits over F. So F is *linearly closed* iff it is r-linearly closed for all $r \in \mathbb{N}$.

Lemma 1.7.32. *Suppose K has asymptotic integration, is* 1*-linearly newtonian, and r-linearly closed. Then K is r-linearly newtonian.*

Proof. Note that K is $\uplambda$-free and d-valued by Proposition 1.7.28. Let $P \in K\{Y\}$ be such that $\operatorname{nmul} P = \deg P = 1$ and $\operatorname{order} P \leqslant r$; by [ADH, 14.2.6] it suffices to show that then P has a zero in $\smallo$. By [ADH, proof of 13.7.10] we can compositionally conjugate, pass to an elementary extension, and multiply by an element of $K^\times$ to arrange that K has small derivation, $P_0 \prec^\flat 1$, and $P_1 \asymp 1$. Let $A := L_P$. The valuation ring of the flattening $(K, v^\flat)$ is 1-linearly surjective by [ADH, 14.2.1], so all operators in $K[\partial]$ of order 1 are neatly surjective in the sense of $(K, v^\flat)$. Since A splits over K, we obtain from [ADH, 5.6.10(ii)] that A is neatly surjective in the sense of $(K, v^\flat)$. As $v^\flat(A) = 0$ and $v^\flat(P_0) > 0$, this gives $y \in K$ with $v^\flat(y) > 0$ such that $P_0 + A(y) = 0$, that is, $P(y) = 0$. $\square$

Using the terminology of K-external zeros, we can add another item to the list of equivalent statements in Proposition 1.7.28:

Lemma 1.7.33. *Suppose K has asymptotic integration. Then we have:*

$$K \text{ is 1-linearly newtonian} \iff K \text{ is } \lambda\text{-free and no } P \in K\{Y\} \text{ with } \deg P = 1 \text{ and } \operatorname{order} P = 1 \text{ has a } K\text{-external zero.}$$

Proof. Suppose K is 1-linearly newtonian. Then by (i) $\Rightarrow$ (iii) in Proposition 1.7.28, K is λ-free, d-valued, 1-linearly surjective, and $\mathrm{I}(K) \subseteq K^\dagger$. Let $P \in K\{Y\}$ where $\deg P = \operatorname{order} P = 1$ and y in an immediate asymptotic extension L of K with $P(y) = 0$. Then [ADH, 9.1.2] and Corollary 1.2.11 give $L^\dagger \cap K = K^\dagger$, so $y \in K$ by Lemmas 1.2.2 and 1.2.3. This gives the direction $\Rightarrow$. The converse follows from Corollary 1.7.22 and (ii) $\Rightarrow$ (i) in Proposition 1.7.28. $\square$

Here is a higher-order version of Lemma 1.7.33:

Lemma 1.7.34. *Suppose K is ω-free. Then*

$$K \text{ is } r\text{-linearly newtonian} \iff \text{no } P \in K\{Y\} \text{ with } \deg P = 1 \text{ and } \operatorname{order} P \leqslant r \text{ has a } K\text{-external zero.}$$

Proof. Suppose K is r-linearly newtonian. Then K is d-valued by Lemma 1.2.9. Let $P \in K\{Y\}$ be of degree 1 and order $\leqslant r$, and let y be in an immediate asymptotic extension L of K with $P(y) = 0$. Then $A(y) = b$ for $A := L_P \in K[\partial]$, $b := -P(0) \in K$. By [ADH, 14.2.2] there is also a $z \in K$ with $A(z) = b$, hence $y - z \in \ker_L A = \ker A$ by [ADH, remarks after 14.2.9] and so $y \in K$. This gives the direction $\Rightarrow$. For the converse note that every quasilinear $P \in K\{Y\}$ has a zero $\widehat{a} \preccurlyeq 1$ in an immediate asymptotic extension of K by (0.7). $\square$

We also have the following r-version of (0.7):

Proposition 1.7.35. *If K is λ-free and no $P \in K\{Y\}^{\neq}$ of order $\leqslant r$ has a K-external zero, then K is ω-free and r-newtonian.*

Proof. The ω-freeness follows as before from [ADH, 11.7.13]. The rest of the proof is as in [ADH, p. 653] with P restricted to have order $\leqslant r$. $\square$

Application to solving asymptotic equations. *Here K is* d*-valued, ω-free, with small derivation, and $\mathfrak{M}$ is a monomial group of K.* See [ADH, 3.3] for "monomial group", and [ADH, 13.8] for "asymptotic equation". We let a, b, y range over K. In addition we fix a $P \in K\{Y\}^{\neq}$ of order $\leqslant r$ and a $\preccurlyeq$-closed set $\mathcal{E} \subseteq K^\times$. (Recall that $r \geqslant 1$.) This gives the asymptotic equation

$$\text{(E)} \qquad P(Y) = 0, \qquad Y \in \mathcal{E}.$$

This gives the following r-version of [ADH, 13.8.8], with basically the same proof:

Proposition 1.7.36. *Suppose Γ is divisible, no $Q \in K\{Y\}^{\neq}$ of order $\leqslant r$ has a K-external zero, $d := \operatorname{ndeg}_{\mathcal{E}} P \geqslant 1$, and there is no $f \in \mathcal{E} \cup \{0\}$ with $\operatorname{mul} P_{+f} = d$. Then* (E) *has an unraveler.*

Here is an r-version of [ADH, 14.3.4] with the same proof:

Lemma 1.7.37. *Suppose K is r-newtonian. Let $g \in K^\times$ be an approximate zero of P with $\operatorname{ndeg} P_{\times g} = 1$. Then there exists $y \sim g$ such that $P(y) = 0$.*

For the next three results we assume the following:

C is algebraically closed, Γ is divisible, and no $Q \in K\{Y\}^{\neq}$ of order $\leqslant r$ has a K-external zero.

These three results are r-versions of [ADH, 14.3.5, 14.3.6, 14.3.7] with the same proofs, using Propositions 1.7.35 and 1.7.36 instead of (0.7) and [ADH, 13.8.8]:

Proposition 1.7.38. *If* $\operatorname{ndeg}_{\mathcal{E}} P > \operatorname{mul}(P) = 0$*, then* (E) *has a solution.*

Corollary 1.7.39. *K is weakly r-differentially closed: for each $Q \in K\{Y\} \setminus K$ of order $\leqslant r$ there is a $y \in K$ with $Q(y) = 0$.*

Corollary 1.7.40. *Suppose $g \in K^{\times}$ is an approximate zero of P. Then $P(y) = 0$ for some $y \sim g$.*

A useful equivalence. *Suppose K is ω-free.* (No small derivation or monomial group assumed.) Recall that $r \geqslant 1$. Here is an r-version of [21, 3.4]:

Corollary 1.7.41. *The following are equivalent:*

(i) *K is r-newtonian;*
(ii) *K is strongly r-newtonian;*
(iii) *no $P \in K\{Y\}^{\neq}$ of order $\leqslant r$ has a K-external zero.*

Proof. Since K is ω-free it has rational asymptotic integration [ADH, p. 515]. Also, if K is 1-newtonian, then K is henselian [ADH, p. 645] and d-valued [ADH, 14.2.5]. For (i) $\Rightarrow$ (ii), use [21, 3.3], for (ii) $\Rightarrow$ (iii), use Lemma 1.7.31, and for (iii) $\Rightarrow$ (i), use Proposition 1.7.35. $\square$

Next an r-version of (0.8):

Corollary 1.7.42. *Suppose K is r-newtonian, Γ is divisible, and C is algebraically closed. Then K is weakly r-differentially closed, so K is $(r+1)$-linearly closed and thus $(r+1)$-linearly newtonian.*

Proof. To show that K is weakly r-differentially closed we arrange by compositional conjugation and passing to a suitable elementary extension that K has small derivation and K has a monomial group. Then K is weakly r-differentially closed by Corollaries 1.7.39 and 1.7.41. The rest uses [ADH, 5.8.9] and Lemma 1.7.32. $\square$

Part 2. The Universal Exponential Extension

Let K be an algebraically closed differential field. In Section 2.2 below we extend K in a canonical way to a differential integral domain $\mathrm{U} = \mathrm{U}_K$ whose differential fraction field has the same constant field C as K, called the *universal exponential extension* of K. (The universal exponential extension of $\mathbb{T}[i]$ appeared in [15] in the guise of "oscillating transseries"; we explain the connection at the end of Section 2.4.) The underlying ring of U is a group ring of a certain abelian group over K, and we therefore first review some relevant basic facts about such group rings in Section 2.1. The main feature of U is that if K is 1-linearly surjective, then each $A \in K[\partial]$ of order $r \in \mathbb{N}$ which splits over K has r many C-linearly independent zeros in U. This is explained in Section 2.4, after some differential-algebraic preliminaries in Section 2.3, where we consider a novel kind of *spectrum* of a linear differential operator over a differential field. In Section 2.5 we introduce for H-asymptotic K with small derivation and asymptotic integration the *ultimate exceptional values* of a given linear differential operator $A \in K[\partial]^{\neq}$. These help to isolate the zeros of A in U much like the exceptional values of A help to locate the zeros of A in immediate asymptotic extensions of K as in Section 1.4.

Of this part, only the construction of U (Sections 2.1 and 2.2) and the definition of the set of ultimate exceptional values and its basic properties (Section 2.5) are used later in this monograph in an essential way. In [6] we discuss the analytic meaning of U when K is the algebraic closure of a Liouville closed Hardy field containing $\mathbb{R}$ as a subfield. In a follow-up paper [7] we use the main theorem of [6] together with the results from the remaining Sections 2.3 and 2.4 to study the solutions of linear differential equations over Hardy fields.

2.1. Some Facts about Group Rings

In this section G is a torsion-free abelian group, written multiplicatively, K is a field, and γ, δ range over G. For use in Section 2.2 below we recall some facts about the group ring $K[G]$: a commutative K-algebra with $1 \neq 0$ that contains G as a subgroup of its multiplicative group $K[G]^{\times}$ and which, as a K-linear space, decomposes as

$$K[G] \;=\; \bigoplus_{\gamma} K\gamma \qquad \text{(internal direct sum).}$$

Hence for any $f \in K[G]$ we have a unique family (f_γ) of elements of K, with $f_\gamma = 0$ for all but finitely many γ, such that

$$f \;=\; \sum_{\gamma} f_\gamma \gamma. \tag{2.1.1}$$

We define the support of $f \in K[G]$ as above by

$$\operatorname{supp}(f) \;:=\; \{\gamma : f_\gamma \neq 0\} \;\subseteq\; G.$$

In the rest of this section f, g, h range over $K[G]$. For any K-algebra R, every group morphism $G \to R^{\times}$ extends uniquely to a K-algebra morphism $K[G] \to R$.

Clearly $K[G]^{\times} \supseteq K^{\times}G$; in fact:

Lemma 2.1.1. *The ring $K[G]$ is an integral domain and $K[G]^{\times} = K^{\times}G$.*

Proof. We take an ordering of G making G into an ordered abelian group; see [ADH, 2.4]. Let $f, g \neq 0$ and set

$$\gamma^- := \min \operatorname{supp}(f),\ \gamma^+ := \max \operatorname{supp}(f), \quad \delta^- := \min \operatorname{supp}(g),\ \delta^+ := \max \operatorname{supp}(g);$$

so $\gamma^- \leqslant \gamma^+$ and $\delta^- \leqslant \delta^+$. We have $(fg)_{\gamma^-\delta^-} = f_{\gamma^-} g_{\delta^-} \neq 0$, and likewise with γ^+, δ^+ in place of γ^-, δ^-. In particular, $fg \neq 0$, showing that $K[G]$ is an integral domain. Now suppose $fg = 1$. Then $\operatorname{supp}(fg) = \{1\}$, hence $\gamma^-\delta^- = 1 = \gamma^+\delta^+$, so $\gamma^- = \gamma^+$, and thus $f \in K^\times G$. □

Lemma 2.1.2. *Suppose K has characteristic 0 and $G \neq \{1\}$. Then the fraction field Ω of $K[G]$ is not algebraically closed.*

Proof. Let $\gamma \in G \setminus \{1\}$ and $n \geqslant 1$. We claim that there is no $y \in \Omega$ with $y^2 = 1 - \gamma^n$. For this, first replace G by its divisible hull to arrange that G is divisible. Towards a contradiction, suppose $f, g \in K[G]^{\neq}$ and $f^2 = g^2(1 - \gamma^n)$. Take a divisible subgroup H of G that is complementary to the smallest divisible subgroup $\gamma^{\mathbb{Q}}$ of G containing γ, so $G = H\gamma^{\mathbb{Q}}$ and $G \cap \gamma^{\mathbb{Q}} = \{1\}$. Then $K[G] \subseteq K(H)[\gamma^{\mathbb{Q}}]$ (inside Ω), so we may replace K, G by $K(H)$, $\gamma^{\mathbb{Q}}$ to arrange $G = \gamma^{\mathbb{Q}}$. For suitable $m \geqslant 1$ we apply the K-algebra automorphism of $K[G]$ given by $\gamma \mapsto \gamma^m$ to arrange $f, g \in K[\gamma, \gamma^{-1}]$ (replacing n by mn). Then replace f, g by $\gamma^m f$, $\gamma^m g$ for suitable $m \geqslant 1$ to arrange $f, g \in K[\gamma]$. Now use that $1 - \gamma$ is a prime divisor of $1 - \gamma^n$ of multiplicity 1 in the UFD $K[\gamma]$ to get a contradiction. □

The K-linear map

$$f \mapsto \operatorname{tr}(f) := f_1 \ :\ K[G] \to K$$

is called the **trace** of $K[G]$. Thus

$$\operatorname{tr}(fg) \ =\ \sum_\gamma f_\gamma g_{\gamma^{-1}}.$$

We claim that $\operatorname{tr} \circ \sigma = \operatorname{tr}$ for every automorphism σ of the K-algebra $K[G]$. This invariance comes from an intrinsic description of $\operatorname{tr}(f)$ as follows: given f we have a unique finite set $U \subseteq K[G]^\times = K^\times G$ such that $f = \sum_{u \in U} u$ and $u_1/u_2 \notin K^\times$ for all distinct $u_1, u_2 \in U$; if $U \cap K^\times = \{c\}$, then $\operatorname{tr}(f) = c$; if $U \cap K^\times = \emptyset$, then $\operatorname{tr}(f) = 0$. If G_0 is a subgroup of G and K_0 is a subfield of K, then $K_0[G_0]$ is a subring of $K[G]$, and the trace of $K[G]$ extends the trace of $K_0[G_0]$.

The automorphisms of $K[G]$. For a commutative group H, written multiplicatively, $\operatorname{Hom}(G, H)$ denotes the set of group morphisms $G \to H$, made into a group by pointwise multiplication. Any $\chi \in \operatorname{Hom}(G, K^\times)$—sometimes called a *character*—gives a K-algebra automorphism $f \mapsto f_\chi$ of $K[G]$ defined by

$$f_\chi \ :=\ \sum_\gamma f_\gamma \chi(\gamma)\gamma. \tag{2.1.2}$$

This yields a group action of $\operatorname{Hom}(G, K^\times)$ on $K[G]$ by K-algebra automorphisms:

$$\operatorname{Hom}(G, K^\times) \times K[G] \to K[G], \qquad (\chi, f) \mapsto f_\chi.$$

Sending $\chi \in \operatorname{Hom}(G, K^\times)$ to $f \mapsto f_\chi$ yields an embedding of the group $\operatorname{Hom}(G, K^\times)$ into the group $\operatorname{Aut}(K[G]|K)$ of automorphisms of the K-algebra $K[G]$; its image is the (commutative) subgroup of $\operatorname{Aut}(K[G]|K)$ consisting of the K-algebra automorphisms σ of $K[G]$ such that $\sigma(\gamma)/\gamma \in K^\times$ for all γ. Identify $\operatorname{Hom}(G, K^\times)$ with its image under this embedding. From $K[G]^\times = K^\times G$ we obtain $\sigma(K^\times G) = K^\times G$

for all $\sigma \in \operatorname{Aut}(K[G]|K)$, and using this one verifies easily that $\operatorname{Hom}(G, K^\times)$ is a normal subgroup of $\operatorname{Aut}(K[G]|K)$. We also have the group embedding

$$\operatorname{Aut}(G) \ \to \ \operatorname{Aut}(K[G]|K)$$

assigning to each $\sigma \in \operatorname{Aut}(G)$ the unique automorphism of the K-algebra $K[G]$ extending σ. Identifying $\operatorname{Aut}(G)$ with its image in $\operatorname{Aut}(K[G]|K)$ via this embedding we have $\operatorname{Hom}(G, K^\times) \cap \operatorname{Aut}(G) = \{\mathrm{id}\}$ and $\operatorname{Hom}(G, K^\times) \cdot \operatorname{Aut}(G) = \operatorname{Aut}(K[G], |K)$ inside $\operatorname{Aut}(K[G]|K)$, and thus $\operatorname{Aut}(K[G]|K) = \operatorname{Hom}(G, K^\times) \rtimes \operatorname{Aut}(G)$, an internal semidirect product of subgroups of $\operatorname{Aut}(K[G]|K)$.

The gaussian extension. *In this subsection $v \colon K^\times \to \Gamma$ is a valuation on the field K.* We extend v to a map $v_{\mathrm{g}} \colon K[G]^{\neq} \to \Gamma$ by setting

$$v_{\mathrm{g}} f \ := \ \min_{\gamma} v f_\gamma \qquad (f \in K[G]^{\neq} \text{ as in (2.1.1)}). \tag{2.1.3}$$

Proposition 2.1.3. *The map $v_{\mathrm{g}} \colon K[G]^{\neq} \to \Gamma$ is a valuation on the domain $K[G]$.*

Proof. We can reduce to the case that G is finitely generated, since $K[G]$ is the directed union of its subrings $K[G_0]$ with G_0 a finitely generated subgroup of G. We then have a group isomorphism $G \to \mathbb{Z}^n$ inducing a K-algebra isomorphism $K[G] \to K[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ (with distinct indeterminates $X_1, \dots, X_n$) under which v_{g} corresponds to the gaussian extension of the valuation of K to $K(X_1, \dots, X_n)$ restricted to its subring $K[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$; see [ADH, 3.1]. $\square$

We call v_{g} the **gaussian extension** of the valuation of K to $K[G]$. We denote by $\preccurlyeq_{\mathrm{g}}$ the dominance relation on $\Omega := \operatorname{Frac}(K[G])$ associated to the extension of v_{g} to a valuation on the field Ω [ADH, (3.1.1)], with corresponding asymptotic relations $\asymp_{\mathrm{g}}$ and $\prec_{\mathrm{g}}$. For the subring $\mathcal{O}[G]$ of $K[G]$ generated by G over $\mathcal{O}$ we have

$$\mathcal{O}[G] \ = \ \{f : f \preccurlyeq_{\mathrm{g}} 1\}.$$

The residue morphism $\mathcal{O} \to \boldsymbol{k} := \mathcal{O}/\scriptstyle{\mathcal{O}}$ extends to a surjective ring morphism $\mathcal{O}[G] \to \boldsymbol{k}[G]$ with $\gamma \mapsto \gamma$ for all γ and whose kernel is the ideal

$$\scriptstyle{\mathcal{O}}\textstyle[G] := \ \{f : f \prec_{\mathrm{g}} 1\}$$

of $\mathcal{O}[G]$. Hence this ring morphism induces an isomorphism $\mathcal{O}[G]/\scriptstyle{\mathcal{O}}\textstyle[G] \cong \boldsymbol{k}[G]$. If G_0 is subgroup of G and K_0 is a valued subfield of K, then the restriction of v_{g} to a valuation on $K_0[G_0]$ is the gaussian extension of the valuation of K_0 to $K_0[G_0]$.

An inner product and two norms. *In the rest of this section H is a real closed subfield of K such that $K = H[\mathrm{i}]$ where $\mathrm{i}^2 = -1$.* In later use H will be a Hardy field, which is why we use the letter H here. Note that the only nontrivial automorphism of the (algebraically closed) field K over H is *complex conjugation*:

$$z = a + b\mathrm{i} \mapsto \overline{z} := a - b\mathrm{i} \qquad (a, b \in H).$$

For f as in (2.1.1) we set

$$f^* \ := \ \sum_{\gamma} \overline{f_\gamma} \gamma^{-1},$$

so $(f^*)^* = f$, and $f \mapsto f^*$ lies in $\operatorname{Aut}\big(K[G]|H\big)$. We define the function

$$(f, g) \mapsto \langle f, g \rangle \ : \ K[G] \times K[G] \to K$$

by

$$\langle f, g\rangle \ := \ \mathrm{tr}\big(fg^*\big) \ = \ \sum_\gamma f_\gamma \overline{g_\gamma}.$$

One verifies easily that this is a "positive definite hermitian form" on the K-linear space $K[G]$: it is additive on the left and on the right, and for all f, g and all $\lambda \in K$: $\langle \lambda f, g\rangle = \lambda\langle f, g\rangle$, $\langle g, f\rangle = \overline{\langle f, g\rangle}$, $\langle f, f\rangle \in H^{\geqslant}$, and $\langle f, f\rangle = 0 \Leftrightarrow f = 0$, and thus also $\langle f, \lambda g\rangle = \overline{\lambda}\langle f, g\rangle$. (Hermitian forms are usually defined only on $\mathbb{C}$-linear spaces and are $\mathbb{C}$-valued, which is why we used quote marks, as we do below for *norm* and *orthonormal basis*; see [17, Chapter XV, §5] for the more general case.) Note:

$$\langle f, gh\rangle \ = \ \mathrm{tr}\big(f(gh)^*\big) \ = \ \big\langle fg^*, h\big\rangle.$$

Lemma 2.1.4. *Let $u, w \in K[G]^\times$. If $u \notin K^\times w$, then $\langle u, w\rangle = 0$, and if $u \in K^\times w$, then $\langle u, w\rangle = uw^*$.*

Proof. Take $a, b \in K^\times$ and γ, δ such that $u = a\gamma$, $w = b\delta$. If $u \notin K^\times w$, then $\gamma \neq \delta$, so $\langle u, w\rangle = 0$. If $u \in K^\times w$, then $\gamma = \delta$, hence $\langle u, w\rangle = a\overline{b} = uw^*$. □

For $z \in K$ we set $|z| := \sqrt{z\overline{z}} \in H^{\geqslant}$, and then define $\|\cdot\|\colon K[G] \to H^{\geqslant}$ by

$$\|f\|^2 \ = \ \langle f, f\rangle \ = \ \sum_\gamma |f_\gamma|^2.$$

As in the case $H = \mathbb{R}$ and $K = \mathbb{C}$ one derives the Cauchy-Schwarz Inequality:

$$|\langle f, g\rangle| \ \leqslant \ \|f\| \cdot \|g\|.$$

Thus $\|\cdot\|$ is a "norm" on the K-linear space $K[G]$: for all f, g and all $\lambda \in K$,

$$\|f + g\| \leqslant \|f\| + \|g\|, \quad \|\lambda f\| = |\lambda| \cdot \|f\|, \quad \|f\| = 0 \Leftrightarrow f = 0.$$

Note that G is an "orthonormal basis" of $K[G]$ with respect to $\langle\ ,\ \rangle$, and $f_\gamma = \langle f, \gamma\rangle$. We also use the function $\|\cdot\|_1\colon K[G] \to H^{\geqslant}$ given by

$$\|f\|_1 \ := \ \sum_\gamma |f_\gamma|,$$

which is a "norm" on $K[G]$ in the sense of obeying the same laws as we mentioned for $\|\cdot\|$. The two "norms" are in some sense equivalent:

$$\|f\| \ \leqslant \ \|f\|_1 \ \leqslant \ \sqrt{n}\|f\| \qquad (n := |\mathrm{supp}(f)|).$$

where the first inequality follows from the triangle inequality for $\|\cdot\|$ and the second is of Cauchy-Schwarz type. Moreover:

Lemma 2.1.5. *Let $u \in K[G]^\times$. Then $\|fu\| = \|f\|\,\|u\|$ and $\|fu\|_1 = \|f\|_1\,\|u\|_1$.*

Proof. We have

$$\|f\gamma\| \ = \ \langle f\gamma, f\gamma\rangle \ = \ \big\langle f\gamma\gamma^*, f\big\rangle \ = \ \big\langle f, f\big\rangle \ = \ \|f\|$$

using $\gamma^* = \gamma^{-1}$. Together with $K[G]^\times = K^\times G$ this yields the first claim; the second claim follows easily from the definition of $\|\cdot\|_1$. □

Corollary 2.1.6. $\|fg\| \leqslant \|f\| \cdot \|g\|_1$ *and* $\|fg\|_1 \leqslant \|f\|_1 \cdot \|g\|_1$.

Proof. By the triangle inequality for $\|\cdot\|$ and the previous lemma,

$$\|fg\| \ \leqslant \ \sum_\gamma \|fg_\gamma\gamma\| \ = \ \sum_\gamma \|f\|\,\|g_\gamma\gamma\| \ = \ \|f\|\sum_\gamma |g_\gamma| \ = \ \|f\|\,\|g\|_1.$$

The inequality involving $\|fg\|_1$ follows likewise. □

In the next lemma we let $\chi \in \operatorname{Hom}(G, K^\times)$; recall from (2.1.2) the automorphism $f \mapsto f_\chi$ of the K-algebra $K[G]$.

Lemma 2.1.7. $(f_\chi)^* = (f^*)_\chi$ *iff* $|\chi(\gamma)| = 1$ *for all* $\gamma \in \operatorname{supp}(f)$.

Proof. Let $a \in K$; then $\big((a\gamma)_\chi\big)^* = \overline{a\chi(\gamma)}\gamma^{-1}$ and $\big((a\gamma)^*\big)_\chi = \overline{a}\chi(\gamma)^{-1}\gamma^{-1}$. $\square$

Corollary 2.1.8. *Let* $\chi \in \operatorname{Hom}(G, K^\times)$ *with* $|\chi(\gamma)| = 1$ *for all* γ. *Then* $\langle f_\chi, g_\chi\rangle = \langle f, g\rangle$ *for all* f, g, *and hence* $\|f_\chi\| = \|f\|$ *for all* f.

Proof. Since $\operatorname{tr} \circ \sigma = \operatorname{tr}$ for every automorphism σ of the K-algebra $K[G]$,

$$\langle f_\chi, g_\chi\rangle \;=\; \operatorname{tr}\big(f_\chi (g_\chi)^*\big) \;=\; \operatorname{tr}\big((fg^*)_\chi\big) \;=\; \operatorname{tr}(fg^*) \;=\; \langle f, g\rangle,$$

where we use Lemma 2.1.7 for the second equality. $\square$

Valuation and norm. Let $v\colon H^\times \to \Gamma$ be a convex valuation on the ordered field H, extended uniquely to a valuation $v\colon K^\times \to \Gamma$ on the field $K = H[i]$, so $a \asymp |a|$ for $a \in K$. (See the remarks before Corollary 1.2.6.) Let $v_{\mathrm{g}}\colon K[G]^{\neq} \to \Gamma$ be the gaussian extension of v, given by (2.1.3).

Lemma 2.1.9. $\|f\|_1 \preccurlyeq 1 \Leftrightarrow f \preccurlyeq_{\mathrm{g}} 1$, *and* $\|f\|_1 \prec 1 \Leftrightarrow f \prec_{\mathrm{g}} 1$.

Proof. Using that the valuation ring of H is convex we have

$$\|f\|_1 = \sum_\gamma |f_\gamma| \preccurlyeq 1 \iff |f_\gamma| \preccurlyeq 1 \text{ for all } \gamma \iff f_\gamma \preccurlyeq 1 \text{ for all } \gamma \iff f \preccurlyeq_{\mathrm{g}} 1.$$

Likewise one shows: $\|f\|_1 \prec 1 \Leftrightarrow f \prec_{\mathrm{g}} 1$. $\square$

Corollary 2.1.10. $\|f\| \asymp \|f\|_1 \asymp_{\mathrm{g}} f$.

Proof. This is trivial for $f = 0$, so assume $f \neq 0$. Take $a \in H^>$ with $a \asymp_{\mathrm{g}} f$, and replace f by f/a, to arrange $f \asymp_{\mathrm{g}} 1$. Then $\|f\| \asymp \|f\|_1 \asymp_{\mathrm{g}} 1$ by Lemma 2.1.9. $\square$

2.2. The Universal Exponential Extension

As in [ADH, 5.9], given a differential ring K, a *differential K-algebra* is a differential ring R with a morphism $K \to R$ of differential rings. If R is a differential ring extension of a differential ring K we consider R as a differential K-algebra via the inclusion $K \to R$.

Exponential extensions. *In this subsection R is a differential ring and K is a differential subring of R.* Call $a \in R$ **exponential over** K if $a' \in aK$. Note that if $a \in R$ is exponential over K, then $K[a]$ is a differential subring of R. If $a \in R$ is exponential over K and $\phi \in K^\times$, then a, as element of the differential ring extension R^ϕ of K^ϕ, is exponential over K^ϕ. Every $c \in C_R$ is exponential over K, and every $u \in K^\times$ is exponential over K. If $a, b \in R$ are exponential over K, then so is ab, and if $a \in R^\times$ is exponential over K, then so is a^{-1}. Hence the units of R that are exponential over K form a subgroup E of the group $R^\times$ of units of R with $E \supseteq C_R^\times \cdot K^\times$; if $R = K[E]$, then we call R **exponential over** K. An **exponential extension of** K is a differential ring extension of K that is exponential over K. If $R = K[E]$ where E is a set of elements of $R^\times$ which are exponential over K, then R is exponential over K. If R is an exponential extension of K and $\phi \in K^\times$, then R^ϕ is an exponential extension of K^ϕ. The following lemma is extracted from the proof of [23, Theorem 1]:

Lemma 2.2.1 (Rosenlicht)**.** *Suppose K is a field and R is an integral domain with differential fraction field F. Let $I \neq R$ be a differential ideal of R, and let $u_1, \dots, u_n \in R^\times$ $(n \geqslant 1)$ be exponential over K with $u_i \notin u_j C_F^\times K^\times$ for $i \neq j$. Then $\sum_i u_i \notin I$.*

Proof. Suppose $u_1, \dots, u_n$ is a counterexample with minimal $n \geqslant 1$. Then $n \geqslant 2$ and $\sum_i u_i' \in I$, so

$$\sum_i u_i' - u_1^\dagger \sum_i u_i \;=\; \sum_{i>1} (u_i/u_1)^\dagger u_i \in I.$$

Hence $(u_i/u_1)^\dagger = 0$ and thus $u_i/u_1 \in C_F^\times$, for all $i > 1$, a contradiction. $\square$

Corollary 2.2.2. *Suppose K is a field and $F = K(E)$ is a differential field extension of K with $C_F = C$, where E is a subgroup of $F^\times$ whose elements are exponential over K. Then $\{y \in F^\times : y \text{ is exponential over } K\} = K^\times E$.*

Proof. Let $y \in F^\times$ be exponential over K. Take K-linearly independent $u_1, \dots, u_n$ in E and $a_1, \dots, a_n, b_1, \dots, b_n \in K$ with $b_j \neq 0$ for some j, such that

$$y \;=\; \Big(\textstyle\sum_i a_i u_i\Big) \Big/ \Big(\textstyle\sum_j b_j u_j\Big).$$

Then $\sum_j b_j y u_j - \sum_i a_i u_i = 0$, and so Lemma 2.2.1 applied with $R = F$, $I = \{0\}$ gives $b_j y u_j \in a_i u_i K^\times$ for some i, j with $a_i, b_j \neq 0$, and thus $y \in K^\times E$. $\square$

Remark. In the context of Corollary 2.2.2, see [23, Theorem 1] for the structure of the group of elements of $F^\times$ exponential over K, for finitely generated E.

Lemma 2.2.3. *Suppose $C_R^\times$ is divisible and E is a subgroup of $R^\times$ containing $C_R^\times$. Then there is a group morphism $e\colon E^\dagger \to E$ such that $e(b)^\dagger = b$ for all $b \in E^\dagger$.*

Proof. We have a short exact sequence of commutative groups

$$1 \to C_R^\times \xrightarrow{\iota} E \xrightarrow{\ell} E^\dagger \to 0,$$

where ι is the natural inclusion and $\ell(a) := a^\dagger$ for $a \in E$. Since $C_R^\times$ is divisible, this sequence splits, which is what we claimed. $\square$

Let E, e, R be as in the previous lemma. Then e is injective, and its image is a complement of $C_R^\times$ in E. Moreover, given also a group morphism $\widetilde{e}\colon E^\dagger \to E$ such that $\widetilde{e}(b)^\dagger = b$ for all $b \in E^\dagger$, the map $b \mapsto e(b)\widetilde{e}(b)^{-1}$ is a group morphism $E^\dagger \to C_R^\times$.

In the rest of this section K is a differential field with algebraically closed constant field C and divisible group $K^\dagger$ of logarithmic derivatives. (These conditions are satisfied if K is an algebraically closed differential field.) In the next subsection we show that up to isomorphism over K there is a unique exponential extension R of K satisfying $C_R = C$ and $(R^\times)^\dagger = K$. By Lemma 2.2.3 we must then have a group embedding $e\colon K \to R^\times$ such that $e(b)^\dagger = b$ for all $b \in K$; this motivates the construction below.

The universal exponential extension. We first describe a certain exponential extension of K. For this, take a **complement** Λ of $K^\dagger$, that is, a $\mathbb{Q}$-linear subspace of K such that $K = K^\dagger \oplus \Lambda$ (internal direct sum of $\mathbb{Q}$-linear subspaces of K). Below λ ranges over Λ. Let $\mathrm{e}(\Lambda)$ be a multiplicatively written abelian group, isomorphic to the additive subgroup Λ of K, with isomorphism $\lambda \mapsto \mathrm{e}(\lambda)\colon \Lambda \to \mathrm{e}(\Lambda)$. Put

$$\mathrm{U} \ := \ K\big[\mathrm{e}(\Lambda)\big],$$

the group ring of $\mathrm{e}(\Lambda)$ over K, an integral domain. As K-linear space,

$$\mathrm{U} \ = \ \bigoplus_{\lambda} K\,\mathrm{e}(\lambda) \qquad \text{(an internal direct sum of K-linear subspaces).}$$

For every $f \in \mathrm{U}$ we have a unique family (f_λ) in K such that

$$f \ = \ \sum_{\lambda} f_\lambda\,\mathrm{e}(\lambda),$$

with $f_\lambda = 0$ for all but finitely many λ; we call (f_λ) the **spectral decomposition** of f (with respect to Λ). We turn U into a differential ring extension of K by

$$\mathrm{e}(\lambda)' \ = \ \lambda\,\mathrm{e}(\lambda) \qquad \text{for all } \lambda.$$

(Think of $\mathrm{e}(\lambda)$ as $\exp(\int \lambda)$.) Thus for $f \in \mathrm{U}$ with spectral decomposition (f_λ),

$$f' \ = \ \sum_{\lambda} \big(f_\lambda' + \lambda f_\lambda\big)\,\mathrm{e}(\lambda),$$

so f' has spectral decomposition $(f_\lambda' + \lambda f_\lambda)$. Note that U is exponential over K by Lemma 2.1.1: $\mathrm{U}^\times = K^\times\,\mathrm{e}(\Lambda)$, so $(\mathrm{U}^\times)^\dagger = K^\dagger + \Lambda = K$.

Example 2.2.4. Let $K = C(\!(t^{\mathbb{Q}})\!)$ be as in Example 1.2.12, so $K^\dagger = (\mathbb{Q} \oplus o)t$. Take a $\mathbb{Q}$-linear subspace Λ_{c} of C with $C = \mathbb{Q} \oplus \Lambda_{\mathrm{c}}$ (internal direct sum of $\mathbb{Q}$-linear subspaces of C), and let

$$K_{\succ} \ := \ \big\{f \in K :\ \operatorname{supp}(f) \succ 1\big\},$$

a C-linear subspace of K. Then $\Lambda := (K_{\succ} \oplus \Lambda_{\mathrm{c}})t$ is a complement to $K^\dagger$, and hence $t^{-1}\Lambda = K_{\succ} \oplus \Lambda_{\mathrm{c}}$ is a complement to $(K^t)^\dagger$ in K^t. Moreover, if $L := \mathrm{P}(C) \subseteq K$ is the differential field of Puiseux series over C and $L_{\succ} := K_{\succ} \cap L$, then $L_{\succ} \oplus \Lambda_{\mathrm{c}}$ is a complement to $(L^t)^\dagger$.

A subgroup Λ_0 of Λ yields a differential subring $K\big[\mathrm{e}(\Lambda_0)\big]$ of U that is exponential over K as well. These differential subrings have a useful property. Recall from [ADH, 4.6] that a differential ring is said to be *simple* if $\{0\}$ is its only proper differential ideal.

Lemma 2.2.5. *Let Λ_0 be a subgroup of Λ. Then the differential subring $K\big[\mathrm{e}(\Lambda_0)\big]$ of U is simple. In particular, the differential ring U is simple.*

Proof. Let $I \neq R$ be a differential ideal of $R := K\big[\mathrm{e}(\Lambda_0)\big]$. Let $f_1, \dots, f_n \in K^\times$ and let $\lambda_1, \dots, \lambda_n \in \Lambda_0$ be distinct such that $f = \sum_{i=1}^n f_i\,\mathrm{e}(\lambda_i) \in I$. If $n \geqslant 1$, then Lemma 2.2.1 yields $i \neq j$ with $\mathrm{e}(\lambda_i)/\mathrm{e}(\lambda_j) = cg$ for some constant c in the differential fraction field of U and some $g \in K^\times$, so by taking logarithmic derivatives, $\lambda_i - \lambda_j \in K^\dagger$ and thus $\lambda_i = \lambda_j$, a contradiction. Thus $f = 0$. $\square$

Corollary 2.2.6. *Any morphism $K\big[\mathrm{e}(\Lambda_0)\big] \to R$ of differential K-algebras, with Λ_0 a subgroup of Λ and R a differential ring extension of K, is injective.*

The differential ring U is the directed union of its differential subrings of the form $\mathrm{U}_0 = K\big[\mathrm{e}(\Lambda_0)\big]$ where Λ_0 is a finitely generated subgroup of Λ. These U_0 are simple by Lemma 2.2.5 and finitely generated as a K-algebra, hence their differential fraction fields have constant field C by [ADH, 4.6.12]. Thus the differential fraction field of U has constant field C.

Lemma 2.2.7. *Suppose R is an exponential extension of K and R_0 is a differential subring of R with $C_R^\times \subseteq C_{R_0}$ and $K \subseteq (R_0^\times)^\dagger$. Then $R_0 = R$.*

Proof. Let E be the group of units of R that are exponential over K; so $R = K[E]$. Given $u \in E$ we have $u^\dagger \in K \subseteq (R_0^\times)^\dagger$, hence we have $u_0 \in R_0^\times$ with $u^\dagger = u_0^\dagger$, so $u = cu_0$ with $c \in C_R^\times \subseteq C_{R_0}$. Thus $E \subseteq R_0$ and so $R_0 = R$. □

Corollary 2.2.8. *Every endomorphism of the differential K-algebra* U *is an automorphism.*

Proof. Injectivity holds by Corollary 2.2.6, and surjectivity by Lemma 2.2.7. □

Every exponential extension of K with constant field C embeds into U, and hence is an integral domain. More precisely:

Lemma 2.2.9. *Let R be an exponential extension of K such that $C_R^\times$ is divisible, and set $\Lambda_0 := \Lambda \cap (R^\times)^\dagger$, a subgroup of Λ. Then there exists a morphism $K\big[\mathrm{e}(\Lambda_0)\big] \to R$ of differential K-algebras. Any such morphism is injective, and if $C_R = C$, then any such morphism is an isomorphism.*

Proof. Let E be as in the proof of Lemma 2.2.7, and let $e_E \colon E^\dagger \to E$ be the map e from Lemma 2.2.3. Since $E^\dagger = K^\dagger + \Lambda_0$ we have

$$E \ = \ C_R^\times\, e_E(E^\dagger) \ = \ C_R^\times\, e_E(K^\dagger)\, e_E(\Lambda_0) \ = \ C_R^\times\, K^\times\, e_E(\Lambda_0). \tag{2.2.1}$$

The group morphism $\mathrm{e}(\lambda_0) \mapsto e_E(\lambda_0) \colon \mathrm{e}(\Lambda_0) \to E$ ($\lambda_0 \in \Lambda_0$) extends uniquely to a K-algebra morphism $\iota \colon K\big[\mathrm{e}(\Lambda_0)\big] \to R = K[E]$. One verifies easily that ι is a differential ring morphism. The injectivity claim follows from Corollary 2.2.6. If $C_R = C$, then $E = K^\times e_E(\Lambda_0)$ by (2.2.1), whence surjectivity. □

Recall that U is an exponential extension of K with $C_{\mathrm{U}} = C$ and $(\mathrm{U}^\times)^\dagger = K$. By Lemma 2.2.9, this property characterizes U up to isomorphism:

Corollary 2.2.10. *If U is an exponential extension of K such that $C_U = C$ and $K \subseteq (U^\times)^\dagger$, then U is isomorphic to* U *as a differential K-algebra.*

Now U is also an exponential extension of K with $C_{\mathrm{U}} = C$ and with the property that every exponential extension R of K with $C_R = C$ embeds into U as a differential K-algebra. This property determines U up to isomorphism as well:

Corollary 2.2.11. *Suppose U is an exponential extension of K with $C_U = C$ such that every exponential extension R of K with $C_R = C$ embeds into U as a differential K-algebra. Then U is isomorphic to* U *as a differential K-algebra.*

Proof. Any embedding $\mathrm{U} \to U$ of differential K-algebras gives $K \subseteq (U^\times)^\dagger$. □

The results above show to what extent U is independent of the choice of Λ. We call U the **universal exponential extension of** K. If we need to indicate the dependence of U on K we denote it by U_K. By [ADH, 5.1.40] every $y \in \mathrm{U} = K\{\mathrm{e}(\Lambda)\}$ satisfies a linear differential equation $A(y) = 0$ where $A \in K[\partial]^{\neq}$; in the next section we isolate conditions on K which ensure that every $A \in K[\partial]^{\neq}$ has a zero $y \in \mathrm{U}^{\times} = K^{\times}\,\mathrm{e}(\Lambda)$.

Corollary 2.2.10 gives for $\phi \in K^{\times}$ an isomorphism $\mathrm{U}_{K^{\phi}} \cong (\mathrm{U}_K)^{\phi}$ of differential K^{ϕ}-algebras. Next we investigate how U_K behaves when passing from K to a differential field extension. Therefore, *in the rest of this subsection L is a differential field extension of K with algebraically closed constant field C_L, and $L^{\dagger}$ is divisible.* The next lemma relates the universal exponential extension U_L of L to U_K:

Lemma 2.2.12. *The inclusion $K \to L$ extends to an embedding $\iota\colon \mathrm{U}_K \to \mathrm{U}_L$ of differential rings. The image of any such embedding ι is contained in $K[E]$ where $E := \{u \in \mathrm{U}_L^{\times} : u^{\dagger} \in K\}$, and if $C_L = C$, then $\iota(\mathrm{U}_K) = K[E]$.*

Proof. The differential subring $R := K[E]$ of U_L is exponential over K with $(R^{\times})^{\dagger} = K$, hence Lemma 2.2.9 gives an embedding $\mathrm{U}_K \to R$ of differential K-algebras. Let $\iota\colon \mathrm{U}_K \to \mathrm{U}_L$ be any embedding of differential K-algebras. Then $\iota\big(\mathrm{e}(\Lambda)\big) \subseteq E$, so $\iota(\mathrm{U}_K) \subseteq R$; if $C_L = C$, then $\iota(\mathrm{U}_K) = R$ by Lemma 2.2.7. □

Corollary 2.2.13. *If $L^{\dagger} \cap K = K^{\dagger}$ and $\iota\colon \mathrm{U}_K \to \mathrm{U}_L$ is an embedding of differential K-algebras, then $L^{\times} \cap \iota(\mathrm{U}_K^{\times}) = K^{\times}$.*

Proof. Assume $L^{\dagger} \cap K = K^{\dagger}$ and identify U_K with a differential K-subalgebra of U_L via an embedding $\mathrm{U}_K \to \mathrm{U}_L$ of differential K-algebras. Let $a \in L^{\times} \cap \mathrm{U}_K^{\times}$; then $a^{\dagger} \in L^{\dagger} \cap K = K^{\dagger}$, so $a = bc$ where $c \in C_L^{\times}$, $b \in K^{\times}$. Now $c = a/b \in C_L^{\times} \cap \mathrm{U}_K^{\times} = C^{\times}$, since U_K has ring of constants C. So $a \in K^{\times}$ as required. □

Suppose $L^{\dagger} \cap K = K^{\dagger}$. Then the subspace $L^{\dagger}$ of the $\mathbb{Q}$-linear space L has a complement $\Lambda_L \supseteq \Lambda$. We fix such Λ_L and extend $\mathrm{e}\colon \Lambda \to \mathrm{e}(\Lambda)$ to a group isomorphism $\Lambda_L \to \mathrm{e}(\Lambda_L)$, also denoted by e, with $\mathrm{e}(\Lambda_L)$ a multiplicatively written commutative group extending $\mathrm{e}(\Lambda)$. Let $\mathrm{U}_L := L\big[\mathrm{e}(\Lambda_L)\big]$ be the corresponding universal exponential extension of L. Then the natural inclusion $\mathrm{U}_K \to \mathrm{U}_L$ is an embedding of differential K-algebras.

Automorphisms of U. These are easy to describe: the beginning of Section 2.1 gives a group embedding

$$\chi \mapsto \sigma_{\chi}\colon\ \mathrm{Hom}(\Lambda, K^{\times}) \to \mathrm{Aut}\big(K[\mathrm{e}(\Lambda)]|K\big)$$

into the group of K-algebra automorphisms of $K\big[\mathrm{e}(\Lambda)\big]$, given by

$$\sigma_{\chi}(f)\ :=\ f_{\chi}\ =\ \sum_{\lambda} f_{\lambda}\chi(\lambda)\,\mathrm{e}(\lambda) \qquad (\chi \in \mathrm{Hom}(\Lambda, K^{\times}),\ f \in K[\mathrm{e}(\Lambda)]).$$

It is easy to check that if $\chi \in \mathrm{Hom}(\Lambda, C^{\times}) \subseteq \mathrm{Hom}(\Lambda, K^{\times})$, then $\sigma_{\chi} \in \mathrm{Aut}_{\partial}(\mathrm{U}|K)$, that is, σ_{χ} is a differential K-algebra automorphism of U. Moreover:

Lemma 2.2.14. *The map $\chi \mapsto \sigma_{\chi}\colon \mathrm{Hom}(\Lambda, C^{\times}) \to \mathrm{Aut}_{\partial}(\mathrm{U}|K)$ is a group isomorphism. Its inverse assigns to any $\sigma \in \mathrm{Aut}_{\partial}(\mathrm{U}|K)$ the function $\chi\colon \Lambda \to C^{\times}$ given by $\chi(\lambda) := \sigma\big(\mathrm{e}(\lambda)\big)\,\mathrm{e}(-\lambda)$. In particular, $\mathrm{Aut}_{\partial}(\mathrm{U}|K)$ is commutative.*

Proof. Let $\sigma \in \mathrm{Aut}_{\partial}(\mathrm{U}|K)$ and let $\chi\colon \Lambda \to \mathrm{U}^{\times}$ be given by $\chi(\lambda) := \sigma\big(\mathrm{e}(\lambda)\big)\,\mathrm{e}(-\lambda)$. Then $\chi(\lambda)^{\dagger} = 0$ for all λ. It follows easily that $\chi \in \mathrm{Hom}(\Lambda, C^{\times})$ and $\sigma_{\chi} = \sigma$. □

The proof of the next result uses that the additive group $\mathbb{Q}$ embeds into $C^{\times}$.

Corollary 2.2.15. *If $f \in \mathrm{U}$ and $\sigma(f) = f$ for all $\sigma \in \mathrm{Aut}_{\partial}(\mathrm{U}|K)$, then $f \in K$.*

Proof. Suppose $f \in U$ and $\sigma(f) = f$ for all $\sigma \in \operatorname{Aut}_\partial(\mathrm{U}|K)$. For $\chi \in \operatorname{Hom}(\Lambda, C^\times)$ we have $f_\chi = f$, that is, $f_\lambda \chi(\lambda) = f_\lambda$ for all λ, so $\chi(\lambda) = 1$ whenever $f_\lambda \neq 0$. Now use that for $\lambda \neq 0$ there exists $\chi \in \operatorname{Hom}(\Lambda, C^\times)$ such that $\chi(\lambda) \neq 1$, so $f_\lambda = 0$. □

Corollary 2.2.16. *Every automorphism of the differential field K extends to an automorphism of the differential ring* U.

Proof. Lemma 2.2.3 yields a group morphism $\mu\colon K \to \mathrm{U}^\times$ such that $\mu(a)^\dagger = a$ for all $a \in K$. Let $\sigma \in \operatorname{Aut}_\partial(K)$. Then σ extends to an endomorphism, denoted also by σ, of the ring U, such that $\sigma\big(\mathrm{e}(\lambda)\big) = \mu\big(\sigma(\lambda)\big)$ for all λ. Then

$$\sigma\big(\mathrm{e}(\lambda)'\big) \ = \ \sigma\big(\lambda\, \mathrm{e}(\lambda)\big) \ = \ \sigma(\lambda)\mu\big(\sigma(\lambda)\big) \ = \ \mu\big(\sigma(\lambda)\big)' \ = \ \sigma\big(\mathrm{e}(\lambda)\big)',$$

hence σ is an endomorphism of the differential ring U. By Lemma 2.2.5, σ is injective, and by Lemma 2.2.7, σ is surjective. □

The real case. *In this subsection $K = H[\mathrm{i}]$ where H is a real closed differential subfield of K and* $\mathrm{i}^2 = -1$. Set $S_C := \big\{c \in C : |c| = 1\big\}$, a subgroup of $C^\times$. Then by Lemmas 2.1.7 and 2.2.14:

Corollary 2.2.17. *For $\sigma \in \operatorname{Aut}_\partial(\mathrm{U}|K)$ we have the equivalence*

$$\sigma(f^*) = \sigma(f)^* \text{ for all } f \in \mathrm{U} \quad \Longleftrightarrow \quad \sigma = \sigma_\chi \text{ for some } \chi \in \operatorname{Hom}(\Lambda, S_C).$$

Corollaries 2.2.17 and 2.1.8 together give:

Corollary 2.2.18. *Let $\sigma \in \operatorname{Aut}_\partial(\mathrm{U}|K)$ satisfy $\sigma(f^*) = \sigma(f)^*$ for all $f \in \mathrm{U}$. Then $\big\langle \sigma(f), \sigma(g)\big\rangle = \langle f, g\rangle$ for all $f, g \in \mathrm{U}$, hence $\|\sigma(f)\| = \|f\|$ for all $f \in \mathrm{U}$.*

Next we consider the subgroup

$$S \ := \ \{a + b\mathrm{i} : \ a, b \in H, \ a^2 + b^2 = 1\}$$

of $K^\times$, which is divisible, hence so is the subgroup $S^\dagger$ of $K^\dagger$. Lemma 1.2.4 yields $K^\dagger = H^\dagger \oplus S^\dagger$ (internal direct sum of $\mathbb{Q}$-linear subspaces of K) and $S^\dagger \subseteq H i$. Thus we can (and do) take the complement Λ of $K^\dagger$ in K so that $\Lambda = \Lambda_{\mathrm{r}} + \Lambda_{\mathrm{i}} i$ where $\Lambda_{\mathrm{r}}, \Lambda_{\mathrm{i}}$ are subspaces of the $\mathbb{Q}$-linear space H with Λ_{r} a complement of $H^\dagger$ in H and $\Lambda_{\mathrm{i}} i$ a complement of $S^\dagger$ in Hi. The automorphism $a + b\mathrm{i} \mapsto \overline{a + b\mathrm{i}} := a - b\mathrm{i}$ $(a, b \in H)$ of the differential field K now satisfies in $\mathrm{U} = K[\mathrm{e}(\Lambda)]$ the identity

$$\mathrm{e}(\overline{\lambda + \mu}) \ = \ \mathrm{e}(\overline{\lambda})\, \mathrm{e}(\overline{\mu}) \qquad (\lambda, \mu \in \Lambda),$$

so it extends to an automorphism $f \mapsto \overline{f}$ of the ring U as follows: for $f \in \mathrm{U}$ with spectral decomposition (f_λ), set

$$\overline{f} \ := \ \sum_\lambda \overline{f_\lambda}\, \mathrm{e}(\overline{\lambda}) \ = \ \sum_\lambda \overline{f_{\overline{\lambda}}}\, \mathrm{e}(\lambda),$$

so $\overline{\mathrm{e}(\lambda)} = \mathrm{e}(\overline{\lambda})$, and $\overline{f}$ has spectral decomposition $(\overline{f_{\overline{\lambda}}})$. We have $\overline{\overline{f}} = f$ for $f \in \mathrm{U}$, and $f \mapsto \overline{f}$ lies in $\operatorname{Aut}_\partial(\mathrm{U}|H)$. If $H^\dagger = H$, then $\Lambda_r = \{0\}$ and hence $\overline{f} = f^*$ for $f \in \mathrm{U}$, where f^* is as defined in Section 2.1. For $f \in \mathrm{U}$ we set

$$\operatorname{Re} f \ := \ \tfrac{1}{2}(f + \overline{f}), \qquad \operatorname{Im} f \ := \ \tfrac{1}{2\mathrm{i}}(f - \overline{f}).$$

(For $f \in K$ these agree with the usual real and imaginary parts of f as an element of $H[\mathrm{i}]$.) Consider the differential H-subalgebra

$$\mathrm{U}_{\mathrm{r}} \ := \ \big\{f \in \mathrm{U} : \overline{f} = f\big\}$$

of U. For $f \in \mathrm{U}$ with spectral decomposition (f_λ) we have $f \in \mathrm{U}_{\mathrm{r}}$ iff $f_{\overline{\lambda}} = \overline{f_\lambda}$ for all λ; in particular $\mathrm{U}_{\mathrm{r}} \cap K = H$. For $f \in \mathrm{U}$ we have $f = (\operatorname{Re} f) + (\operatorname{Im} f)i$ with $\operatorname{Re} f, \operatorname{Im} f \in \mathrm{U}_{\mathrm{r}}$, hence

$$\mathrm{U} \;=\; \mathrm{U}_{\mathrm{r}} \oplus \mathrm{U}_{\mathrm{r}} i \qquad \text{(internal direct sum of } H\text{-linear subspaces).}$$

Let D be a subfield of H (not necessarily the constant field of H), so $D[\mathrm{i}]$ is a subfield of K. Let V be a $D[\mathrm{i}]$-linear subspace of U; then $V_{\mathrm{r}} := V \cap \mathrm{U}_{\mathrm{r}}$ is a D-linear subspace of V. If $\overline{V} = V$ (that is, V is closed under $f \mapsto \overline{f}$), then $\operatorname{Re} f, \operatorname{Im} f \in V_{\mathrm{r}}$ for all $f \in V$, hence $V = V_{\mathrm{r}} \oplus V_{\mathrm{r}} i$ (internal direct sum of D-linear subspaces of V), so any basis of the D-linear space V_{r} is a basis of the $D[\mathrm{i}]$-linear space V.

Suppose now that $V = \bigoplus_\lambda V_\lambda$ (internal direct sum of subspaces of V) where V_λ is for each λ a $D[\mathrm{i}]$-linear subspace of $K\,\mathrm{e}(\lambda)$. Then $\overline{V} = V$ iff $V_{\overline{\lambda}} = \overline{V_\lambda}$ for all λ. Moreover:

Lemma 2.2.19. *Assume $H = H^\dagger$, $V_0 = \{0\}$, and $\overline{V} = V$. Let $\mathcal{V} \subseteq \mathrm{U}^\times$ be a basis of the subspace $\sum_{\operatorname{Im}\lambda>0} V_\lambda$ of V. Then the maps $v \mapsto \operatorname{Re} v$, $v \mapsto \operatorname{Im} v \colon \mathcal{V} \to V_{\mathrm{r}}$ are injective, $\operatorname{Re}\mathcal{V}$ and $\operatorname{Im}\mathcal{V}$ are disjoint, and $\operatorname{Re}\mathcal{V} \cup \operatorname{Im}\mathcal{V}$ is a basis of V_{r}.*

Proof. Note that $\Lambda = \Lambda_{\mathrm{i}} i$. Let μ range over $\Lambda_{\mathrm{i}}^{>}$ and set $\mathcal{V}_\mu = \mathcal{V} \cap K^\times \mathrm{e}(\mu i)$, a basis of the $D[\mathrm{i}]$-linear space $V_{\mu \mathrm{i}}$. Then $\mathcal{V} = \bigcup_\mu \mathcal{V}_\mu$, a disjoint union. For $v \in \mathcal{V}_\mu$ we have $v = a\,\mathrm{e}(\mu i)$ with $a = a_v \in K^\times$, so

$$\operatorname{Re} v \;=\; \tfrac{a}{2}\,\mathrm{e}(\mu i) + \tfrac{\overline{a}}{2}\,\mathrm{e}(-\mu i), \qquad \operatorname{Im} v \;=\; \tfrac{a}{2i}\,\mathrm{e}(\mu i) - \tfrac{\overline{a}}{2\mathrm{i}}\,\mathrm{e}(-\mu i),$$

from which it is clear that the two maps $\mathcal{V} \to V_{\mathrm{r}}$ in the statement of the lemma are injective. It is also easy to check that $\operatorname{Re}\mathcal{V}$ and $\operatorname{Im}\mathcal{V}$ are disjoint.

As $\mathcal{V}$ is a basis of the $D[\mathrm{i}]$-linear space $\sum_\mu V_{\mu\mathrm{i}} = \sum_{\operatorname{Im}\lambda>0} V_\lambda$, its set of conjugates $\overline{\mathcal{V}}$ is a basis of the $D[\mathrm{i}]$-linear space $\sum_\mu \overline{V_{\mu\mathrm{i}}} = \sum_\mu V_{-\mu\mathrm{i}} = \sum_{\operatorname{Im}\lambda<0} V_\lambda$, and so $\mathcal{V} \cup \overline{\mathcal{V}}$ (a disjoint union) is a basis of V. Thus $\operatorname{Re}\mathcal{V} \cup \operatorname{Im}\mathcal{V}$ is a basis of V as well. As $\operatorname{Re}\mathcal{V} \cup \operatorname{Im}\mathcal{V}$ is contained in V_{r}, it is a basis of the D-linear space V_{r}. $\square$

If $H = H^\dagger$, then $V := \sum_{\lambda\neq 0} K\,\mathrm{e}(\lambda)$ gives $\overline{V} = V$, so Lemma 2.2.19 gives then for $D := H$ the basis of the H-linear space V_{r} consisting of the elements

$$\operatorname{Re}\bigl(\mathrm{e}(\lambda)\bigr) \;=\; \tfrac{1}{2}\bigl(\mathrm{e}(\lambda) + \mathrm{e}(\overline{\lambda})\bigr), \qquad \operatorname{Im}\bigl(\mathrm{e}(\lambda)\bigr) \;=\; \tfrac{1}{2\mathrm{i}}\bigl(\mathrm{e}(\lambda) - \mathrm{e}(\overline{\lambda})\bigr) \qquad (\operatorname{Im}\lambda > 0).$$

Corollary 2.2.20. *Suppose $H = H^\dagger$. Set $\mathrm{c}(\lambda) := \operatorname{Re}\bigl(\mathrm{e}(\lambda)\bigr)$ and $\mathrm{s}(\lambda) := \operatorname{Im}\bigl(\mathrm{e}(\lambda)\bigr)$, for $\operatorname{Im}\lambda > 0$. Then for $V := \sum_{\lambda\neq 0} K\,\mathrm{e}(\lambda)$ we have $\mathrm{U}_{\mathrm{r}} = H + V_{\mathrm{r}}$, so*

$$\mathrm{U}_{\mathrm{r}} \;=\; H \oplus \bigoplus_{\operatorname{Im}\lambda>0} \bigl(H\,\mathrm{c}(\lambda) \oplus H\,\mathrm{s}(\lambda)\bigr) \quad \textit{(internal direct sum of } H\textit{-linear subspaces),}$$

and thus $\mathrm{U}_{\mathrm{r}} = H\bigl[\,\mathrm{c}(\Lambda_{\mathrm{i}}^{>} i) \cup \mathrm{s}(\Lambda_{\mathrm{i}}^{>} i)\bigr]$.

2.3. The Spectrum of a Differential Operator

In this section K is a differential field, a, b range over K, and A, B over $K[\partial]$. This and the next two sections are mainly differential-algebraic in nature, and deal with splittings of linear differential operators. In the present section we introduce the concept of *eigenvalue* of A and the *spectrum* of A (the collection of its eigenvalues). In Section 2.4 we show how the eigenvalues of A relate to the behavior of A over the universal exponential extension of K.

Twisting. Let L be a differential field extension of K with $L^\dagger \supseteq K$. Let $u \in L^\times$ be such that $u^\dagger = a \in K$. Then the twist $A_{\ltimes u} = u^{-1}Au$ of A by u has the same order as A and coefficients in K [ADH, 5.8.8], and only depends on a, not on u or L; in fact, $\operatorname{Ri}(A_{\ltimes u}) = \operatorname{Ri}(A)_{+a}$ [ADH, 5.8.5]. Hence for each a we may define

$$A_a \ := \ A_{\ltimes u} \ = \ u^{-1}Au \in K[\partial]$$

where $u \in L^\times$ is arbitrary with $u^\dagger = a$. The map $A \mapsto A_{\ltimes u}$ is an automorphism of the ring $K[\partial]$ that is the identity on K (with inverse $B \mapsto B_{\ltimes u^{-1}}$); so $A \mapsto A_a$ is an automorphism of the ring $K[\partial]$ that is the identity on K (with inverse $B \mapsto B_{-a}$). Note that $\partial_a = \partial + a$, and that

$$(a, A) \mapsto A_a \ : \ K \times K[\partial] \to K[\partial]$$

is an action of the additive group of K on the set $K[\partial]$, in particular, $A_a = A$ for $a = 0$. For $b \neq 0$ we have $(A_a)_{\ltimes b} = A_{a+b^\dagger}$.

Eigenvalues. *In the rest of this section $A \neq 0$ and $r := \operatorname{order}(A)$.* We call

$$\operatorname{mult}_a(A) \ := \ \dim_C \ker_K A_a \in \{0, \dots, r\}$$

the **multiplicity** of A at a. If $B \neq 0$, then $\operatorname{mult}_a(B) \leqslant \operatorname{mult}_a(AB)$, as well as

$$\operatorname{mult}_a(AB) \ \leqslant \ \operatorname{mult}_a(A) + \operatorname{mult}_a(B), \tag{2.3.1}$$

with equality if and only if $B_a(K) \supseteq \ker_K A_a$; see [ADH, remarks before 5.1.12]. For $u \in K^\times$ we have an isomorphism

$$y \mapsto yu \ : \ \ker_K A_{\ltimes u} \to \ker_K A$$

of C-linear spaces, hence

$$\operatorname{mult}_a(A) \ = \ \operatorname{mult}_b(A) \qquad \text{whenever } a - b \in K^\dagger.$$

Thus we may define the **multiplicity** of A at the element $[a] := a + K^\dagger$ of $K/K^\dagger$ as $\operatorname{mult}_{[a]}(A) := \operatorname{mult}_a(A)$.

In the rest of this section α ranges over $K/K^\dagger$. We say that α is an **eigenvalue** of A if $\operatorname{mult}_\alpha(A) \geqslant 1$. Thus for $B \neq 0$: if α is an eigenvalue of B of multiplicity μ, then α is an eigenvalue of AB of multiplicity $\geqslant \mu$; if α is an eigenvalue of AB, then it is an eigenvalue of A or of B; and if $B_a(K) \supseteq \ker_K(A_a)$, then $\alpha = [a]$ is an eigenvalue of AB if and only if it is an eigenvalue of A or of B.

Example 2.3.1. Suppose $A = \partial - a$. Then for each element $u \neq 0$ in a differential field extension of K with $b := u^\dagger \in K$ we have $A_b = A_{\ltimes u} = \partial - (a - b)$, so $\operatorname{mult}_b(A) \geqslant 1$ iff $a - b \in K^\dagger$. Hence the only eigenvalue of A is $[a]$.

The **spectrum** of A is the set $\Sigma(A) = \Sigma_K(A)$ of its eigenvalues. Thus $\Sigma(A) = \emptyset$ if $r = 0$, and for $b \neq 0$ we have $\operatorname{mult}_a(A) = \operatorname{mult}_a(bA) = \operatorname{mult}_a(A_{\ltimes b})$, so A, bA, and $Ab = bA_{\ltimes b}$ all have the same spectrum. By [ADH, 5.1.21] we have

$$\Sigma(A) = \big\{\alpha : A \in K[\partial](\partial - a) \text{ for some } a \text{ with } [a] = \alpha\big\}. \tag{2.3.2}$$

Hence for irreducible A: $\Sigma(A) \neq \emptyset \ \Leftrightarrow \ r = 1$. From (2.3.1) we obtain:

Lemma 2.3.2. *Suppose $B \neq 0$ and set $s := \operatorname{order} B$. Then*

$$\operatorname{mult}_\alpha(B) \ \leqslant \ \operatorname{mult}_\alpha(AB) \ \leqslant \ \operatorname{mult}_\alpha(A) + \operatorname{mult}_\alpha(B),$$

where the second inequality is an equality if K is s-linearly surjective. Hence

$$\Sigma(B) \ \subseteq \ \Sigma(AB) \ \subseteq \ \Sigma(A) \cup \Sigma(B).$$

If K is s-linearly surjective, then $\Sigma(AB) = \Sigma(A) \cup \Sigma(B)$.

Example. For $n \geqslant 1$ we have $\Sigma\big((\partial - a)^n\big) = \big\{[a]\big\}$. (By induction on n, using Example 2.3.1 and Lemma 2.3.2.)

It follows from Lemma 2.3.2 that A has at most r eigenvalues. More precisely:

Lemma 2.3.3. *We have $\sum_\alpha \operatorname{mult}_\alpha(A) \leqslant r$. If $\sum_\alpha \operatorname{mult}_\alpha(A) = r$, then A splits over K; the converse holds if $r = 1$ or K is 1-linearly surjective.*

Proof. By induction on r. The case $r = 0$ is obvious, so suppose $r > 0$. We may also assume $\Sigma(A) \neq \emptyset$: otherwise $\sum_\alpha \operatorname{mult}_\alpha(A) = 0$ and A does not split over K. Now (2.3.2) gives a, B with $A = B(\partial - a)$. By Example 2.3.1 we have $\Sigma(\partial - a) = \big\{[a]\big\}$ and $\operatorname{mult}_a(\partial - a) = 1$. By the inductive hypothesis applied to B and the second inequality in Lemma 2.3.2 we thus get $\sum_\alpha \operatorname{mult}_\alpha(A) \leqslant r$.

Suppose that $\sum_\alpha \operatorname{mult}_\alpha(A) = r$. Then $\sum_\alpha \operatorname{mult}_\alpha(B) = r - 1$ by Lemma 2.3.2 and the inductive hypothesis applied to B. Therefore B splits over K, again by the inductive hypothesis, and so does A. Finally, if K is 1-linearly surjective and A splits over K, then we arrange that B splits over K, so $\sum_\alpha \operatorname{mult}_\alpha(B) = r - 1$ by the inductive hypothesis, hence $\sum_\alpha \operatorname{mult}_\alpha(A) = r$ by Lemma 2.3.2. □

Section 2.4 gives a more explicit proof of Lemma 2.3.3, under additional hypotheses on K. Next, let L be a differential field extension of K. Then $\operatorname{mult}_a(A)$ does not strictly decrease in passing from K to L [ADH, 4.1.13]. Hence the group morphism

$$a + K^\dagger \mapsto a + L^\dagger \colon K/K^\dagger \to L/L^\dagger$$

restricts to a map $\Sigma_K(A) \to \Sigma_L(A)$; in particular, if $\Sigma_K(A) \neq \emptyset$, then $\Sigma_L(A) \neq \emptyset$. If $L^\dagger \cap K = K^\dagger$, then $|\Sigma_K(A)| \leqslant |\Sigma_L(A)|$, and $\sum_\alpha \operatorname{mult}_\alpha(A)$ also does not strictly decrease if K is replaced by L.

Lemma 2.3.4. *Let $a_1, \dots, a_r \in K$ and*

$$A \ = \ (\partial - a_r) \cdots (\partial - a_1), \qquad \sum_\alpha \operatorname{mult}_\alpha(A) \ = \ r.$$

Then the spectrum of A is $\big\{[a_1], \dots, [a_r]\big\}$, and for all α,

$$\operatorname{mult}_\alpha(A) \ = \ \big|\big\{i \in \{1, \dots, r\} : \ \alpha = [a_i]\big\}\big|.$$

Proof. Let i range over $\{1, \dots, r\}$. By Lemma 2.3.2 and Example 2.3.1,

$$\operatorname{mult}_\alpha(A) \ \leqslant \ \sum_i \operatorname{mult}_\alpha(\partial - a_i) \ = \ \big|\big\{i : \alpha = [a_i]\big\}\big|$$

and hence

$$r \ = \ \sum_\alpha \operatorname{mult}_\alpha(A) \ \leqslant \ \sum_\alpha \big|\big\{i : \alpha = [a_i]\big\}\big| \ = \ r.$$

Thus for each α we have $\operatorname{mult}_\alpha(A) = \big|\big\{i : \alpha = [a_i]\big\}\big|$ as required. □

Recall from [ADH, 5.1.8] that $D^* \in K[\partial]$ denotes the *adjoint* of $D \in K[\partial]$, and that the map $D \mapsto D^*$ is an involution of the ring $K[\partial]$ with $a^* = a$ for all a and $\partial^* = -\partial$. If A splits over K, then so does A^*. Furthermore, $(A_a)^* = (A^*)_{-a}$ for all a. By Lemmas 2.3.3 and 2.3.4:

Corollary 2.3.5. *Suppose K is 1-linearly surjective and $\sum_\alpha \operatorname{mult}_\alpha(A) = r$. Then $\operatorname{mult}_\alpha(A) = \operatorname{mult}_{-\alpha}(A^*)$ for all α. In particular, the map $\alpha \mapsto -\alpha$ restricts to a bijection $\Sigma(A) \to \Sigma(A^*)$.*

Let $\phi \in K^\times$. Then $(A^\phi)_a = (A_{\phi a})^\phi$ and hence

$$\operatorname{mult}_a(A^\phi) \;=\; \operatorname{mult}_{\phi a}(A),$$

so the group isomorphism

$$[a] \mapsto [\phi a] \;:\; K^\phi/\phi^{-1}K^\dagger \to K/K^\dagger \tag{2.3.3}$$

maps $\Sigma(A^\phi)$ onto $\Sigma(A)$.

Note that $K[\partial]/K\partial]A$ as a K-linear space has dimension $r = \operatorname{order} A$. Recall from [ADH, 5.1] that A and $B \neq 0$ are said to *have the same type* if the (left) $K[\partial]$-modules $K[\partial]/K[\partial]A$ and $K[\partial]/K[\partial]B$ are isomorphic (and so $\operatorname{order} B = r$). By [ADH, 5.1.19]:

Lemma 2.3.6. *The operators A and $B \neq 0$ have the same type iff* $\operatorname{order} B = r$ *and there is $R \in K[\partial]$ of order $< r$ with $1 \in K[\partial]R + K[\partial]A$ and $BR \in K[\partial]A$.*

Hence if A, B have the same type, then they also have the same type as elements of $L[\partial]$, for any differential field extension L of K. Since $B \mapsto B_a$ is an automorphism of the ring $K[\partial]$, Lemma 2.3.6 and [ADH, 5.1.20] yield:

Lemma 2.3.7. *If A and $B \neq 0$ have the same type, then so do A_a, B_a, for all a, and thus A, B have the same eigenvalues, with same multiplicity.*

By this lemma the spectrum of A depends only on the type of A, that is, on the isomorphism type of the $K[\partial]$-module $K[\partial]/K[\partial]A$, suggesting one might try to associate a spectrum to each differential module over K. (Recall from [ADH, 5.5] that a differential module over K is a $K[\partial]$-module of finite dimension as K-linear space.) We do not develop this point of view further in the present monograph, where our focus is on linear differential operators. (There will be more on this in [7].) But we remark here that it motivates the terminology of "eigenvalues" originating in the case of the differential field of Puiseux series over $\mathbb{C}$ treated in [20].

2.4. Eigenvalues and Splittings

In this section K is a differential field such that C is algebraically closed and $K^\dagger$ is divisible. We let A, B range over $K[\partial]$, and we assume $A \neq 0$ and set $r := \operatorname{order} A$.

Spectral decomposition of differential operators. Fix a complement Λ of the subspace $K^\dagger$ of the $\mathbb{Q}$-linear space K, let $\mathrm{U} := K\big[\mathrm{e}(\Lambda)\big]$ be the universal exponential extension of K, let Ω be the differential fraction field of the differential K-algebra U, and let λ range over Λ. Then

$$A_\lambda \;=\; A_{\ltimes \mathrm{e}(\lambda)} \;=\; \mathrm{e}(-\lambda)A\,\mathrm{e}(\lambda) \in K[\partial].$$

Moreover, for every $a \in K$ there is a unique λ with $a - \lambda \in K^\dagger$, so $\operatorname{mult}_{[a]}(A) = \operatorname{mult}_\lambda(A)$. Call λ an **eigenvalue** of A with respect to our complement Λ of $K^\dagger$ in K

if $[\lambda]$ is an eigenvalue of A; thus the group isomorphism $\lambda \mapsto [\lambda]\colon \Lambda \to K/K^\dagger$ maps the set of eigenvalues of A with respect to Λ onto the spectrum of A. For $f \in \mathrm{U}$ with spectral decomposition (f_λ) we have

$$A(f) \;=\; \sum_\lambda A_\lambda(f_\lambda)\, \mathrm{e}(\lambda),$$

so $A(\mathrm{U}^\times) \subseteq \mathrm{U}^\times \cup \{0\}$. We call the family (A_λ) the **spectral decomposition** of A (with respect to Λ). Given a C-linear subspace V of U, we set $V_\lambda := V \cap K\,\mathrm{e}(\lambda)$, a C-linear subspace of V; the sum $\sum_\lambda V_\lambda$ is direct. For $V := \mathrm{U}$ we have $\mathrm{U}_\lambda = K\,\mathrm{e}(\lambda)$, and $\mathrm{U} = \bigoplus_\lambda \mathrm{U}_\lambda$ with $A(\mathrm{U}_\lambda) \subseteq \mathrm{U}_\lambda$ for all λ. Taking $V := \ker_{\mathrm{U}} A$, we obtain $V_\lambda = (\ker_K A_\lambda)\,\mathrm{e}(\lambda)$ and hence $\dim_C V_\lambda = \mathrm{mult}_\lambda(A)$, and $V = \bigoplus_\lambda V_\lambda$. Thus

$$|\Sigma(A)| \;\leqslant\; \sum_\lambda \mathrm{mult}_\lambda(A) \;=\; \dim_C \ker_{\mathrm{U}} A \;\leqslant\; r. \tag{2.4.1}$$

Moreover:

Lemma 2.4.1. *The C-linear space $\ker_{\mathrm{U}} A$ has a basis contained in $\mathrm{U}^\times = K^\times\,\mathrm{e}(\Lambda)$.*

Example. We have a C-algebra isomorphism $P(Y) \mapsto P(\partial)\colon C[Y] \to C[\partial]$. Suppose $A \in C[\partial] \subseteq K[\partial]$, let $P(Y) \in C[Y]$, $P(\partial) = A$, and let $c_1, \dots, c_n \in C$ be the distinct zeros of P, of respective multiplicities $m_1, \dots, m_n \in \mathbb{N}^{\geqslant 1}$ (so $r = \deg P = m_1 + \cdots + m_n$). Suppose also $C \subseteq \Lambda$, and $x \in K$ satisfies $x' = 1$. (This holds in Example 2.2.4.) Then the $x^i\,\mathrm{e}(c_j) \in \mathrm{U}$ ($1 \leqslant j \leqslant n$, $0 \leqslant i < m_j$) form a basis of the C-linear space $\ker_{\mathrm{U}} A$ by [ADH, 5.1.18]. So the eigenvalues of A with respect to Λ are $c_1, \dots, c_n$, with respective multiplicities $m_1, \dots, m_n$.

Corollary 2.4.2. *Suppose $\dim_C \ker_{\mathrm{U}} A = r \geqslant 1$ and $A = \partial^r + a_{r-1}\partial^{r-1} + \cdots + a_0$ where $a_0, \dots, a_{r-1} \in K$. Then*

$$\sum_\lambda \mathrm{mult}_\lambda(A)\lambda \;\equiv\; -a_{r-1} \mod K^\dagger.$$

In particular, $\sum_\lambda \mathrm{mult}_\lambda(A)\lambda = 0$ iff $a_{r-1} \in K^\dagger$.

Proof. Take a basis $y_1, \dots, y_r$ of $\ker_{\mathrm{U}} A$ with $y_j = f_j\,\mathrm{e}(\lambda_j)$, $f_j \in K^\times$, $\lambda_j \in \Lambda$. The Wronskian matrix $\mathrm{Wr}(y_1, \dots, y_r)$ of $(y_1, \dots, y_r)$ [ADH, p. 206] equals

$$\mathrm{Wr}(y_1, \dots, y_r) \;=\; M \begin{pmatrix} \mathrm{e}(\lambda_1) & & \\ & \ddots & \\ & & \mathrm{e}(\lambda_r) \end{pmatrix} \qquad \text{where } M \in \mathrm{GL}_n(K).$$

Then $w := \mathrm{wr}(y_1, \dots, y_r) = \det \mathrm{Wr}(y_1, \dots, y_r) \neq 0$ by [ADH, 4.1.13] and

$$-a_{r-1} \;=\; w^\dagger \;=\; (\det M)^\dagger + \lambda_1 + \cdots + \lambda_r$$

where we used [ADH, 4.1.17] for the first equality. □

If A splits over K, then so does A_λ. Moreover, if $A_\lambda(K) = K$, then $A(\mathrm{U}_\lambda) = \mathrm{U}_\lambda$: for $f, g \in K$ with $A_\lambda(f) = g$ we have $A\big(f\,\mathrm{e}(\lambda)\big) = g\,\mathrm{e}(\lambda)$. Thus:

Lemma 2.4.3. *Suppose K is r-linearly surjective, or K is 1-linearly surjective and A splits over K. Then $A(\mathrm{U}_\lambda) = \mathrm{U}_\lambda$ for all λ and hence $A(\mathrm{U}) = \mathrm{U}$.*

In the next subsection we study the connection between splittings of A and bases of the C-linear space $\ker_{\mathrm{U}} A$ in more detail.

Constructing splittings and bases. Recall that order $A = r \in \mathbb{N}$. Set $\mathrm{U} = \mathrm{U}_K$, so $\mathrm{U}^\times = K^\times \mathrm{e}(\Lambda)$. Let $y_1, \dots, y_r \in \mathrm{U}^\times$. We construct a sequence $A_0, \dots, A_n$ of monic operators in $K[\partial]$ with $n \leqslant r$ as follows. First, set $A_0 := 1$. Next, given $A_0, \dots, A_{i-1}$ in $K[\partial]^{\neq}$ $(1 \leqslant i \leqslant r)$, set $f_i := A_{i-1}(y_i)$; if $f_i \neq 0$, then $f_i \in \mathrm{U}^\times$, so $f_i^\dagger \in K$, and the next term in the sequence is

$$A_i \ := \ (\partial - a_i)A_{i-1}, \qquad a_i \ := \ f_i^\dagger,$$

whereas if $f_i = 0$, then $n := i - 1$ and the construction is finished.

Lemma 2.4.4. $\ker_{\mathrm{U}} A_i = Cy_1 \oplus \cdots \oplus Cy_i$ *(internal direct sum) for* $i = 0, \dots, n$.

Proof. By induction on $i \leqslant n$. The case $i = 0$ being trivial, suppose $1 \leqslant i \leqslant n$ and the claim holds for $i-1$ in place of i. Then $A_{i-1}(y_i) = f_i \neq 0$, hence $y_i \notin \ker_{\mathrm{U}} A_{i-1} = Cy_1 \oplus \cdots \oplus Cy_{i-1}$, and $A_i = (\partial - f_i^\dagger)A_{i-1}$, so by [ADH, 5.1.14(i)] we have $\ker_{\mathrm{U}} A_i = \ker_{\mathrm{U}} A_{i-1} \oplus Cy_i = Cy_1 \oplus \cdots \oplus Cy_i$. $\square$

We denote the tuple $(a_1, \dots, a_n) \in K^n$ just constructed by $\operatorname{split}(y_1, \dots, y_r)$, so $A_n = (\partial - a_n) \cdots (\partial - a_1)$. Suppose $r \geqslant 1$. Then $n \geqslant 1$, $a_1 = y_1^\dagger$, $A_1 = \partial - a_1$, $A_1(y_2), \dots, A_1(y_n) \in \mathrm{U}^\times$, and we have

$$(a_2, \dots, a_n) \ = \ \operatorname{split}\big(A_1(y_2), \dots, A_1(y_n)\big).$$

By Lemma 2.4.4, $n \leqslant r$ is maximal such that $y_1, \dots, y_n$ are C-linearly independent. In particular, $y_1, \dots, y_r$ are C-linearly independent iff $n = r$.

Corollary 2.4.5. *If $A(y_i) = 0$ for $i = 1, \dots, n$, then $A \in K[\partial]A_n$. Thus if $n = r$ and $A(y_i) = 0$ for $i = 1, \dots, r$, then $A = a(\partial - a_r) \cdots (\partial - a_1)$ where $a \in K^\times$.*

This follows from [ADH, 5.1.15(i)] and Lemma 2.4.4.

Suppose that H is a differential subfield of K and $y_1^\dagger, \dots, y_r^\dagger \in H$. Then we have $\operatorname{split}(y_1, \dots, y_r) \in H^n$: use that $y' \in Hy$ with $y \in U$ gives $y^{(m)} \in Hy$ for all m, so $B(y) \in Hy$ for all $B \in H[\partial]$, hence for such B, if $f := B(y) \neq 0$, then $f^\dagger \in H$.

Corollary 2.4.6. *Suppose* $\dim_C \ker_{\mathrm{U}} A = r$. *Then* $\ker_{\mathrm{U}} A = \ker_\Omega A$ *and A splits over K. If $A = (\partial - a_r) \cdots (\partial - a_1)$, $a_1, \dots, a_r \in K$, then the spectrum of A is $\big\{[a_1], \dots, [a_r]\big\}$, and for all $\alpha \in K/K^\dagger$,*

$$\operatorname{mult}_\alpha(A) \ = \ \big|\big\{i \in \{1, \dots, r\} : \ \alpha = [a_i]\big\}\big|.$$

Proof. A splits over K by Lemma 2.4.1 and Corollary 2.4.5. The rest follows from Lemma 2.3.4 in view of $\sum_\lambda \operatorname{mult}_\lambda(A) = \dim_C \ker_{\mathrm{U}} A$. $\square$

Conversely, we can associate to a given splitting of A over K a basis of $\ker_{\mathrm{U}} A$ consisting of r elements of $\mathrm{U}^\times$, provided K is 1-linearly surjective when $r \geqslant 2$:

Lemma 2.4.7. *Assume K is 1-linearly surjective in case $r \geqslant 2$. Let*

$$A \ = \ (\partial - a_r) \cdots (\partial - a_1) \qquad \text{where } a_i = b_i^\dagger + \lambda_i,\ b_i \in K^\times,\ \lambda_i \in \Lambda\ (i = 1, \dots, r).$$

Then there are C-linearly independent $y_1, \dots, y_r \in \ker_{\mathrm{U}} A$ with $y_i \in K^\times \mathrm{e}(\lambda_i)$ for $i = 1, \dots, r$ and $\operatorname{split}(y_1, \dots, y_r) = (a_1, \dots, a_r)$.

Proof. By induction on r. The case $r=0$ is trivial, and for $r=1$ we can take $y_1 = b_1\,\mathrm{e}(\lambda_1)$. Let $r \geqslant 2$ and suppose inductively that for

$$B \ := \ (\partial - a_r)\cdots(\partial - a_2)$$

we have C-linearly independent $z_2,\dots,z_r \in \ker_{\mathrm{U}} B$ with $z_i \in K^\times\,\mathrm{e}(\lambda_i)$ for $i=2,\dots,r$ and $\operatorname{split}(z_2,\dots,z_r) = (a_2,\dots,a_r)$. For $i=2,\dots,r$, Lemma 2.4.3 gives $y_i \in K^\times\,\mathrm{e}(\lambda_i)$ with $(\partial - a_1)(y_i) = z_i$. Set $y_1 := b_1\,\mathrm{e}(\lambda_1)$, so $\ker_{\mathrm{U}}(\partial - a_1) = Cy_1$. Then $y_1,\dots,y_r \in \ker_{\mathrm{U}} A$ are C-linearly independent such that $y_i \in K^\times\,\mathrm{e}(\lambda_i)$ for $i=1,\dots,r$, and one verifies easily that $\operatorname{split}(y_1,\dots,y_r) = (a_1,\dots,a_r)$. □

Corollary 2.4.8. *Assume K is 1-linearly surjective when $r \geqslant 2$. Then*

$$A \text{ splits over } K \iff \dim_C \ker_{\mathrm{U}} A \ = \ r.$$

Remark. If $\dim_C \ker_{\mathrm{U}} A = r$ and $\lambda_1,\dots,\lambda_d$ are the eigenvalues of A with respect to Λ, then the differential subring $K\big[\mathrm{e}(\lambda_1),\mathrm{e}(-\lambda_1),\dots,\mathrm{e}(\lambda_d),\mathrm{e}(-\lambda_d)\big]$ of U is the Picard-Vessiot ring for A over K; see [20, Section 1.3]. If K is linearly closed and linearly surjective, then U is by Corollary 2.4.8 the universal Picard-Vessiot ring of the differential field K as defined in [20, Chapter 10]. Our construction of U above is modeled on the description of the universal Picard-Vessiot ring of the algebraic closure of $C(\!(t)\!)$ given in [20, Chapter 3].

Recalling our convention that $r = \operatorname{order} A$, here is a complement to Lemma 2.4.1:

Corollary 2.4.9. *Let V be a C-linear subspace of* U *with $r = \dim_C V$. Then there is at most one monic A with $V = \ker_{\mathrm{U}} A$. Moreover, the following are equivalent:*

(i) *$V = \ker_{\mathrm{U}} A$ for some monic A that splits over K;*
(ii) *$V = \ker_{\mathrm{U}} B$ for some $B \neq 0$;*
(iii) *$V = \sum_\lambda V_\lambda$;*
(iv) *V has a basis contained in* $\mathrm{U}^\times$.

Proof. The first claim follows from [ADH, 5.1.15] applied to the differential fraction field of U in place of K. The implication (i) ⇒ (ii) is clear, (ii) ⇒ (iii) was noted before Lemma 2.4.1, and (iii) ⇒ (iv) is obvious. For (iv) ⇒ (i), let $y_1,\dots,y_r \in \mathrm{U}^\times$ be a basis of V. Then $\operatorname{split}(y_1,\dots,y_r) = (a_1,\dots,a_r) \in K^r$, so $V = \ker_{\mathrm{U}} A$ for $A = (\partial - a_r)\cdots(\partial - a_1)$ by Lemma 2.4.4, so (i) holds. □

Let $y_1,\dots,y_r \in \mathrm{U}^\times$ and $(a_1,\dots,a_n) := \operatorname{split}(y_1,\dots,y_r)$. We finish this subsection with some remarks about $(a_1,\dots,a_n)$ for use in [7]. Let $A_0,\dots,A_n \in K[\partial]$ be as above and recall that $n \leqslant r$ is maximal such that $y_1,\dots,y_n$ are C-linearly independent.

Lemma 2.4.10. *Assume $n = r$. Let $z_1,\dots,z_r \in \mathrm{U}^\times$. The following are equivalent:*

(i) *$z_1,\dots,z_r$ are C-linearly independent and $(a_1,\dots,a_r) = \operatorname{split}(z_1,\dots,z_r)$;*
(ii) *for $i=1,\dots,r$ there are $c_{ii}, c_{i,i-1},\dots,c_{i1} \in C$ such that*

$$z_i \ = \ c_{ii}y_i + c_{i,i-1}y_{i-1} + \cdots + c_{i1}y_1 \ \textit{and} \ c_{ii} \neq 0.$$

Proof. The case $r=0$ is trivial. Let $r=1$. If (i) holds, then $y_1^\dagger = a_1 = z_1^\dagger$, hence $z_1 \in C^\times y_1$, so (ii) holds. The converse is obvious. Let $r \geqslant 2$, and assume (i) holds. Put $\widetilde{y}_i := A_1(y_i)$ and $\widetilde{z}_i := A_1(z_i)$ for $i=2,\dots,r$. Then

$$\operatorname{split}(\widetilde{y}_2,\dots,\widetilde{y}_r) \ = \ (a_2,\dots,a_r) \ = \ \operatorname{split}(\widetilde{z}_2,\dots,\widetilde{z}_r),$$

so we can assume inductively to have $c_{ij} \in C$ $(2 \leqslant j \leqslant i \leqslant r)$ with

$$\widetilde{z}_i \ = \ c_{ii}\widetilde{y}_i + c_{i,i-1}\widetilde{y}_{i-1} + \cdots + c_{i2}\widetilde{y}_2 \quad \text{and} \quad c_{ii} \neq 0 \qquad (2 \leqslant i \leqslant r).$$

Hence for $2 \leqslant i \leqslant r$,

$$z_i \in c_{ii}y_i + c_{i,i-1}y_{i-1} + \cdots + c_{i2}y_2 + \ker_{\mathrm{U}} A_1.$$

Now use $\ker_{\mathrm{U}} A_1 = Cy_1$ to conclude (ii). For the converse, let $c_{ij} \in C$ be as in (ii). Then clearly $z_1, \dots, z_r$ are C-linearly independent. Let $(b_1, \dots, b_r) := \operatorname{split}(z_1, \dots, z_r)$ and $B_{r-1} := (\partial - b_{r-1}) \cdots (\partial - b_1)$. Then $a_r = f_r^\dagger$ where $f_r = A_{r-1}(y_r) \neq 0$, and $b_r = g_r^\dagger$ where $g_r := B_{r-1}(z_r) \neq 0$. Now inductively we have $a_j = b_j$ for $j = 1, \dots, r-1$, so $A_{r-1} = B_{r-1}$, and $A_{r-1}(y_i) = 0$ for $i = 1, \dots, r-1$ by Lemma 2.4.4. Hence $g_r = c_{rr}f_r$, and thus $a_r = b_r$. □

Lemma 2.4.11. *Let $z \in \mathrm{U}^\times$. Then* $\operatorname{split}(y_1 z, \dots, y_r z) = (a_1 + z^\dagger, \dots, a_n + z^\dagger)$.

Proof. Since for $m \leqslant r$, the units $y_1 z, \dots, y_m z$ of U are C-linearly independent iff $y_1, \dots, y_m$ are C-linearly independent, we see that the tuples $\operatorname{split}(y_1 z, \dots, y_r z)$ and $\operatorname{split}(y_1, \dots, y_r)$ have the same length n. Let $(b_1, \dots, b_n) := \operatorname{split}(y_1 z, \dots, y_r z)$; we show $(b_1, \dots, b_n) = (a_1 + z^\dagger, \dots, a_n + z^\dagger)$ by induction on n. The case $n = 0$ is obvious, so suppose $n \geqslant 1$. Then $a_1 = y_1^\dagger$ and $b_1 = (y_1 z)^\dagger = a_1 + z^\dagger$ as required. By remarks following the proof of Lemma 2.4.4 we have

$$(a_2, \dots, a_n) \ = \ \operatorname{split}\big(A_1(y_2), \dots, A_1(y_n)\big) \qquad \text{where } A_1 := \partial - a_1.$$

Now $B_1 := \partial - b_1 = (A_1)_{\ltimes z^{-1}}$, so likewise

$$(b_2, \dots, b_n) \ = \ \operatorname{split}\big(B_1(y_2 z), \dots, B_1(y_n z)\big) \ = \ \operatorname{split}\big(A_1(y_2)z, \dots, A_1(y_n)z\big).$$

Hence $b_2 = a_2 + z^\dagger, \dots, b_n = a_n + z^\dagger$ by our inductive hypothesis. □

For $f \in \partial K$ we let $\int f$ denote an element of K such that $(\int f)' = f$.

Lemma 2.4.12. *Let $g_1, \dots, g_r \in K^\times$ and*

$$A \ = \ g_1 \cdots g_r (\partial g_r^{-1})(\partial g_{r-1}^{-1}) \cdots (\partial g_1^{-1}),$$

and suppose the integrals below can be chosen such that

$$y_1 \ = \ g_1, \quad y_2 \ = \ g_1 \textstyle\int g_2, \quad \dots, \quad y_r \ = \ g_1 \int \big(g_2 \int g_3 (\cdots (g_{r-1} \int g_r) \cdots)\big),$$

Then $y_1, \dots, y_r \in K^\times$, $n = r$, and $a_i = (g_1 \cdots g_i)^\dagger$ for $i = 1, \dots, r$.

Proof. Let $b_i := (g_1 \cdots g_i)^\dagger$ for $i = 1, \dots, r$. By induction on $i = 0, \dots, r$ we show $n \geqslant i$ and $(a_1, \dots, a_i) = (b_1, \dots, b_i)$. This is clear for $i = 0$, so suppose $i \in \{1, \dots, r\}$, $n \geqslant i - 1$, and $(a_1, \dots, a_{i-1}) = (b_1, \dots, b_{i-1})$. Then

$$A_{i-1} = (\partial - a_{i-1}) \cdots (\partial - a_1) = (\partial - b_{i-1}) \cdots (\partial - b_1) = g_1 \cdots g_{i-1} (\partial g_{i-1}^{-1}) \cdots (\partial g_1^{-1}),$$

using Lemma 1.1.3 for the last equality. So $A_{i-1}(y_i) = g_1 \cdots g_i \neq 0$, and thus $n \geqslant i$ and $a_i = A_{i-1}(y_i)^\dagger = b_i$. □

The case of real operators. We now continue the subsection *The real case* of Section 2.2. Thus $K = H[\mathrm{i}]$ where H is a real closed differential subfield of K and $\mathrm{i}^2 = -1$, and $\Lambda = \Lambda_{\mathrm{r}} + \Lambda_{\mathrm{i}}\mathrm{i}$ where Λ_{r}, Λ_{i} are subspaces of the $\mathbb{Q}$-linear space H. The complex conjugation automorphism $z \mapsto \overline{z}$ of the differential field K extends uniquely to an automorphism $B \mapsto \overline{B}$ of the ring $K[\partial]$ with $\overline{\partial} = \partial$. We have $\overline{A(f)} = \overline{A}(\overline{f})$ for $f \in \mathrm{U}$, from which it follows that $\dim_C \ker_K A = \dim_C \ker_K \overline{A}$, $(\overline{A})_\lambda = \overline{(A_{\overline{\lambda}})}$, $\operatorname{mult}_\lambda \overline{A} = \operatorname{mult}_{\overline{\lambda}} A$, and $f \mapsto \overline{f} \colon \mathrm{U} \to \mathrm{U}$ restricts to a C_H-linear bijection $\ker_{\mathrm{U}} A \to \ker_{\mathrm{U}} \overline{A}$.

In the rest of this subsection we assume $H = H^\dagger$ (so $\Lambda = \Lambda_{\mathrm{i}}\mathrm{i}$) and $A \in H[\partial]$ (and by earlier conventions, $A \neq 0$ and $r := \operatorname{order} A$). Then $A = \overline{A}$, hence for all λ we have $A_{\overline{\lambda}} = \overline{A_\lambda}$ and $\operatorname{mult}_\lambda A = \operatorname{mult}_{\overline{\lambda}} A$. Thus with μ ranging over $\Lambda_{\mathrm{i}}^{>}$:

$$\sum_\lambda \operatorname{mult}_\lambda(A) \;=\; \operatorname{mult}_0(A) + 2\sum_\mu \operatorname{mult}_{\mu\mathrm{i}}(A).$$

Note that 0 is an eigenvalue of A iff $\ker_H A \neq \{0\}$.

Let $V := \ker_{\mathrm{U}} A$, a subspace of the C-linear space U with $\overline{V} = V$ and $\dim_C V \leqslant r$. Recall that we have the differential H-subalgebra $\mathrm{U}_{\mathrm{r}} = \{f \in \mathrm{U} : \overline{f} = f\}$ of U and the C_H-linear subspace $V_{\mathrm{r}} = \ker_{\mathrm{U}_{\mathrm{r}}} A$ of U_{r}. Now $V = V_{\mathrm{r}} \oplus V_{\mathrm{r}}\mathrm{i}$ (internal direct sum of C_H-linear subspaces), so $\dim_C V = \dim_{C_H} V_{\mathrm{r}}$. Combining Lemma 2.4.1 and the remarks preceding it with Lemma 2.2.19 and its proof yields:

Corollary 2.4.13. *The C-linear space V has a basis*

$$a_1\,\mathrm{e}(\mu_1\mathrm{i}),\ \overline{a_1}\,\mathrm{e}(-\mu_1\mathrm{i}),\ \dots,\ a_m\,\mathrm{e}(\mu_m\mathrm{i}),\ \overline{a_m}\,\mathrm{e}(-\mu_m\mathrm{i}),\ h_1,\ \dots,\ h_n \qquad (2m+n \leqslant r),$$

where $a_1,\dots,a_m \in K^\times$, $\mu_1,\dots,\mu_m \in \Lambda_{\mathrm{i}}^{>}$, $h_1,\dots,h_n \in H^\times$. For such a basis,

$$\operatorname{Re}\big(a_1\,\mathrm{e}(\mu_1\mathrm{i})\big), \operatorname{Im}\big(a_1\,\mathrm{e}(\mu_1\mathrm{i})\big),\ \dots,\ \operatorname{Re}\big(a_m\,\mathrm{e}(\mu_m\mathrm{i})\big), \operatorname{Im}\big(a_m\,\mathrm{e}(\mu_m\mathrm{i})\big),\ h_1,\ \dots,\ h_n$$

is a basis of the C_H-linear space V_{r}, and $h_1,\dots,h_n$ is a basis of the C_H-linear subspace $\ker_H A = V \cap H$ of H.

Using $H = H^\dagger$, arguments as in the proof of Lemma 2.4.7 show:

Lemma 2.4.14. *Assume H is 1-linearly surjective when $r \geqslant 2$. Let $a_1,\dots,a_r \in H$ be such that $A = (\partial - a_r)\cdots(\partial - a_1)$. Then the C_H-linear space $\ker_H A$ has a basis $y_1,\dots,y_r$ such that $\operatorname{split}(y_1,\dots,y_r) = (a_1,\dots,a_r)$.*

Recall from Lemma 2.3.3 that if $r = 1$ or K is 1-linearly surjective, then

$$A \text{ splits over } K \quad\Longleftrightarrow\quad \sum_\lambda \operatorname{mult}_\lambda(A) = r.$$

Now $\operatorname{mult}_\lambda(A) = \operatorname{mult}_{\overline{\lambda}}(A)$ for all λ, so if $\operatorname{mult}_\lambda(A) = r \geqslant 1$, then $\lambda = 0$. Also, for $W := V \cap K = \ker_K A$ and $W_{\mathrm{r}} := W \cap \mathrm{U}_{\mathrm{r}}$ we have $W_{\mathrm{r}} = \ker_H A$ and

$$W \;=\; W_{\mathrm{r}} \oplus W_{\mathrm{r}}\mathrm{i} \qquad \text{(internal direct sum of } C_H\text{-linear subspaces)},$$

so $\operatorname{mult}_0(A) = \dim_C \ker_K A = \dim_{C_H} \ker_H A$. If $y_1,\dots,y_r$ is a basis of the C_H-linear space $\ker_H A$, then $\operatorname{split}(y_1,\dots,y_r) \in H^r$ in reversed order is a splitting of A over H by Corollary 2.4.5. These remarks and Lemma 2.4.14 now yield:

Corollary 2.4.15. *If $\operatorname{mult}_0(A) = r$, then A splits over H. The converse holds if H is 1-linearly surjective or $r = 1$.*

Corollary 2.4.16. *Suppose $r \geqslant 1$, and K is 1-linearly surjective if $r \geqslant 2$. Then*

$$A \text{ splits over } H \iff \operatorname{mult}_0(A) = r \iff |\Sigma(A)| = 1.$$

We now focus on the order 2 case:

Lemma 2.4.17. *Suppose $r = 2$ and A splits over K but not over H. Then*

$$\dim_C \ker_{\mathrm{U}} A \ = \ 2.$$

If H is 1-linearly surjective, then A has two distinct eigenvalues.

Proof. We can assume A is monic, so $A = (\partial - f)(\partial - g)$ with $f, g \in K$ and $g = a + b\mathrm{i}$, $a, b \in H$, $b \neq 0$. Then $g = d^\dagger + \mu\mathrm{i}$ with $d \in K^\times$ and $\mu \in \Lambda_{\mathrm{i}}$, and so $d\,\mathrm{e}(\mu\mathrm{i}) \in \ker_{\mathrm{U}} A$. From $A = \overline{A}$ we obtain $\overline{d}\,\mathrm{e}(-\mu\mathrm{i}) \in \ker_{\mathrm{U}} A$. These two elements of $\ker_{\mathrm{U}} A$ are C-linearly independent, since

$$d\,\mathrm{e}(\mu\mathrm{i})/\overline{d}\,\mathrm{e}(-\mu\mathrm{i}) \ = \ (d/\overline{d})\,\mathrm{e}(2\mu\mathrm{i}) \notin C :$$

this is clear if $\mu \neq 0$, and if $\mu = 0$, then $d^\dagger = g$, so $(d/\overline{d})^\dagger = g - \overline{g} = 2b\mathrm{i} \neq 0$, and hence $d/\overline{d} \notin C$. Thus $\dim_C \ker_{\mathrm{U}} A = 2$, and $\mu\mathrm{i}$, $-\mu\mathrm{i}$ are eigenvalues of A with respect to Λ. Now assume H is 1-linearly surjective. Then we claim that $\mu \neq 0$. To see this note that [ADH, 5.1.21, 5.2.10] and the assumption that A does not split over H yield $\dim_{C_H} \ker_H A = \dim_C \ker_K A = 0$, hence $g \notin K^\dagger$ and thus $\mu\mathrm{i} = g - d^\dagger \neq 0$. $\square$

Combining Lemmas 2.4.14 and 2.4.17 yields:

Corollary 2.4.18. *If H is 1-linearly surjective, A has order 2, and A splits over K, then* $\dim_C \ker_{\mathrm{U}} A \ = \ 2$.

In the rest of this subsection H is 1-linearly surjective and $A = 4\partial^2 + f$, $f \in H$. Let $\omega \colon H \to H$ and $\sigma \colon H^\times \to H$ be as in (0.3) and (0.4). Then by (0.5) and (0.6):

$$\begin{aligned} A \text{ splits over } H &\iff f \in \omega(H), \\ A \text{ splits over } K &\iff f \in \sigma(H^\times) \cup \omega(H). \end{aligned}$$

If A splits over H, then $\Sigma(A) = \{0\}$ and $\operatorname{mult}_0(A) = 2$, by Corollary 2.4.16. Suppose A splits over K but not over H, and let $y \in H^\times$ satisfy $\sigma(y) = f \notin \omega(H)$. Then by [ADH, p. 262] we have $A = 4(\partial + g)(\partial - g)$ where $g = \frac{1}{2}(-y^\dagger + y\mathrm{i})$. Hence the two distinct eigenvalues of A are $(y/2)\mathrm{i} + K^\dagger$ and $-(y/2)\mathrm{i} + K^\dagger$.

The case of oscillating transseries. We now apply the results above to the algebraically closed differential field $K = \mathbb{T}[\mathrm{i}]$. Note that $\mathbb{T}[\mathrm{i}]$ has constant field $\mathbb{C}$ and extends the (real closed) differential field $\mathbb{T}$ of transseries. After (0.10) in the introduction, we already remarked:

Lemma 2.4.19. $\mathbb{T}[\mathrm{i}]$ *is linearly closed and linearly surjective.*

Now applying Corollary 2.4.8 and Lemma 2.4.1 to $\mathbb{T}[\mathrm{i}]$ gives:

Corollary 2.4.20. *For $K = \mathbb{T}[\mathrm{i}]$, there are $\mathbb{C}$-linearly independent units $y_1, \dots, y_r$ of $\mathrm{U}_{\mathbb{T}[\mathrm{i}]}$ with $A(y_1) = \cdots = A(y_r) = 0$.*

Next we describe another incarnation of $\mathrm{U}_{\mathbb{T}[\mathrm{i}]}$, namely as a ring $\mathbb{O}$ of "oscillating" transseries. Towards this goal we first note that by [ADH, 11.5.1, 11.8.2] we have

$$\begin{aligned} \mathrm{I}(\mathbb{T}) \ &= \ \big\{ y \in \mathbb{T} : \ y \preccurlyeq f' \text{ for some } f \prec 1 \text{ in } \mathbb{T} \big\} \\ &= \ \big\{ y \in \mathbb{T} : \ y \prec 1/(\ell_0 \cdots \ell_n) \text{ for all } n \big\}, \end{aligned}$$

so a complement $\Lambda_{\mathbb{T}}$ of $\mathrm{I}(\mathbb{T})$ in $\mathbb{T}$ is given by

$$\Lambda_{\mathbb{T}} := \big\{y \in \mathbb{T} : \operatorname{supp}(y) \succ 1/(\ell_0 \cdots \ell_{n-1}\ell_n^2) \text{ for all } n\big\}.$$

Since $\mathbb{T}^\dagger = \mathbb{T}$ and $\mathrm{I}\big(\mathbb{T}[i]\big) \subseteq \mathbb{T}[i]^\dagger$ we have $\mathbb{T}[i]^\dagger = \mathbb{T} \oplus \mathrm{I}(\mathbb{T})i$ by Lemmas 1.2.4 and 1.2.13. We now take $\Lambda = \Lambda_{\mathbb{T}} i$ as our complement Λ of $\mathbb{T}[i]^\dagger$ in $\mathbb{T}[i]$ and explain how the universal exponential extension U of $\mathbb{T}[i]$ for this Λ was introduced in [15, Section 7.7] in a different way. Let

$$\mathbb{T}_{\succ} := \{f \in \mathbb{T} : \operatorname{supp} f \succ 1\},$$

and similarly with $\prec$ in place of $\succ$; then $\mathbb{T}_{\prec} = \mathcal{o}_{\mathbb{T}}$ and $\mathbb{T}_{\succ}$ are $\mathbb{R}$-linear subspaces of $\mathbb{T}$, and $\mathbb{T}$ decomposes as an internal direct sum

$$\mathbb{T} = \mathbb{T}_{\succ} \oplus \mathbb{R} \oplus \mathbb{T}_{\prec} \tag{2.4.2}$$

of $\mathbb{R}$-linear subspaces of $\mathbb{T}$. Let $\mathrm{e}^{\mathrm{i}\mathbb{T}_{\succ}} = \{\mathrm{e}^{\mathrm{i}f} : f \in \mathbb{T}_{\succ}\}$ be a multiplicative copy of the additive group $\mathbb{T}_{\succ}$, with isomorphism $f \mapsto \mathrm{e}^{\mathrm{i}f}$. Then we have the group ring

$$\mathbb{O} := K\big[\mathrm{e}^{\mathrm{i}\mathbb{T}_{\succ}}\big]$$

of $\mathrm{e}^{\mathrm{i}\mathbb{T}_{\succ}}$ over $K = \mathbb{T}[i]$. We make $\mathbb{O}$ into a differential ring extension of K by

$$(\mathrm{e}^{\mathrm{i}f})' = \mathrm{i}f'\,\mathrm{e}^{\mathrm{i}f} \qquad (f \in \mathbb{T}_{\succ}).$$

Hence $\mathbb{O}$ is an exponential extension of K. The elements of $\mathbb{O}$ are called *oscillating transseries.* For each $f \in \mathbb{T}$ there is a unique $g \in \mathbb{T}$, to be denoted by $\int f$, such that $g' = f$ and g has constant term $g_1 = 0$. The injective map $\int : \mathbb{T} \to \mathbb{T}$ is $\mathbb{R}$-linear; we use this map to show that U and $\mathbb{O}$ are disguised versions of each other:

Proposition 2.4.21. *There is a unique isomorphism* $\mathrm{U} = K\big[\mathrm{e}(\Lambda)\big] \to \mathbb{O}$ *of differential K-algebras sending* $\mathrm{e}(hi)$ *to* $\mathrm{e}^{\mathrm{i}\int h}$ *for all* $h \in \Lambda_{\mathbb{T}}$.

This requires the next lemma. We assume familiarity with [ADH, Appendix A], especially with the ordered group G^{LE} (a subgroup of $\mathbb{T}^\times$) of logarithmic-exponential monomials and its subgroup $G^{\mathrm{E}} = \bigcup_n G_n$ of exponential monomials.

Lemma 2.4.22. *If* $\mathfrak{m} \in G^{\mathrm{LE}}$ *and* $\mathfrak{m} \succ 1$, *then* $\operatorname{supp} \mathfrak{m}' \subseteq \Lambda_{\mathbb{T}}$.

Proof. We first prove by induction on n a fact about elements of G^{E}:

$$\text{if } \mathfrak{m} \in G_n,\ \mathfrak{m} \succ 1, \text{ then } \operatorname{supp} \mathfrak{m}' \succ 1/x.$$

For $r \in \mathbb{R}^>$ we have $(x^r)' = rx^{r-1} \succ 1/x$, so the claim holds for $n = 0$. Suppose the claim holds for a certain n. Now $G_{n+1} = G_n \exp(A_n)$, G_n is a convex subgroup of G_{n+1}, and

$$A_n = \big\{f \in \mathbb{R}[[G_n]] : \operatorname{supp} f \succ G_{n-1}\big\} \qquad (\text{where } G_{-1} := \{1\}).$$

Let $\mathfrak{m} = \mathfrak{n}\exp(a) \in G_{n+1}$ where $\mathfrak{n} \in G_n$, $a \in A_n$; then

$$\mathfrak{m} \succ 1 \iff a > 0, \text{ or } a = 0,\ \mathfrak{n} \succ 1.$$

Suppose $\mathfrak{m} \succ 1$. If $a = 0$, then $\mathfrak{m} = \mathfrak{n}$, and we are done by inductive hypothesis, so assume $a > 0$. Then $\mathfrak{m}' = (\mathfrak{n}' + \mathfrak{n}a')\exp(a)$ and $(\mathfrak{n}' + \mathfrak{n}a') \in \mathbb{R}[[G_n]]$, a differential subfield of $\mathbb{T}$, and $\exp(a) > \mathbb{R}[[G_n]]$, hence $\operatorname{supp} \mathfrak{m}' \succ 1 \succ 1/x$ as required.

Next, suppose $\mathfrak{m} \in G^{\mathrm{LE}}$ and $\mathfrak{m} \succ 1$. Take $n \geqslant 1$ such that $\mathfrak{m}{\uparrow}^n \in G^{\mathrm{E}}$. We have $(\mathfrak{m}{\uparrow}^n)' = (\mathfrak{m}' \cdot \ell_0\ell_1 \cdots \ell_{n-1}){\uparrow}^n$. For $\mathfrak{n} \in \operatorname{supp} \mathfrak{m}'$ and using $\mathfrak{m}{\uparrow}^n \succ 1$ this gives

$$(\mathfrak{n} \cdot \ell_0\ell_1 \cdots \ell_{n-1}){\uparrow}^n \succ 1/x$$

by what we proved for monomials in G^{E}. Applying $\downarrow_n$ this yields $\mathfrak{n} \succ 1/(\ell_0\ell_1\cdots\ell_n)$, hence $\mathfrak{n} \in \Lambda_{\mathbb{T}}$ as claimed. $\square$

Proof of Proposition 2.4.21. Applying ∂ to the decomposition (2.4.2) gives

$$\mathbb{T} \;=\; \partial(\mathbb{T}_{\succ}) \oplus \partial(\mathbb{T}_{\prec}).$$

Now $\partial(\mathbb{T}_{\succ}) \subseteq \Lambda_{\mathbb{T}}$ by Lemma 2.4.22, and $\partial(\mathbb{T}_{\prec}) \subseteq \mathrm{I}(\mathbb{T})$, and so these two inclusions are equalities. Thus $\int \Lambda_{\mathbb{T}} = \mathbb{T}_{\succ}$, from which the proposition follows. $\square$

Proposition 2.4.23. *There is a unique group morphism* $\exp\colon K = \mathbb{T}[i] \to \mathbb{O}^{\times}$ *that extends the given exponential maps* $\exp\colon \mathbb{T} \to \mathbb{T}^{\times}$ *and* $\exp\colon \mathbb{C} \to \mathbb{C}^{\times}$*, and such that* $\exp(if) = \mathrm{e}^{if}$ *for all* $f \in \mathbb{T}_{\succ}$ *and* $\exp(\varepsilon) = \sum_n \frac{\varepsilon^n}{n!}$ *for all* $\varepsilon \in \mathcal{O}$*. It is surjective, has kernel* $2\pi i\mathbb{Z} \subseteq \mathbb{C}$*, and satisfies* $\exp(f)' = f'\exp(f)$ *for all* $f \in K$.

Proof. The first statement follows easily from the decompositions

$$K \;=\; \mathbb{T} \oplus i\mathbb{T} \;=\; \mathbb{T} \oplus i\mathbb{T}_{\succ} \oplus i\mathbb{R} \oplus i\mathcal{O}_{\mathbb{T}}, \qquad \mathbb{C} \;=\; \mathbb{R} \oplus i\mathbb{R}, \qquad \mathcal{O} \;=\; \mathcal{O}_{\mathbb{T}} \oplus i\mathcal{O}_{\mathbb{T}}$$

of K, $\mathbb{C}$, and $\mathcal{O} = \mathcal{O}_K$ as internal direct sums of $\mathbb{R}$-linear subspaces. Next,

$$\mathbb{O}^{\times} \;=\; K^{\times}\,\mathrm{e}^{i\mathbb{T}_{\succ}} \;=\; \mathbb{T}^{>}\cdot S_{\mathbb{C}}\cdot(1+\mathcal{O})\cdot \mathrm{e}^{i\mathbb{T}_{\succ}}, \qquad S_{\mathbb{C}} \;:=\; \big\{z \in \mathbb{C} :\ |z| = 1\big\},$$

by Lemmas 2.1.1 and 1.2.4, and Corollary 1.2.7. Now $\mathbb{T}^{>} = \exp(\mathbb{T})$ and $S_{\mathbb{C}} = \exp(i\mathbb{R})$, so surjectivity follows from $\exp(\mathcal{O}) = 1 + \mathcal{O}$, a consequence of the well-known bijectivity of the map $\varepsilon \mapsto \sum_n \frac{\varepsilon^n}{n!}\colon \mathcal{O} \to 1+\mathcal{O}$, whose inverse is given by

$$1+\delta \mapsto \log(1+\delta) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\delta^n \qquad (\delta \in \mathcal{O}).$$

That the kernel is $2\pi i\mathbb{Z}$ follows from the initial decomposition of the additive group of K as $\mathbb{T} \oplus i\mathbb{T}_{\succ} \oplus i\mathbb{R} \oplus i\mathcal{O}_{\mathbb{T}}$. The identity $\exp(f)' = f'\exp(f)$ for $f \in K$ follows from it being satisfied for $f \in \mathbb{T}$, $f \in i\mathbb{T}_{\succ}$, $f \in \mathbb{C}$, and $f \in \mathcal{O}$. $\square$

To integrate oscillating transseries, note first that the $\mathbb{R}$-linear operator $\int\colon \mathbb{T} \to \mathbb{T}$ extends uniquely to a $\mathbb{C}$-linear operator $\int\colon \mathbb{T}[i] \to \mathbb{T}[i]$. This in turn extends uniquely to a $\mathbb{C}$-linear operator $\int\colon \mathbb{O} \to \mathbb{O}$ such that $(\int \Phi)' = \Phi$ for all $\Phi \in \mathbb{O}$ and $\int \mathbb{T}[i]\,\mathrm{e}^{\phi i} \subseteq \mathbb{T}[i]\,\mathrm{e}^{\phi i}$ for all $\phi \in \mathbb{T}_{\succ}$: given $\phi \in \mathbb{T}_{\succ}^{\neq}$ and $g \in \mathbb{T}[i]$, there is a unique $f \in \mathbb{T}[i[$ such that $(f\,\mathrm{e}^{\phi i})' = g\,\mathrm{e}^{\phi i}$: existence holds because $y' + y\phi' i = g$ has a solution in $\mathbb{T}[i]$, the latter being linearly surjective, and uniqueness holds by Lemma 1.2.3 applied to $K = L = \mathbb{T}[i]$, because $\phi' i \notin \mathbb{T}[i]^{\dagger}$ in view of remarks preceding Lemma 1.2.13.

The operator $\int$ is a right-inverse of the linear differential operator ∂ on $\mathbb{O}$. To extend this to other linear differential operators, make the subgroup $G^{\mathbb{O}} := G^{\mathrm{LE}}\,\mathrm{e}^{i\mathbb{T}_{\succ}}$ of $\mathbb{O}^{\times}$ into an ordered group so that the ordered subgroup G^{LE} of $\mathbb{T}^{>}$ is a convex ordered subgroup of $G^{\mathbb{O}}$ and $\mathrm{e}^{i\phi} \succ G^{\mathrm{LE}}$ for $\phi > 0$ in $\mathbb{T}_{\succ}$. (Possible in only one way.) Next, extend the natural inclusion $\mathbb{T}[i] \to \mathbb{C}[[G^{\mathrm{LE}}]]$ to a $\mathbb{C}$-algebra embedding $\mathbb{O} \to \mathbb{C}[[G^{\mathbb{O}}]]$ by sending $\mathrm{e}^{i\phi} \in \mathbb{O}$ to $\mathrm{e}^{i\phi} \in G^{\mathbb{O}} \subseteq \mathbb{C}[[G^{\mathbb{O}}]]$. Identify $\mathbb{O}$ with a subalgebra of $\mathbb{C}[[G^{\mathbb{O}}]]$ via this embedding, so $\operatorname{supp} f \subseteq G^{\mathbb{O}}$ for $f \in \mathbb{O}$. It makes the Hahn space $\mathbb{C}[[G^{\mathbb{O}}]]$ over $\mathbb{C}$ an immediate extension of its valued subspace $\mathbb{O}$. The latter is in particular also a Hahn space over $\mathbb{C}$.

Let $A \in \mathbb{T}[i][\partial]^{\neq}$. Then $A(\mathbb{O}) = \mathbb{O}$ by Lemmas 2.4.3, 2.4.19, and Proposition 2.4.21. The proof of [ADH, 2.3.22] now gives for each $g \in \mathbb{O}$ a unique element $f =: A^{-1}(g) \in \mathbb{O}$ with $A(f) = g$ and $\operatorname{supp}(f) \cap \mathfrak{d}\big(\ker^{\neq}_{\mathbb{O}} A\big) = \emptyset$. This requirement on $\operatorname{supp} A^{-1}(g)$ yields a $\mathbb{C}$-linear operator A^{-1} on $\mathbb{O}$ with $A \circ A^{-1} = \mathrm{id}_{\mathbb{O}}$; we call it the **distinguished** right-inverse of the operator A on $\mathbb{O}$. With this definition ∂^{-1} is the operator $\int$ on $\mathbb{O}$ specified earlier.

In the next section we explore various valuations on universal exponential extensions (such as $\mathbb{O}$) with additional properties.

2.5. Valuations on the Universal Exponential Extension

In this section K is a valued differential field with algebraically closed constant field $C \subseteq \mathcal{O}$ and divisible group $K^{\dagger}$ of logarithmic derivatives. Then $\Gamma = v(K^{\times})$ is also divisible, since we have a group isomorphism

$$va \mapsto a^{\dagger} + (\mathcal{O}^{\times})^{\dagger} \ :\ \Gamma \to K^{\dagger}/(\mathcal{O}^{\times})^{\dagger} \qquad (a \in K^{\times}).$$

Let Λ be a complement of the $\mathbb{Q}$-linear subspace $K^{\dagger}$ of K, let λ range over Λ, let $\mathrm{U} = K\big[\mathrm{e}(\Lambda)\big]$ be the universal exponential extension of K constructed in Section 2.2 and set $\Omega := \operatorname{Frac}(\mathrm{U})$. Thus Ω is a differential field with constant field C.

The gaussian extension. We equip U with the gaussian extension v_{g} of the valuation of K as defined in Section 2.1; so for $f \in \mathrm{U}$ with spectral decomposition (f_λ):

$$v_{\mathrm{g}}(f) \ =\ \min_{\lambda} v(f_\lambda),$$

and hence

$$v_{\mathrm{g}}(f') \ =\ \min_{\lambda} v(f_\lambda' + \lambda f_\lambda).$$

The field Ω with the valuation extending v_{g} is a valued differential field extension of K, but it can happen that K has small derivation, whereas Ω does not:

Example. Let $K = C(\!(t^{\mathbb{Q}})\!)$ and Λ be as in Example 2.2.4, so $t \prec 1 \prec x = t^{-1}$ and $t' = -t^2$. Then K is d-valued of H-type with small derivation, but in Ω with the above valuation,

$$t\,\mathrm{e}(x) \ \prec\ 1, \qquad \big(t\,\mathrm{e}(x)\big)' \ =\ -t^2\,\mathrm{e}(x) + \mathrm{e}(x) \ \sim\ \mathrm{e}(x) \ \asymp\ 1.$$

To obtain an example where $K = H[i]$ for a Liouville closed H-field H and $i^2 = -1$, take $K := \mathbb{T}[i]$ and $\Lambda := \Lambda_{\mathbb{T}} i$ as at the end of Section 2.4. Now $x \in \Lambda_{\mathbb{T}}$ and in Ω equipped with the above valuation we have for $t := x^{-1}$:

$$t\,\mathrm{e}(xi) \ \prec\ 1, \qquad \big(t\,\mathrm{e}(xi)\big)' \ =\ -t^2\,\mathrm{e}(xi) + i\,\mathrm{e}(xi) \ \sim\ i\,\mathrm{e}(xi) \ \asymp\ 1,$$

so $\big(t\,\mathrm{e}(xi)\big)' \not\prec t^{\dagger}$, hence Ω is neither asymptotic nor has small derivation.

However, we show next that under certain assumptions on K with small derivation, Ω has also a valuation which does make Ω a valued differential field extension of K with small derivation. For this we rely on results from [ADH, 10.4]. Although such a valuation is less canonical than v_{g}, it is useful for harnessing the finiteness statements about the set $\mathscr{E}^{\mathrm{e}}(A)$ of eventual exceptional values of $A \in K[\partial]^{\neq}$ from Section 1.4 to obtain similar facts about the set of *ultimate exceptional values* of A introduced later in this section.

Spectral extensions. *In this subsection K is* d*-valued of H-type with $\Gamma \neq \{0\}$ and with small derivation.*

Lemma 2.5.1. *The valuation of K extends to a valuation on the field Ω that makes Ω a* d*-valued extension of K of H-type with small derivation.*

Proof. Applying [ADH, 10.4.7] to an algebraic closure of K gives a d-valued algebraically closed extension L of K of H-type with small derivation and $C_L = C$ such that $L^\dagger \supseteq K$. Let $E := \{y \in L^\times : y^\dagger \in K\}$, so E is a subgroup of $L^\times$, $E^\dagger = K$, and $K[E]$ is an exponential extension of K with $C_{K[E]} = C$. Then Corollary 2.2.10 gives an embedding $\mathrm{U} \to L$ of differential K-algebras with image $K[E]$, which extends to an embedding $\Omega \to L$ of differential fields. Using this embedding to transfer the valuation of L to Ω gives a valuation as required. $\square$

A **spectral extension** of the valuation of K to Ω is a valuation on the field Ω with the properties stated in Lemma 2.5.1. If K is $\boldsymbol{\omega}$-free, then so is Ω equipped with any spectral extension of the valuation of K, by [ADH, 13.6] (and then Ω has rational asymptotic integration by [ADH, 11.7]). We do not know whether this goes through with "$\boldsymbol{\lambda}$-free" instead of "$\boldsymbol{\omega}$-free". Here is something weaker:

Lemma 2.5.2. *Suppose K is algebraically closed and $\boldsymbol{\lambda}$-free. Then some spectral extension of the valuation of K to Ω makes Ω a* d*-valued field with divisible value group and asymptotic integration.*

Proof. Take L, E and an embedding $\Omega \to L$ as in the proof of Lemma 2.5.1. Use this embedding to identify Ω with a differential subfield of L, so $U = K[E]$ and $\Omega = K(E)$, and equip Ω with the spectral extension of the valuation of K obtained by restricting the valuation of L to Ω. Since L is algebraically closed, E is divisible, and $\Gamma_L = \Gamma + v(E)$ by [ADH, 10.4.7(iv)]. So $\Gamma_\Omega = \Gamma_L$ is divisible. Let $a \in K^\times$, $y \in E$. Then $K(y)$ has asymptotic integration by Proposition 1.3.12, hence $v(ay) \in (\Gamma^{\neq}_{K(y)})' \subseteq (\Gamma^{\neq}_{\Omega})'$. Thus Ω has asymptotic integration. $\square$

In the rest of this subsection Ω is equipped with a spectral extension v (with value group Γ_Ω) of the valuation of K. The proof of Lemma 2.5.1 and [ADH, 10.4.7] show that we can choose v so that $\Psi_\Omega \subseteq \Gamma$; but under suitable hypotheses on K, this is automatic:

Lemma 2.5.3. *Suppose K has asymptotic integration and $\mathrm{I}(K) \subseteq K^\dagger$. Then $\Psi_\Omega \subseteq \Gamma$, the group morphism*

$$\lambda \mapsto v\big(\mathrm{e}(\lambda)\big) \ : \ \Lambda \to \Gamma_\Omega \tag{2.5.1}$$

is injective, and Γ_Ω is divisible with $\Gamma_\Omega = \Gamma \oplus v\big(\mathrm{e}(\Lambda)\big)$ (internal direct sum of $\mathbb{Q}$-linear subspaces of Γ_Ω). Moreover, $\Psi_\Omega = \Psi^\downarrow$ in Γ.

Proof. For $a \in K^\times$ we have $\big(a\,\mathrm{e}(\lambda)\big)^\dagger = a^\dagger + \lambda \in K$, and if $a\,\mathrm{e}(\lambda) \asymp 1$, then

$$a^\dagger + \lambda \ = \ \big(a\,\mathrm{e}(\lambda)\big)^\dagger \in (\mathcal{O}_\Omega^\times)^\dagger \cap K \ \subseteq \ \mathrm{I}(\Omega) \cap K \ = \ \mathrm{I}(K),$$

so $\lambda \in \Lambda \cap \big(\mathrm{I}(K) + K^\dagger\big) = \Lambda \cap K^\dagger = \{0\}$ and $a \asymp 1$. Thus for $a_1, a_2 \in K^\times$ and distinct $\lambda_1, \lambda_2 \in \Lambda$ we have $a_1\,\mathrm{e}(\lambda_1) \not\asymp a_2\,\mathrm{e}(\lambda_2)$, and so for $f \in \mathrm{U}$ with spectral decomposition (f_λ) we have $vf = \min_\lambda v\big(f_\lambda\,\mathrm{e}(\lambda)\big)$. Hence

$$\Psi_\Omega \ \subseteq \big\{v(a^\dagger + \lambda) : \ a \in K^\times, \ \lambda \in \Lambda\big\} \ = \ v(K) \ = \ \Gamma_\infty,$$

the map (2.5.1) is injective and $\Gamma\cap v\big(\mathrm{e}(\Lambda)\big)=\{0\}$, and so $\Gamma_\Omega=\Gamma\oplus v\big(\mathrm{e}(\Lambda)\big)$ (internal direct sum of subgroups of Γ_Ω). Since Γ and Λ are divisible, so is Γ_Ω. Now $\Psi_\Omega=\Psi^{\downarrow}$ follows from $K=(\mathrm{U}^\times)^\dagger\subseteq\Omega^\dagger$ and K having asymptotic integration. □

We can now improve on Lemma 2.4.1:

Corollary 2.5.4. *Suppose K has asymptotic integration and $\mathrm{I}(K)\subseteq K^\dagger$, and let $A\in K[\partial]^{\neq}$. Then the C-linear space $\ker_{\mathrm{U}}A$ has a basis $\mathcal{B}\subseteq\mathrm{U}^\times$ such that v is injective on $\mathcal{B}$ and $v(\mathcal{B})=v(\ker^{\neq}_{\mathrm{U}}A)$, and thus $|v(\ker^{\neq}_{\mathrm{U}}A)|=\dim_C\ker_{\mathrm{U}}A$.*

Proof. By [ADH, 5.6.6] we have a basis $\mathcal{B}_\lambda$ of the C-linear space $\ker_K A_\lambda$ such that v is injective on $\mathcal{B}_\lambda$ and $v(\mathcal{B}_\lambda)=v(\ker^{\neq}_K A_\lambda)$. Then $\mathcal{B}:=\bigcup_\lambda\mathcal{B}_\lambda\,\mathrm{e}(\lambda)$ is a basis of $\ker_{\mathrm{U}}A$. It has the desired properties by Lemma 2.5.3. □

Corollary 2.5.5. *Suppose K is $\uplambda$-free and $\mathrm{I}(K)\subseteq K^\dagger$. Then Ω has asymptotic integration, and so its H-asymptotic couple is closed by Lemma 2.5.3.*

Proof. By Lemma 2.5.3, $\Gamma_\Omega=\Gamma+v\big(\mathrm{e}(\Lambda)\big)$. Using Proposition 1.3.12 as in the proof of Lemma 2.5.2, with $\mathrm{e}(\Lambda)$ in place of E, shows Ω has asymptotic integration. □

Ultimate exceptional values. *In this subsection K is H-asymptotic with small derivation and asymptotic integration.* Also $A\in K[\partial]^{\neq}$ and $r:=\operatorname{order}(A)$, and γ ranges over $\Gamma=v(K^\times)$. We have $v(\ker^{\neq}A_\lambda)\subseteq\mathscr{E}^{\mathrm{e}}(A_\lambda)$, so if λ is an eigenvalue of A with respect to λ, then $\mathscr{E}^{\mathrm{e}}(A_\lambda)\neq\emptyset$. We call the elements of the set

$$\mathscr{E}^{\mathrm{u}}(A)\ =\ \mathscr{E}^{\mathrm{u}}_K(A)\ :=\ \bigcup_\lambda\mathscr{E}^{\mathrm{e}}(A_\lambda)\ =\ \big\{\gamma:\ \operatorname{nwt}_{A_\lambda}(\gamma)\geqslant 1\text{ for some }\lambda\big\}$$

the **ultimate exceptional values of** A with respect to Λ. The definition of $\mathscr{E}^{\mathrm{u}}_K(A)$ involves our choice of Λ, but we are leaving this implicit to avoid complicated notation. In Section 4.4 we shall restrict K and Λ so that $\mathscr{E}^{\mathrm{u}}(A)$ does not depend any longer on the choice of Λ. There we shall use the following observation:

Lemma 2.5.6. *Let $a,b\in K$ be such that $a-b\in(\mathcal{O}^\times)^\dagger$. Then for all γ we have $\operatorname{nwt}_{A_a}(\gamma)=\operatorname{nwt}_{A_b}(\gamma)$; in particular, $\mathscr{E}^{\mathrm{e}}(A_a)=\mathscr{E}^{\mathrm{e}}(A_b)$.*

Proof. Use that if $u\in\mathcal{O}^\times$ and $a-b=u^\dagger$, then $A_a=(A_b)_{\ltimes u}$. □

Corollary 2.5.7. *Let Λ^* be a complement of the $\mathbb{Q}$-linear subspace $K^\dagger$ of K and let $\lambda\mapsto\lambda^*\colon\Lambda\to\Lambda^*$ be the group isomorphism with $\lambda-\lambda^*\in K^\dagger$ for all λ. If $\lambda-\lambda^*\in(\mathcal{O}^\times)^\dagger$ for all λ, then $\operatorname{nwt}_{A_\lambda}(\gamma)=\operatorname{nwt}_{A_{\lambda^*}}(\gamma)$ for all γ, so $\mathscr{E}^{\mathrm{u}}(A)=\bigcup_\lambda\mathscr{E}^{\mathrm{e}}(A_{\lambda^*})$.*

Remark 2.5.8. For $a\in K^\times$ we have $\mathscr{E}^{\mathrm{u}}(aA)=\mathscr{E}^{\mathrm{u}}(A)$ and $\mathscr{E}^{\mathrm{u}}(Aa)=\mathscr{E}^{\mathrm{u}}(A)-va$. Note also that $\mathscr{E}^{\mathrm{e}}(A)=\mathscr{E}^{\mathrm{e}}(A_0)\subseteq\mathscr{E}^{\mathrm{u}}(A)$. Let $\phi\in K^\times$ be active in K, and set $\lambda^\phi:=\phi^{-1}\lambda$. Then $\Lambda^\phi:=\phi^{-1}\Lambda$ is a complement of the $\mathbb{Q}$-linear subspace $(K^\phi)^\dagger=\phi^{-1}K^\dagger$ of K^ϕ, and $(A^\phi)_{\lambda^\phi}=(A_\lambda)^\phi$. Hence $\mathscr{E}^{\mathrm{u}}_K(A)$ agrees with the set $\mathscr{E}^{\mathrm{u}}_{K^\phi}(A^\phi)$ of ultimate exceptional values of A^ϕ with respect to Λ^ϕ.

Remark 2.5.9. Suppose L is an H-asymptotic extension of K with asymptotic integration and algebraically closed constant field C_L such that $L^\dagger$ is divisible, and Ψ is cofinal in Ψ_L or K is $\uplambda$-free. Then $\mathscr{E}^{\mathrm{e}}(A_\lambda)=\mathscr{E}^{\mathrm{e}}_L(A_\lambda)\cap\Gamma$, by Lemma 1.4.1 and Corollary 1.7.10. Hence if $\Lambda_L\supseteq\Lambda$ is a complement of the subspace $L^\dagger$ of the $\mathbb{Q}$-linear space L, and $\mathscr{E}^{\mathrm{u}}_L(A)$ is the set of ultimate exceptional values of A (viewed as an element of $L[\partial]$) with respect to Λ_L, then $\mathscr{E}^{\mathrm{u}}(A)\subseteq\mathscr{E}^{\mathrm{u}}_L(A)$. (Note that such a complement Λ_L exists iff $L^\dagger\cap K=K^\dagger$.)

In the rest of this subsection we equip U with the gaussian extension v_{g} of the valuation of K. Recall that we have a decomposition $\ker_{\mathrm{U}} A = \bigoplus_\lambda (\ker A_\lambda)\,\mathrm{e}(\lambda)$ of the C-linear space $\ker_{\mathrm{U}} A$ as an internal direct sum of subspaces, and hence

$$(2.5.2) \qquad v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) \;=\; \bigcup_\lambda v(\ker^{\neq} A_\lambda) \;\subseteq\; \bigcup_\lambda \mathscr{E}^{\mathrm{e}}(A_\lambda) \;=\; \mathscr{E}^{\mathrm{u}}(A).$$

Here are some consequences:

Lemma 2.5.10. *Suppose K is r-linearly newtonian. Then $v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) = \mathscr{E}^{\mathrm{u}}(A)$.*

Proof. By Proposition 1.4.2 we have $v(\ker^{\neq} A_\lambda) = \mathscr{E}^{\mathrm{e}}(A_\lambda)$ for each λ. Therefore $v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) = \mathscr{E}^{\mathrm{u}}(A)$ by (2.5.2). $\square$

Lemma 2.5.11. *Suppose K is* d*-valued. Then $|v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A)| \leqslant \dim_C \ker_{\mathrm{U}} A \leqslant r$.*

Proof. By [ADH, 5.6.6(i)] applied to A_λ in place of A we have

$$|v(\ker^{\neq} A_\lambda)| \;=\; \dim_C \ker A_\lambda \;=\; \mathrm{mult}_\lambda(A) \qquad \text{for all } \lambda$$

and thus by (2.5.2),

$$|v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A)| \;\leqslant\; \sum_\lambda |v(\ker^{\neq} A_\lambda)| \;=\; \sum_\lambda \mathrm{mult}_\lambda(A) \;=\; \dim_C \ker_{\mathrm{U}} A \;\leqslant\; r$$

as claimed. $\square$

Lemma 2.5.12. *Suppose $\mathrm{I}(K) \subseteq K^\dagger$ and $r = 1$. Then*

$$v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) \;=\; \mathscr{E}^{\mathrm{u}}(A), \qquad |\mathscr{E}^{\mathrm{u}}(A)| \;=\; 1.$$

Proof. Arrange $A = \partial - g$, $g \in K$, and take $f \in K^\times$ and λ such that $g = f^\dagger + \lambda$. Then $u := f\,\mathrm{e}(\lambda) \in \mathrm{U}^\times$ satisfies $A(u) = 0$, hence $\ker^{\neq}_{\mathrm{U}} A = Cu$ and thus $v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) = \{vf\}$. By Lemma 1.4.9 we have $v(\ker^{\neq} A_\lambda) = \mathscr{E}^{\mathrm{e}}(A_\lambda)$ for all λ and hence $v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) = \mathscr{E}^{\mathrm{u}}(A)$ by (2.5.2). $\square$

Corollary 2.5.13. *If $\mathrm{I}(K) \subseteq K^\dagger$ and $a \in K^\times$, then $\mathscr{E}^{\mathrm{e}}(\partial - a^\dagger) = \mathscr{E}^{\mathrm{u}}(\partial - a^\dagger) = \{va\}$.*

Proposition 2.5.15 below partly extends Lemma 2.5.12.

Spectral extensions and ultimate exceptional values. *In this subsection K is* d*-valued of H-type with small derivation, asymptotic integration, and $\mathrm{I}(K) \subseteq K^\dagger$.* Also $A \in K[\partial]^{\neq}$ has order r and γ ranges over Γ.

Suppose Ω is equipped with a spectral extension v of the valuation of K. Let $g \in K^\times$ with $vg = \gamma$. The Newton weight of $A_\lambda g \in K[\partial]$ does not change in passing from K to Ω, since Ψ is cofinal in Ψ_Ω by Lemma 2.5.3; see [ADH, 11.1]. Thus

$$\mathrm{nwt}_{A_\lambda}(\gamma) \;=\; \mathrm{nwt}(A_\lambda g) \;=\; \mathrm{nwt}\big(Ag\,\mathrm{e}(\lambda)\big) \;=\; \mathrm{nwt}_A\big(v(g\,\mathrm{e}(\lambda)\big) \;=\; \mathrm{nwt}_A\big(\gamma + v(\mathrm{e}(\lambda))\big).$$

In particular, using $\Gamma_\Omega = \Gamma \oplus v\big(\mathrm{e}(\Lambda)\big)$,

$$(2.5.3) \qquad \mathscr{E}^{\mathrm{e}}_\Omega(A) \;=\; \bigcup_\lambda \mathscr{E}^{\mathrm{e}}(A_\lambda) + v\big(\mathrm{e}(\lambda)\big) \qquad \text{(a disjoint union).}$$

Thus $\mathscr{E}^{\mathrm{u}}(A) = \pi\big(\mathscr{E}^{\mathrm{e}}_\Omega(A)\big)$ where $\pi\colon \Gamma_\Omega \to \Gamma$ is given by $\pi\big(\gamma + v\big(\mathrm{e}(\lambda)\big)\big) = \gamma$.

Lemma 2.5.14. *We have* $\dim_C \ker_{\mathrm{U}} A \leqslant \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)|$, *and*

$$\dim_C \ker_{\mathrm{U}} A \ = \ \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| \iff v(\ker^{\neq} A_\lambda) \ = \ \mathscr{E}^{\mathrm{e}}(A_\lambda) \text{ for all } \lambda.$$

Moreover, if $\dim_C \ker_{\mathrm{U}} A = \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)|$, *then* $v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) = \mathscr{E}^{\mathrm{u}}(A)$.

Proof. Clearly, $\dim_C \ker_{\mathrm{U}} A \leqslant \dim_C \ker_\Omega A$. Equip Ω with a spectral extension of the valuation of K. Then $\dim_C \ker_\Omega A = |v(\ker^{\neq}_\Omega A)|$ and $v(\ker^{\neq}_\Omega A) \subseteq \mathscr{E}^{\mathrm{e}}_\Omega(A)$ by [ADH, 5.6.6(i)] and [ADH, p. 481], respectively, applied to Ω in the role of K. Also $|\mathscr{E}^{\mathrm{e}}_\Omega(A)| = \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)|$ (a sum of cardinals) by the remarks preceding the lemma. This yields the first claim of the lemma.

Next, note that $v(\ker^{\neq} A_\lambda) \subseteq \mathscr{E}^{\mathrm{e}}(A_\lambda)$ for all λ. Hence from (2.5.3) and

$$v(\ker^{\neq}_{\mathrm{U}} A) \ = \ \bigcup_\lambda v(\ker^{\neq} A_\lambda) + v\big(\mathrm{e}(\lambda)\big) \qquad \text{(a disjoint union)}$$

we obtain:

$$v(\ker^{\neq}_{\mathrm{U}} A) = \mathscr{E}^{\mathrm{e}}_\Omega(A) \quad \iff \quad v(\ker^{\neq} A_\lambda) \ = \ \mathscr{E}^{\mathrm{e}}(A_\lambda) \text{ for all } \lambda.$$

Also $|v(\ker^{\neq}_{\mathrm{U}} A)| = \dim_C \ker_{\mathrm{U}} A$ by [ADH, 2.3.13], and

$$v(\ker^{\neq}_{\mathrm{U}} A) \ \subseteq \ v(\ker^{\neq}_\Omega A) \ \subseteq \ \mathscr{E}^{\mathrm{e}}_\Omega(A), \qquad |\mathscr{E}^{\mathrm{e}}_\Omega(A)| \ = \ \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)|.$$

This yields the displayed equivalence.

Suppose $\dim_C \ker_{\mathrm{U}} A = \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)|$; we need to show $v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) = \mathscr{E}^{\mathrm{u}}(A)$. We have $\pi\big(\mathscr{E}^{\mathrm{e}}_\Omega(A)\big) = \mathscr{E}^{\mathrm{u}}(A)$ for the above projection map $\pi\colon \Gamma_\Omega \to \Gamma$, so it is enough to show $\pi\big(v(\ker^{\neq}_{\mathrm{U}} A)\big) = v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A)$. For that, note that for $\mathcal{B} \subseteq K^\times \mathrm{e}(\Lambda)$ in Corollary 2.5.4 we have

$$\pi\big(v(\ker^{\neq}_{\mathrm{U}} A)\big) \ = \ \pi(v\mathcal{B}) \ = \ v_{\mathrm{g}}(\mathcal{B}) \ = \ v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A),$$

using for the last equality the details in the proof of Corollary 2.5.4. □

Proposition 2.5.15. *Suppose K is ω-free. Then* $\operatorname{nwt}_{A_\lambda}(\gamma) = 0$ *for all but finitely many pairs* (γ, λ) *and*

$$|\mathscr{E}^{\mathrm{u}}(A)| \ \leqslant \ \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| \ = \ \sum_{\gamma,\lambda} \operatorname{nwt}_{A_\lambda}(\gamma) \ \leqslant \ r.$$

If $\dim_C \ker_{\mathrm{U}} A = r$, *then* $\sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| = r$ *and* $v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) = \mathscr{E}^{\mathrm{u}}(A)$.

Proof. Equip Ω with a spectral extension v of the valuation of K. Then Ω is ω-free, so $\sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| = |\mathscr{E}^{\mathrm{e}}_\Omega(A)| \leqslant r$ by the remarks preceding Lemma 2.5.14 and Corollary 1.4.5 applied to Ω in place of K. These remarks also give $\operatorname{nwt}_{A_\lambda}(\gamma) = 0$ for all but finitely many pairs (γ, λ), and so

$$\sum_{\gamma,\lambda} \operatorname{nwt}_{A_\lambda}(\gamma) \ = \ \sum_{\gamma,\lambda} \operatorname{nwt}_A\big(\gamma + v(\mathrm{e}(\lambda)\big) \ = \ |\mathscr{E}^{\mathrm{e}}_\Omega(A)| \ \leqslant \ r.$$

Corollary 1.4.5 applied to A_λ in place of A yields $|\mathscr{E}^{\mathrm{e}}(A_\lambda)| = \sum_\gamma \operatorname{nwt}_{A_\lambda}(\gamma)$ and so $\sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| = \sum_{\gamma,\lambda} \operatorname{nwt}_{A_\lambda}(\gamma)$. This proves the first part (including the display). The rest follows from this and Lemma 2.5.14. □

In the next lemma (to be used in the proof of Proposition 3.1.27), as well as in Corollary 2.5.21, L is a d-valued H-asymptotic extension of K with algebraically closed constant field and asymptotic integration (so L has small derivation), such that $L^\dagger$ is divisible, $L^\dagger \cap K = K^\dagger$, and $\mathrm{I}(L) \subseteq L^\dagger$. We also fix there a complement Λ_L of the $\mathbb{Q}$-linear subspace $L^\dagger$ of L with $\Lambda \subseteq \Lambda_L$. Let $\mathrm{U}_L = L\big[\mathrm{e}(\Lambda_L)\big]$ be the corresponding universal exponential extension of L containing $\mathrm{U} = K\big[\mathrm{e}(\Lambda)\big]$ as a differential subring, as described in the remarks following Corollary 2.2.13, with differential fraction field Ω_L.

Lemma 2.5.16. *Assume $C_L = C$. Let Ω_L be equipped with a spectral extension of the valuation of L, and take Ω as a valued subfield of Ω_L; so the valuation of Ω is a spectral extension of the valuation of K. Suppose Ψ is cofinal in Ψ_L or K is $\uplambda$-free. Then $\mathscr{E}^{\mathrm{e}}_{\Omega_L}(A) \cap \Gamma_\Omega \ = \ \mathscr{E}^{\mathrm{e}}_{\Omega}(A)$.*

Proof. Let μ range over Λ_L. We have

$$\Gamma_{\Omega_L} \ = \ \Gamma_L \oplus v\big(\mathrm{e}(\Lambda_L)\big), \qquad \Gamma_\Omega \ = \ \Gamma \oplus v\big(\mathrm{e}(\Lambda)\big)$$

by Lemma 2.5.3 and

$$\mathscr{E}^{\mathrm{e}}_{\Omega_L} \ = \ \bigcup_\mu \mathscr{E}^{\mathrm{e}}_L(A_\mu) + v\big(\mathrm{e}(\mu)\big), \qquad \mathscr{E}^{\mathrm{e}}_{\Omega} \ = \ \bigcup_\lambda \mathscr{E}^{\mathrm{e}}(A_\lambda) + v\big(\mathrm{e}(\lambda)\big)$$

by (2.5.3). Hence

$$\mathscr{E}^{\mathrm{e}}_{\Omega_L}(A) \cap \Gamma_\Omega \ = \ \bigcup_\lambda \big(\mathscr{E}^{\mathrm{e}}_L(A_\lambda) \cap \Gamma\big) + v\big(\mathrm{e}(\lambda)\big) \ = \ \bigcup_\lambda \mathscr{E}^{\mathrm{e}}(A_\lambda) + v\big(\mathrm{e}(\lambda)\big) \ = \ \mathscr{E}^{\mathrm{e}}_{\Omega}(A),$$

where we used the injectivity of $\mu \mapsto v\big(\mathrm{e}(\mu)\big)$ for the first equality and Remark 2.5.9 for the second. $\square$

The material in the rest of this section is not used later in this monograph.

Terminal operators. *In this subsection we continue to work in the setting of the previous subsection.* Call A **terminal** with respect to Λ if $\sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| = r$. We omit "with respect to Λ" if it is clear from the context what Λ is. In Section 4.4 we shall restrict K, Λ anyway so that this dependence on Λ disappears. Recall also that for a given spectral extension of the valuation of K to Ω we have $|\mathscr{E}_\Omega(A)| = \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)|$. If A is terminal and $\phi \in K^\times$ is active in K, then $A^\phi \in K^\phi[\updelta]$ is terminal with respect to Λ^ϕ (cf. remarks after Corollary 2.5.7). If A is terminal and $a \in K^\times$, then aA is terminal. If $r = 0$, then A is terminal.

Lemma 2.5.17. *If $r = 1$, then A is terminal.*

Proof. Assume $r = 1$. Then $\dim_C \ker_{\mathrm{U}} A = 1$, so $\sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| \geqslant 1$ by Lemma 2.5.14. Equip Ω with a spectral extension of the valuation of K. Then Ω is ungrounded by Lemma 2.5.3, and $r = 1$ gives $|\mathscr{E}_\Omega(A)| \leqslant 1$. Now use $|\mathscr{E}_\Omega(A)| = \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)|$. $\square$

Lemma 2.5.18. *Suppose A and $B \in K[\partial]^{\neq}$ are terminal, and each operator B_λ is asymptotically surjective. Then AB is terminal.*

Proof. Use that $(AB)_\lambda = A_\lambda B_\lambda$, and that $|\mathscr{E}^{\mathrm{e}}(A_\lambda B_\lambda)| = |\mathscr{E}^{\mathrm{e}}(A_\lambda)| + |\mathscr{E}^{\mathrm{e}}(B_\lambda)|$ by Corollary 1.4.19. $\square$

Thus if A is terminal and $a \in K^\times$, then aA, Aa, and $A_{\ltimes a}$ are terminal. From Lemmas 2.5.17, 2.5.18, and Corollary 1.4.25 we conclude:

Corollary 2.5.19. *If K is λ-free and A splits over K, then A is terminal.*

Corollary 2.5.20. *Suppose K is ω-free and $B \in K[\partial]^{\neq}$. Then A and B are terminal iff AB is terminal.*

Proof. The "only if" part follows from Lemma 2.5.18 and Corollary 1.4.26. For the "if" part, use Corollary 1.4.19 and Proposition 2.5.15. $\square$

Corollary 2.5.21. *Suppose A is terminal, Ψ is cofinal in Ψ_L or K is λ-free, and L is ω-free. Then, with respect to the complement Λ_L of $L^\dagger$ in L, we have:*

(i) *as an element of $L[\partial]$, A is terminal;*
(ii) *$\mathscr{E}^{\mathrm{e}}(A_\mu) = \emptyset$ for all $\mu \in \Lambda_L \setminus \Lambda$;*
(iii) *$\mathscr{E}^{\mathrm{e}}(A_\lambda) = \mathscr{E}^{\mathrm{e}}_L(A_\lambda)$ for all λ; and*
(iv) *$\mathscr{E}^{\mathrm{u}}(A) = \mathscr{E}^{\mathrm{u}}_L(A)$.*

Proof. By the remarks after Corollary 2.5.7 we have $\mathscr{E}^{\mathrm{e}}(A_\lambda) \subseteq \mathscr{E}^{\mathrm{e}}_L(A_\lambda)$ for each λ, and so with μ ranging over Λ_L, by Proposition 2.5.15 applied to L in place of K, we have $r = \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| \leqslant \sum_\mu |\mathscr{E}^{\mathrm{e}}_L(A_\mu)| \leqslant r$. This yields (i)–(iv). $\square$

In [8] we shall study other situations where A is terminal.

The real case. *In this subsection H is a real closed H-field with small derivation, asymptotic integration, and $H^\dagger = H$; also $K = H[\mathrm{i}]$, $\mathrm{i}^2 = -1$, for our valued differential field K. We also assume $\mathrm{I}(H)\mathrm{i} \subseteq K^\dagger$.* Then K is d-valued of H-type with small derivation, asymptotic integration, $K^\dagger = H + \mathrm{I}(H)\mathrm{i}$, and $\mathrm{I}(K) \subseteq K^\dagger$. Note that H and thus K is λ-free by [ADH, remark after 11.6.2, and 11.6.8]. Let A in $K[\partial]^{\neq}$ have order r and let γ range over Γ.

Lemma 2.5.22. *If the real closed H-asymptotic extension F of H has asymptotic integration and convex valuation ring, then $L^\dagger \cap K = K^\dagger$ for the algebraically closed H-asymptotic field extension $L := F[\mathrm{i}]$ of K.*

Proof. Use Corollary 1.2.15 and earlier remarks in the same subsection. $\square$

Corollary 2.5.23. *The H-field H has an H-closed extension F with $C_F = C_H$, and for any such F, the algebraically closed* d*-valued field extension $L := F[\mathrm{i}]$ of H-type of K is ω-free with $C_L = C$, $\mathrm{I}(L) \subseteq L^\dagger$, and $L^\dagger \cap K = K^\dagger$.*

Proof. Use [ADH, 16.4.1, 9.1.2] to extend H to an ω-free H-field with the same constant field as H, next use [ADH, 11.7.23] to pass to its real closure, and then use [ADH, 14.5.9] to extend further to an H-closed F, still with the same constant field as H. For any such F, the d-valued field $L := F[\mathrm{i}]$ of H-type is ω-free by [ADH, 11.7.23] and newtonian by (0.10). Hence $\mathrm{I}(L) \subseteq L^\dagger$ by Lemma 1.2.9, and $L^\dagger \cap K = K^\dagger$ by Lemma 2.5.22. $\square$

This leads to a variant of Proposition 2.5.15:

Proposition 2.5.24. *The conclusion of Proposition 2.5.15 holds. In particular:*

$$\dim_C \ker_{\mathrm{U}} A = r \implies A \text{ is terminal.}$$

Proof. Corollary 2.5.23 gives an H-closed extension F of H with $C_F = C_H$, so $L := F[\mathrm{i}]$ is ω-free, $C_L = C$, $\mathrm{I}(L) \subseteq L^\dagger$, and $L^\dagger \cap K = K^\dagger$. Take a complement $\Lambda_L \supseteq \Lambda$ of the subspace $L^\dagger$ of the $\mathbb{Q}$-linear space L. By Remark 2.5.9 we have $\mathscr{E}^{\mathrm{e}}(A_\lambda) = \mathscr{E}^{\mathrm{e}}_L(A_\lambda) \cap \Gamma$. Hence Proposition 2.5.15 applied to K, Λ replaced by L, Λ_L, respectively, and A viewed as element of $L[\partial]$, yields $\sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| \leqslant r$. Corollary 1.7.10

applied to A_λ in place of A gives $|\mathscr{E}^{\mathrm{e}}(A_\lambda)| = \sum_\gamma \operatorname{nwt}_{A_\lambda}(\gamma)$. This yields the conclusion of Proposition 2.5.15 as in the proof of that proposition. $\square$

Let now F be a Liouville closed H-field extension of H and suppose $\mathrm{I}(L) \subseteq L^\dagger$ where $L := F[\mathrm{i}]$. Lemma 2.5.22 yields $L^\dagger \cap K = K^\dagger$, so L is as described just before Lemma 2.5.16, and we have a complement $\Lambda_L \supseteq \Lambda$ of the subspace $L^\dagger$ of the $\mathbb{Q}$-linear space L. Note that if A splits over K, then A is terminal by Corollary 2.5.19.

Corollary 2.5.25. *Suppose A is terminal. Then, with respect to the complement Λ_L of $L^\dagger$ in L, the conclusions* (i)–(iv) *of Corollary 2.5.21 hold.*

Proof. By the remarks after Corollary 2.5.7 we have $\mathscr{E}^{\mathrm{e}}(A_\lambda) \subseteq \mathscr{E}^{\mathrm{e}}_L(A_\lambda)$ for all λ, and so with μ ranging over Λ_L, Proposition 2.5.24 applied to L in place of K, we have $r = \sum_\lambda |\mathscr{E}^{\mathrm{e}}(A_\lambda)| \leqslant \sum_\mu |\mathscr{E}^{\mathrm{e}}_L(A_\mu)| \leqslant r$. This yields (i)–(iv). $\square$

Part 3. Normalizing Holes and Slots

In this introduction K is an H-asymptotic field with small derivation and rational asymptotic integration. In Section 3.2 we introduce *holes* in K: A *hole in K* is a triple $(P, \mathfrak{m}, \widehat{a})$ with $P \in K\{Y\} \setminus K$, $\mathfrak{m} \in K^\times$, and $\widehat{a} \in \widehat{K} \setminus K$ for some immediate asymptotic extension $\widehat{K}$ of K, such that $\widehat{a} \prec \mathfrak{m}$ and $P(\widehat{a}) = 0$. The main goal of Part 3 is a normalization theorem, namely Theorem 3.3.33, that allows us to transform under reasonable conditions a hole $(P, \mathfrak{m}, \widehat{a})$ in K into a "normal" hole; this helps to pin down the location of $\widehat{a}$ relative to K. The notion of $(P, \mathfrak{m}, \widehat{a})$ being *normal* involves the linear part of the differential polynomial $P_{\times\mathfrak{m}}$, in particular the *span* of this linear part. We introduce the span in the preliminary Section 3.1. In Section 3.4 we study *isolated* holes $(P, \mathfrak{m}, \widehat{a})$ in K, which under reasonable conditions ensure the uniqueness of the isomorphism type of $K\langle\widehat{a}\rangle$ as a valued differential field over K; see Proposition 3.4.9. In Section 3.5 we focus on holes $(P, \mathfrak{m}, \widehat{a})$ in K where $\operatorname{order} P = \deg P = 1$. For technical reasons we actually work in Part 3 also with *slots* in K, which are a bit more general than holes in K.

First some notational conventions. Let Γ be an ordered abelian group. For $\gamma, \delta \in \Gamma$ with $\gamma \neq 0$ the expression "$\delta = o(\gamma)$" means "$n|\delta| < |\gamma|$ for all $n \geqslant 1$" according to [ADH, 2.4], but here we find it convenient to extend this to $\gamma = 0$, in which case "$\delta = o(\gamma)$" means "$\delta = 0$". Suppose $\Gamma = v(E^\times)$ is the value group of a valued field E and $\mathfrak{m} \in E^\times$. Then we denote the archimedean class $[v\mathfrak{m}] \subseteq \Gamma$ of $v\mathfrak{m} \in \Gamma$ by just $[\mathfrak{m}]$. Suppose $\mathfrak{m} \not\asymp 1$. Then we have a proper convex subgroup

$$\Delta(\mathfrak{m}) \ := \ \big\{\gamma \in \Gamma : \gamma = o(v\mathfrak{m})\big\} \ = \ \big\{\gamma \in \Gamma : [\gamma] < [\mathfrak{m}]\big\},$$

of Γ. If $\mathfrak{m} \asymp_{\Delta(\mathfrak{m})} \mathfrak{n} \in E$, then $0 \neq \mathfrak{n} \not\asymp 1$ and $\Delta(\mathfrak{m}) = \Delta(\mathfrak{n})$. In particular, if $\mathfrak{m} \asymp \mathfrak{n} \in E$, then $0 \neq \mathfrak{n} \not\asymp 1$ and $\Delta(\mathfrak{m}) = \Delta(\mathfrak{n})$. Note that for $f, g \in E$ the meaning of "$f \preccurlyeq_{\Delta(\mathfrak{m})} g$" does not change in passing to a valued field extension of E, although $\Delta(\mathfrak{m})$ can increase as a subgroup of the value group of the extension.

3.1. The Span of a Linear Differential Operator

In this section K is a valued differential field with small derivation and $\Gamma := v(K^\times)$. We let a, b, sometimes subscripted, range over K, and $\mathfrak{m}$, $\mathfrak{n}$ over $K^\times$. Consider a linear differential operator

$$A \ = \ a_0 + a_1\partial + \cdots + a_r\partial^r \in K[\partial], \qquad a_r \neq 0.$$

We shall use below the quantities $\operatorname{dwm}(A)$ and $\operatorname{dwt}(A)$ defined in [ADH, 5.6]. We also introduce a measure $\mathfrak{v}(A)$ for the "lopsidedness" of A as follows:

$$\mathfrak{v}(A) \ := \ a_r/a_m \in \ K^\times \qquad \text{where } m := \operatorname{dwt}(A).$$

So $a_r \asymp \mathfrak{v}(A)A$ and $\mathfrak{v}(A) \preccurlyeq 1$, with

$$\mathfrak{v}(A) \asymp 1 \quad\Longleftrightarrow\quad \operatorname{dwt}(A) = r \quad\Longleftrightarrow\quad \mathfrak{v}(A) = 1.$$

Also note that $\mathfrak{v}(aA) = \mathfrak{v}(A)$ for $a \neq 0$. Moreover,

$$\mathfrak{v}(A_{\ltimes\mathfrak{n}})A_{\ltimes\mathfrak{n}} \ \asymp \ a_r \ \asymp \ \mathfrak{v}(A)A$$

since $A_{\ltimes\mathfrak{n}} = a_r\partial^r + \text{lower order terms in } \partial$.

Example. $\mathfrak{v}(a + \partial) = 1$ if $a \preccurlyeq 1$, and $\mathfrak{v}(a + \partial) = 1/a$ if $a \succ 1$.

We call $\mathfrak{v}(A)$ the **span** of A. We are mainly interested in the valuation of $\mathfrak{v}(A)$. This is related to the gaussian valuation $v(A)$ of A: if A is monic, then $v\big(\mathfrak{v}(A)\big) = -v(A)$. An important property of the span of A is that its valuation is not affected by small additive perturbations of A:

Lemma 3.1.1. *Suppose $B \in K[\partial]$, $\operatorname{order}(B) \leqslant r$ and $B \prec \mathfrak{v}(A)A$. Then:*

(i) $A + B \sim A$, $\operatorname{dwm}(A+B) = \operatorname{dwm}(A)$, *and* $\operatorname{dwt}(A+B) = \operatorname{dwt}(A)$*;*
(ii) $\operatorname{order}(A+B) = r$ *and* $\mathfrak{v}(A+B) \sim \mathfrak{v}(A)$*.*

Proof. From $B \prec \mathfrak{v}(A)A$ and $\mathfrak{v}(A) \preccurlyeq 1$ we obtain $B \prec A$, and thus (i). Set $m := \operatorname{dwt}(A)$, let i range over $\{0,\dots,r\}$, and let $B = b_0 + b_1\partial + \cdots + b_r\partial^r$. Then $a_i \preccurlyeq a_m$ and $b_i \prec \mathfrak{v}(A)A \asymp a_r \preccurlyeq a_m$. Therefore, if $a_i \asymp a_m$, then $a_i + b_i \sim a_i$, and if $a_i \prec a_m$, then $a_i + b_i \prec a_m$. Hence $\mathfrak{v}(A+B) = (a_r+b_r)/(a_m+b_m) \sim a_r/a_m = \mathfrak{v}(A)$. $\square$

For $b \neq 0$, the valuation of $\mathfrak{v}(Ab)$ only depends on vb; it is enough to check this for $b \asymp 1$. More generally:

Lemma 3.1.2. *Let $B \in K[\partial]^{\neq}$ and $b \asymp B$. Then $\mathfrak{v}(AB) \asymp \mathfrak{v}(Ab)\mathfrak{v}(B)$.*

Proof. Let $B = b_0 + b_1\partial + \cdots + b_s\partial^s$, $b_s \neq 0$. Then

$$AB \;=\; a_r b_s \partial^{r+s} + \text{lower order terms in } \partial,$$

so by [ADH, 5.6.1(ii)] for $\gamma = 0$:

$$\begin{aligned} v\big(\mathfrak{v}(AB)\big) \;&=\; v(a_r b_s) - v(AB) \;=\; v(a_r b_s) - v(Ab) \\ &=\; v(a_r b) - v(Ab) + v(b_s) - v(B) \\ &=\; v\big(\mathfrak{v}(Ab)\mathfrak{v}(B)\big). \end{aligned}$$

$\square$

Corollary 3.1.3. *Let $B \in K[\partial]^{\neq}$. If $\mathfrak{v}(AB) = 1$, then $\mathfrak{v}(A) = \mathfrak{v}(B) = 1$. The converse holds if B is monic.*

This is clear from from Lemma 3.1.2, and in turn gives:

Corollary 3.1.4. *Suppose $A = a(\partial - b_1)\cdots(\partial - b_r)$. Then*

$$\mathfrak{v}(A) = 1 \quad\Longleftrightarrow\quad b_1,\dots,b_r \preccurlyeq 1.$$

Remark. Suppose $K = C(\!(t)\!)$ with the t-adic valuation and derivation $\partial = t\frac{d}{dt}$. In the literature, A is called *regular singular* if $\mathfrak{v}(A) = 1$, and *irregular singular* if $\mathfrak{v}(A) \prec 1$; see [20, Definition 3.14].

Lemma 3.1.5. *Let $B \in K[\partial]^{\neq}$. Then $\mathfrak{v}(AB) \preccurlyeq \mathfrak{v}(B)$, and if B is monic, then $\mathfrak{v}(AB) \preccurlyeq \mathfrak{v}(A)$.*

Proof. Lemma 3.1.2 and $\mathfrak{v}(Ab) \preccurlyeq 1$ for $b \neq 0$ yields $\mathfrak{v}(AB) \preccurlyeq \mathfrak{v}(B)$. Suppose B is monic, so $v(B) \leqslant 0$. To show $\mathfrak{v}(AB) \preccurlyeq \mathfrak{v}(A)$ we arrange that A is also monic. Then AB is monic, and $\mathfrak{v}(AB) \preccurlyeq \mathfrak{v}(A)$ is equivalent to $v(AB) \leqslant v(A)$. Now

$$v(AB) \;=\; v_{AB}(0) \;=\; v_A\big(v_B(0)\big) \;=\; v_A\big(v(B)\big) \;\leqslant\; v_A(0) \;=\; v(A)$$

by [ADH, 4.5.1(iii), 5.6.1(ii)]. $\square$

Corollary 3.1.6. *If $A = a(\partial - b_1)\cdots(\partial - b_r)$, then $b_1,\dots,b_r \preccurlyeq \mathfrak{v}(A)^{-1}$.*

Let Δ be a convex subgroup of Γ, let $\dot{\mathcal{O}}$ be the valuation ring of the coarsening v_Δ of the valuation v of K by Δ, with maximal ideal $\dot{\smallo}$, and $\dot{K} = \dot{\mathcal{O}}/\dot{\smallo}$ be the valued differential residue field of v_Δ. The residue morphism $\dot{\mathcal{O}} \to \dot{K}$ extends to the ring morphism $\dot{\mathcal{O}}[\partial] \to \dot{K}[\partial]$ with $\partial \mapsto \partial$. If $A \in \dot{\mathcal{O}}[\partial]$ and $\dot{A} \neq 0$, then $\operatorname{dwm}(\dot{A}) = \operatorname{dwm}(A)$ and $\operatorname{dwt}(\dot{A}) = \operatorname{dwt}(A)$. We set $\mathfrak{v} := \mathfrak{v}(A)$.

Lemma 3.1.7. *If $A \in \dot{\mathcal{O}}[\partial]$ and $\operatorname{order}(\dot{A}) = r$, then $\mathfrak{v}(\dot{A}) = \dot{\mathfrak{v}}$.*

Behavior of the span under twisting. Recall that $o(\gamma) := 0 \in \Gamma$ for $\gamma = 0 \in \Gamma$. With this convention, here is a consequence of [ADH, 6.1.3]:

Lemma 3.1.8. *Let $B \in K[\partial]^{\neq}$. Then $v(AB) = v(A) + v(B) + o\big(v(B)\big)$.*

Proof. Take b with $b \asymp B$. Then

$$v(AB) = v_{AB}(0) = v_A\big(v_B(0)\big) = v_A(vb) = v(Ab)$$

by [ADH, 5.6.1(ii)]. Moreover, $v(Ab) = v(A) + vb + o(vb)$, by [ADH, 6.1.3]. ☐

We have $\mathfrak{v}(A_{\ltimes \mathfrak{n}}) = \mathfrak{v}(A\mathfrak{n})$, so $v(A_{\ltimes \mathfrak{n}}) = v(A) + o(v\mathfrak{n})$ by Lemma 3.1.8. Moreover:

Lemma 3.1.9. $v\big(\mathfrak{v}(A\mathfrak{n})\big) = v\big(\mathfrak{v}(A)\big) + o(v\mathfrak{n})$.

Proof. Replacing A by $a_r^{-1}A$ we arrange A is monic, so $A_{\ltimes \mathfrak{n}}$ is monic, and thus

$$v\big(\mathfrak{v}(A\mathfrak{n})\big) \;=\; v\big(\mathfrak{v}(A_{\ltimes \mathfrak{n}})\big) = -v(A_{\ltimes \mathfrak{n}}) = -v(A) + o(v\mathfrak{n}) = v\big(\mathfrak{v}(A)\big) + o(v\mathfrak{n})$$

by remarks preceding the lemma. ☐

Recall: we denote the archimedean class $[v\mathfrak{n}] \subseteq \Gamma$ by $[\mathfrak{n}]$. Lemma 3.1.9 yields:

Corollary 3.1.10. $\big[\mathfrak{v}(A)\big] < [\mathfrak{n}] \iff \big[\mathfrak{v}(A\mathfrak{n})\big] < [\mathfrak{n}]$.

Under suitable conditions on K we can say more about the valuation of $\mathfrak{v}(A_{\ltimes \mathfrak{n}})$: Lemma 3.1.12 below.

Lemma 3.1.11. *Let $\mathfrak{n}^\dagger \succcurlyeq 1$ and $\mathfrak{m}_0, \dots, \mathfrak{m}_r \in K^\times$ be such that*

$$v(\mathfrak{m}_i) + v(A) \;=\; \min_{i \leqslant j \leqslant r} v(a_j) + (j-i)v(\mathfrak{n}^\dagger).$$

Then with $m := \operatorname{dwt}(A)$ we have

$$\mathfrak{m}_0 \;\succcurlyeq\; \cdots \;\succcurlyeq\; \mathfrak{m}_r \quad \text{and} \quad (\mathfrak{n}^\dagger)^m \;\preccurlyeq\; \mathfrak{m}_0 \;\preccurlyeq\; (\mathfrak{n}^\dagger)^r.$$

(In particular, $[\mathfrak{m}_0] \leqslant [\mathfrak{n}^\dagger]$, with equality if $m > 0$.)

Proof. From $v(\mathfrak{n}^\dagger) \leqslant 0$ we obtain $v(\mathfrak{m}_0) \leqslant \cdots \leqslant v(\mathfrak{m}_r)$. We have $0 \leqslant v(a_j/a_m)$ for $j = 0, \dots, r$ and so

$$rv(\mathfrak{n}^\dagger) \;\leqslant\; \min_{0 \leqslant j \leqslant r} v(a_j/a_m) + jv(\mathfrak{n}^\dagger) \;=\; v(\mathfrak{m}_0) \;\leqslant\; mv(\mathfrak{n}^\dagger)$$

as required. ☐

Lemma 3.1.12. *Suppose $\partial\mathcal{O} \subseteq \smallo$. Then*

$$\mathfrak{n}^\dagger \preccurlyeq 1 \implies v(A_{\ltimes \mathfrak{n}}) = v(A), \qquad \mathfrak{n}^\dagger \succ 1 \implies |v(A_{\ltimes \mathfrak{n}}) - v(A)| \leqslant -rv(\mathfrak{n}^\dagger).$$

Proof. Let $R := \operatorname{Ri} A$. Then $v(A_{\ltimes\mathfrak{n}}) = v(R_{+\mathfrak{n}^\dagger})$ by [ADH, 5.8.11]. If $\mathfrak{n}^\dagger \preccurlyeq 1$, then $v(R_{+\mathfrak{n}^\dagger}) = v(R)$ by [ADH, 4.5.1(i)], hence $v(A_{\ltimes\mathfrak{n}}) = v(R) = v(A)$ by [ADH, 5.8.10]. Now suppose $\mathfrak{n}^\dagger \succ 1$. *Claim*: $v(A_{\ltimes\mathfrak{n}}) - v(A) \geqslant rv(\mathfrak{n}^\dagger)$. To prove this claim we replace A by $a^{-1}A$, where $a \asymp A$, to arrange $A \asymp 1$. Let i, j range over $\{0,\dots,r\}$. We have $R_{+\mathfrak{n}^\dagger} = \sum_i b_i R_i$ where

$$b_i \;=\; \sum_{j\geqslant i} \binom{j}{i} a_j R_{j-i}(\mathfrak{n}^\dagger).$$

Take $\mathfrak{m}_i \in K^\times$ as in Lemma 3.1.11. By Lemma 1.1.17 we have $R_n(\mathfrak{n}^\dagger) \sim (\mathfrak{n}^\dagger)^n$ for all n; hence $v(b_i) \geqslant v(\mathfrak{m}_i)$ for all i. Thus

$$v(A_{\ltimes\mathfrak{n}}) - v(A) \;=\; v(A_{\ltimes\mathfrak{n}}) \;=\; v(R_{+\mathfrak{n}^\dagger}) \;\geqslant\; \min_i v(b_i) \;\geqslant\; v(\mathfrak{m}_0) \;\geqslant\; rv(\mathfrak{n}^\dagger)$$

by Lemma 3.1.11, proving our claim. Applying this claim with $A_{\ltimes\mathfrak{n}}$, $\mathfrak{n}^{-1}$ in place of A, $\mathfrak{n}$ also yields $v(A_{\ltimes\mathfrak{n}}) - v(A) \leqslant -rv(\mathfrak{n}^\dagger)$, thus $|v(A_{\ltimes\mathfrak{n}}) - v(A)| \leqslant -rv(\mathfrak{n}^\dagger)$. $\square$

Remark. Suppose that $\partial\mathcal{O} \subseteq \smallo$ and $\mathfrak{n}^\dagger \succ 1$. Then Lemma 3.1.12 improves on Lemma 3.1.9, since $v(\mathfrak{n}^\dagger) = o(v\mathfrak{n})$ by [ADH, 6.4.1(iii)].

Lemma 3.1.13. *Suppose $\partial\mathcal{O} \subseteq \smallo$ and $\mathfrak{n}^\dagger \preccurlyeq \mathfrak{v}(A)^{-1}$. Let $B \in K[\partial]$ and $s \in \mathbb{N}$ be such that* $\operatorname{order}(B) \leqslant s$ *and $B \prec \mathfrak{v}(A)^{s+1}A$. Then $B_{\ltimes\mathfrak{n}} \prec \mathfrak{v}(A_{\ltimes\mathfrak{n}})A_{\ltimes\mathfrak{n}}$.*

Proof. We may assume $B \neq 0$ and $s = \operatorname{order}(B)$. It suffices to show $B_{\ltimes\mathfrak{n}} \prec \mathfrak{v}(A)A$. If $\mathfrak{n}^\dagger \preccurlyeq 1$, then Lemma 3.1.12 applied to B in place of A yields $B_{\ltimes\mathfrak{n}} \asymp B \prec \mathfrak{v}(A)A$. Suppose $\mathfrak{n}^\dagger \succ 1$. Then Lemma 3.1.12 gives $|v(B_{\ltimes\mathfrak{n}}) - v(B)| \leqslant -sv(\mathfrak{n}^\dagger) \leqslant sv(\mathfrak{v}(A))$ and hence $B_{\ltimes\mathfrak{n}} \preccurlyeq \mathfrak{v}(A)^{-s}B \prec \mathfrak{v}(A)A$. $\square$

If $\partial\mathcal{O} \subseteq \smallo$, then we have functions $\operatorname{dwm}_A, \operatorname{dwt}_A \colon \Gamma \to \mathbb{N}$ as defined in [ADH, 5.6]. Combining Lemmas 3.1.1 and 3.1.13 yields a variant of [ADH, 6.1.7]:

Corollary 3.1.14. *Suppose $\partial\mathcal{O} \subseteq \smallo$ and $\mathfrak{n}^\dagger \preccurlyeq \mathfrak{v}(A)^{-1}$. Let $B \in K[\partial]$ be such that* $\operatorname{order}(B) \leqslant r$ *and $B \prec \mathfrak{v}(A)^{r+1}A$. Then* $\operatorname{dwm}_{A+B}(v\mathfrak{n}) = \operatorname{dwm}_A(v\mathfrak{n})$ *and* $\operatorname{dwt}_{A+B}(v\mathfrak{n}) = \operatorname{dwt}_A(v\mathfrak{n})$. *In particular,*

$$v\mathfrak{n} \in \mathscr{E}(A+B) \iff v\mathfrak{n} \in \mathscr{E}(A).$$

About $A(\mathfrak{n}^q)$ and $A\mathfrak{n}^q$. Suppose $\mathfrak{m}^l = \pm\mathfrak{n}^k$ where $k, l \in \mathbb{Z}$, $l \neq 0$. Then $\mathfrak{m}^\dagger = q\mathfrak{n}^\dagger$ with $q = k/l \in \mathbb{Q}$. In particular, if K is real closed or algebraically closed, then for any $\mathfrak{n}$ and $q \in \mathbb{Q}$ we have $\mathfrak{m}^\dagger = q\mathfrak{n}^\dagger$ for some $\mathfrak{m}$.

Below in this subsection K is d*-valued and $\mathfrak{n}$ is such that for all $q \in \mathbb{Q}^>$ we are given an element of $K^\times$, denoted by $\mathfrak{n}^q$ for suggestiveness, with $(\mathfrak{n}^q)^\dagger = q\mathfrak{n}^\dagger$.*

Let $q \in \mathbb{Q}^>$; then $v(\mathfrak{n}^q) = qv(\mathfrak{n})$: to see this we may arrange that K is algebraically closed by [ADH, 10.1.23], and hence contains an $\mathfrak{m}$ such that $v\mathfrak{m} = q\,v\mathfrak{n}$ and $\mathfrak{m}^\dagger = q\mathfrak{n}^\dagger = (\mathfrak{n}^q)^\dagger$, and thus $v(\mathfrak{n}^q) = v\mathfrak{m} = q\,v\mathfrak{n}$.

Lemma 3.1.15. *Suppose $\mathfrak{n}^\dagger \succcurlyeq 1$. Then for all but finitely many $q \in \mathbb{Q}^>$,*

$$v\big(A(\mathfrak{n}^q)\big) \;=\; v(\mathfrak{n}^q) + \min_j v(a_j) + jv(\mathfrak{n}^\dagger).$$

Proof. Let $q \in \mathbb{Q}^>$ and take $b_0,\dots,b_r \in K$ with $A\mathfrak{n}^q = b_0 + b_1\partial + \cdots + b_r\partial^r$. Then

$$b_0 \;=\; A(\mathfrak{n}^q) \;=\; \mathfrak{n}^q\big(a_0R_0(q\mathfrak{n}^\dagger) + a_1R_1(q\mathfrak{n}^\dagger) + \cdots + a_rR_r(q\mathfrak{n}^\dagger)\big).$$

Let i, j range over $\{0,\dots,r\}$. By Lemma 1.1.17, $R_i(q\mathfrak{n}^\dagger) \sim q^i(\mathfrak{n}^\dagger)^i$ for all i. Take $\mathfrak{m}$ (independent of q) such that $v(\mathfrak{m}) = \min_j v(a_j) + jv(\mathfrak{n}^\dagger)$, and let I be the nonempty set of i with $\mathfrak{m} \asymp a_i(\mathfrak{n}^\dagger)^i$. For $i \in I$ we take $c_i \in C^\times$ such that $a_i(\mathfrak{n}^\dagger)^i \sim c_i\mathfrak{m}$, and set $R := \sum_{i\in I} c_i Y^i \in C[Y]^{\neq}$. Therefore, if $R(q) \neq 0$, then

$$\sum_{i\in I} a_i R_i(q\mathfrak{n}^\dagger) \ \sim\ \mathfrak{m}R(q).$$

Assume $R(q) \neq 0$ in what follows. Then

$$\sum_{i=0}^{r} a_i R_i(q\mathfrak{n}^\dagger) \ \sim\ \sum_{i\in I} a_i R_i(q\mathfrak{n}^\dagger) \ \sim\ \mathfrak{m}R(q) \ \asymp\ \mathfrak{m},$$

hence $b_0 \asymp \mathfrak{m}\mathfrak{n}^q$, in particular, $b_0 \neq 0$. ☐

Lemma 3.1.16. *Assume $\mathfrak{n}^\dagger \succcurlyeq 1$ and $[\mathfrak{v}] < [\mathfrak{n}]$ for $\mathfrak{v} := \mathfrak{v}(A)$. Then $\big[\mathfrak{v}(A\mathfrak{n}^q)\big] < [\mathfrak{n}]$ for all $q \in \mathbb{Q}^>$, and for all but finitely many $q \in \mathbb{Q}^>$ we have $\mathfrak{v}(A\mathfrak{n}^q) \preccurlyeq \mathfrak{v}$, and thus $[\mathfrak{v}] \leqslant \big[\mathfrak{v}(A\mathfrak{n}^q)\big]$.*

Proof. Let $q \in \mathbb{Q}^>$. Then $[\mathfrak{v}] < [\mathfrak{n}] = [\mathfrak{n}^q]$, so $[\mathfrak{v}(A\mathfrak{n}^q)] < [\mathfrak{n}^q] = [\mathfrak{n}]$ by Corollary 3.1.10. To show the second part, let $m = \operatorname{dwt}(A)$. Replacing A by $a_m^{-1}A$ we arrange $a_m = 1$, so $a_r = \mathfrak{v}$, $A \asymp 1$. Take $b_0,\dots,b_r$ with $A\mathfrak{n}^q = b_0 + b_1\partial + \cdots + b_r\partial^r$. As in the proof of Lemma 3.1.15 we obtain an $\mathfrak{m}$ and a polynomial $R(Y) \in C[Y]^{\neq}$ (both independent of q) such that $v(\mathfrak{m}) = \min_j v(a_j) + jv(\mathfrak{n}^\dagger)$, and $b_0 \asymp \mathfrak{m}\mathfrak{n}^q$ if $R(q) \neq 0$. Assume $R(q) \neq 0$ in what follows; we show that then $\mathfrak{v}(A\mathfrak{n}^q) \preccurlyeq \mathfrak{v}$. For $n := \operatorname{dwt}(A\mathfrak{n}^q)$,

$$b_0\mathfrak{v}(A\mathfrak{n}^q) \ \preccurlyeq\ b_n\mathfrak{v}(A\mathfrak{n}^q) \ =\ b_r \ =\ \mathfrak{n}^q\mathfrak{v},$$

hence $\mathfrak{v}(A\mathfrak{n}^q) \preccurlyeq \mathfrak{v}/\mathfrak{m}$. It remains to note that $\mathfrak{m} \succcurlyeq a_m(\mathfrak{n}^\dagger)^m = (\mathfrak{n}^\dagger)^m \succcurlyeq 1$. ☐

Lemma 3.1.17. *Assume $\mathfrak{n}^\dagger \succcurlyeq 1$ and $\mathfrak{m}$ satisfies*

$$v\mathfrak{m} + v(A) \ =\ \min_{0\leqslant j\leqslant r} v(a_j) + jv(\mathfrak{n}^\dagger).$$

Then $[\mathfrak{m}] \leqslant [\mathfrak{n}^\dagger]$, with equality if $\operatorname{dwt}(A) > 0$, and for all but finitely many $q \in \mathbb{Q}^>$,

$$A\mathfrak{n}^q \ \asymp\ \mathfrak{m}\,\mathfrak{n}^q A, \qquad \mathfrak{v}(A)/\mathfrak{v}(A\mathfrak{n}^q) \ \asymp\ \mathfrak{m}.$$

Proof. Replacing A by $a_m^{-1}A$ where $m = \operatorname{dwt}(A)$ we arrange $a_m = 1$, so $a_r = \mathfrak{v} := \mathfrak{v}(A)$ and $A \asymp 1$. Let i, j range over $\{0,\dots,r\}$. Let $q \in \mathbb{Q}^>$, and take $b_i \in K$ such that $A\mathfrak{n}^q = \sum_i b_i\partial^i$. By [ADH, (5.1.3)] we have

$$b_i \ =\ \frac{1}{i!}A^{(i)}(\mathfrak{n}^q) \ =\ \mathfrak{n}^q\frac{1}{i!}\operatorname{Ri}(A^{(i)})(q\mathfrak{n}^\dagger) \ =\ \mathfrak{n}^q\sum_{j\geqslant i}\binom{j}{i}a_jR_{j-i}(q\mathfrak{n}^\dagger).$$

Take $\mathfrak{m}_i \in K^\times$ as in Lemma 3.1.11. Then $\mathfrak{m}_0 \asymp \mathfrak{m}$ (so $[\mathfrak{m}] \leqslant [\mathfrak{n}^\dagger]$, with equality if $m > 0$), and $\mathfrak{m}_r \asymp \mathfrak{v}$. Lemma 3.1.15 applied to $A^{(i)}/i!$ instead of A gives that for all but finitely $q \in \mathbb{Q}^>$ we have $b_i \asymp \mathfrak{m}_i\mathfrak{n}^q$ for all i. Assume that $q \in \mathbb{Q}^>$ has this property. From $v(\mathfrak{m}) = v(\mathfrak{m}_0) \leqslant \cdots \leqslant v(\mathfrak{m}_r) = v(\mathfrak{v})$ we obtain

$$v(\mathfrak{m}) + qv(\mathfrak{n}) \ =\ v(b_0) \ \leqslant\ \cdots\ \leqslant\ v(b_r) \ =\ v(\mathfrak{v}) + q\,v(\mathfrak{n}).$$

With $n = \operatorname{dwt}(A\mathfrak{n}^q)$ this gives $v(b_0) = \cdots = v(b_n) = v(A\mathfrak{n}^q)$. Thus

$$\mathfrak{v}(A\mathfrak{n}^q) \ =\ b_r/b_n \ \asymp\ b_r/b_0 \ \asymp\ (\mathfrak{n}^q\mathfrak{v})/(\mathfrak{n}^q\mathfrak{m}) \ =\ \mathfrak{v}/\mathfrak{m}$$

as claimed. ☐

The next lemma (not used later) is a more precise version of Lemma 3.1.17, but with an additional hypothesis on $\mathfrak{n}^\dagger$:

Lemma 3.1.18. *Assume* $\mathfrak{n}^\dagger \succ \mathfrak{v}(A)^{-1}$. *Then*

$$A(\mathfrak{n}) \sim A\mathfrak{n} \sim a_r\mathfrak{n}(\mathfrak{n}^\dagger)^r \sim a_r\mathfrak{n}^{(r)}, \qquad \mathfrak{v}(A\mathfrak{n}) \sim (\mathfrak{n}^\dagger)^{-r}.$$

Proof. Let i, j range over $\{0,\dots,r\}$ and take $b_i \in K$ such that $A\mathfrak{n} = \sum_i b_i\partial^i$, so $b_i = \mathfrak{n}\sum_{j\geqslant i}\binom{j}{i}a_jR_{j-i}(\mathfrak{n}^\dagger)$. By Lemma 1.1.16 we have $R_{j-i}(\mathfrak{n}^\dagger) \sim (\mathfrak{n}^\dagger)^{j-i}$ for $i \leqslant j$. From $\mathfrak{n}^\dagger \succ \mathfrak{v}^{-1} \succcurlyeq 1$ we get for $i \leqslant j < r$:

$$a_r(\mathfrak{n}^\dagger)^{r-i} \ \succcurlyeq\ a_r\mathfrak{n}^\dagger(\mathfrak{n}^\dagger)^{j-i} \ \succ\ a_r\mathfrak{v}^{-1}(\mathfrak{n}^\dagger)^{j-i} \ \succcurlyeq\ a_j(\mathfrak{n}^\dagger)^{j-i}.$$

Therefore $b_i \sim \mathfrak{n}\binom{r}{i}a_r(\mathfrak{n}^\dagger)^{r-i}$, from which the first displayed equivalences follow. Now $\operatorname{dwt}(A\mathfrak{n}) = 0$ and so $\mathfrak{v}(A\mathfrak{n}) = b_r/b_0 = (\mathfrak{n}a_r)/A(\mathfrak{n}) \sim (\mathfrak{n}^\dagger)^{-r}$ as claimed. □

Let $\mathfrak{v} \in K^\times$ with $\mathfrak{v} \not\asymp 1$; so we have the proper convex subgroup of Γ given by

$$\Delta(\mathfrak{v}) \ =\ \big\{\gamma\in\Gamma :\ \gamma = o(v\mathfrak{v})\big\} \ =\ \big\{\gamma\in\Gamma :\ [\gamma] < [\mathfrak{v}]\big\}.$$

If K is H-asymptotic, then we also have the convex subgroup

$$\Delta \ =\ \big\{\gamma\in\Gamma :\ \gamma^\dagger > v(\mathfrak{v}^\dagger)\big\}$$

of Γ with $\Delta \subseteq \Delta(\mathfrak{v})$. If K is H-asymptotic of Hardy type (Section 1.2), then we have $\Delta = \Delta(\mathfrak{v})$, and hence the relations $\preccurlyeq_{\Delta(\mathfrak{v})}$, $\prec_{\Delta(\mathfrak{v})}$, $\asymp_{\Delta(\mathfrak{v})}$ agree with $\preccurlyeq_{\mathfrak{v}}$, $\prec_{\mathfrak{v}}$, $\asymp_{\mathfrak{v}}$, respectively, from [ADH, p. 407].

Corollary 3.1.19. *Suppose* $\mathfrak{n}^\dagger \succcurlyeq 1$ *and* $[\mathfrak{n}^\dagger] < [\mathfrak{v}]$ *where* $\mathfrak{v} := \mathfrak{v}(A)$ (*so* $0 \neq \mathfrak{v} \prec 1$). *Let* $B \in K[\partial]$ *and* $w \geqslant r$ *be such that* $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^w A$. *Then for all but finitely many* $q \in \mathbb{Q}^>$ *we have* $\mathfrak{w} := \mathfrak{v}(A\mathfrak{n}^q) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$ *and* $B\mathfrak{n}^q \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^w A\mathfrak{n}^q$.

Proof. The case $B = 0$ is trivial, so assume $B \neq 0$. Take $\mathfrak{m}$ as in Lemma 3.1.17, and take $\mathfrak{m}_B$ likewise with B in place of A. By this lemma, $[\mathfrak{m}], [\mathfrak{m}_B] \leqslant [\mathfrak{n}^\dagger] < [\mathfrak{v}]$, hence $\mathfrak{m}, \mathfrak{m}_B \asymp_{\Delta(\mathfrak{v})} 1$. Moreover, for all but finitely many $q \in \mathbb{Q}^>$ we have $A\mathfrak{n}^q \asymp \mathfrak{m}\mathfrak{n}^q A$, $B\mathfrak{n}^q \asymp \mathfrak{m}_B\mathfrak{n}^q B$, and $\mathfrak{v}/\mathfrak{w} \asymp \mathfrak{m}$ where $\mathfrak{w} := \mathfrak{v}(A\mathfrak{n}^q)$; assume that $q \in \mathbb{Q}^>$ has these properties. Then $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^w A$ yields

$$B\mathfrak{n}^q \ \asymp\ \mathfrak{m}_B\mathfrak{n}^q B \ \prec_{\Delta(\mathfrak{v})} \mathfrak{m}\mathfrak{n}^q\mathfrak{v}^w A \ \asymp\ \mathfrak{v}^w A\mathfrak{n}^q.$$

Now $\mathfrak{m} \asymp_{\Delta(\mathfrak{v})} 1$ gives $\mathfrak{v} \asymp_{\Delta(\mathfrak{v})} \mathfrak{w}$, hence $B\mathfrak{n}^q \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^w A\mathfrak{n}^q$. □

The behavior of the span under compositional conjugation. If K is H-asymptotic with asymptotic integration, then $\Psi \cap \Gamma^> \neq \emptyset$, but it is convenient not to require "asymptotic integration" in some lemmas below. Instead: *In this subsection K is H-asymptotic and ungrounded with $\Psi \cap \Gamma^> \neq \emptyset$.* We let ϕ, $\mathfrak{v}$ range over $K^\times$. We say that ϕ is *active* if ϕ is active in K. Recall from [ADH, pp. 290–292] that $\mathfrak{d}$ denotes the derivation $\phi^{-1}\partial$ of K^ϕ, and that

$$A^\phi \ =\ a_r\phi^r\mathfrak{d}^r + \text{lower order terms in } \mathfrak{d}. \tag{3.1.1}$$

Lemma 3.1.20. *Suppose* $\mathfrak{v} := \mathfrak{v}(A) \prec^\flat 1$ *and* $\phi \preccurlyeq 1$ *is active. Then*

$$A \ \asymp_{\Delta(\mathfrak{v})}\ A^\phi, \qquad \mathfrak{v} \ \asymp_{\Delta(\mathfrak{v})}\ \mathfrak{v}(A^\phi) \ \prec^\flat\ 1, \qquad \mathfrak{v}, \mathfrak{v}(A^\phi) \ \prec^\flat_\phi\ 1.$$

Proof. From $\phi^\dagger \prec 1 \preccurlyeq \mathfrak{v}^\dagger$ we get $[\phi] < [\mathfrak{v}]$, so $\phi \asymp_{\Delta(\mathfrak{v})} 1$. Hence $A^\phi \asymp_{\Delta(\mathfrak{v})} A$ by [ADH, 11.1.4]. For the rest we can arrange $A \asymp 1$, so $A^\phi \asymp_{\Delta(\mathfrak{v})} 1$ and $\mathfrak{v} \asymp a_r$. In view of (3.1.1) this yields $\mathfrak{v}(A^\phi) \asymp_{\Delta(\mathfrak{v})} a_r\phi^r \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. So $\mathfrak{v}(A^\phi)^\dagger \asymp \mathfrak{v}^\dagger \succcurlyeq 1$, which gives $\mathfrak{v}(A^\phi) \prec^\flat 1$, and also $\mathfrak{v}, \mathfrak{v}(A^\phi) \prec^\flat_\phi 1$. □

Lemma 3.1.21. *If* $\operatorname{nwt}(A) = r$, *then* $\mathfrak{v}(A^\phi) = 1$ *eventually, and if* $\operatorname{nwt}(A) < r$, *then* $\mathfrak{v}(A^\phi) \prec^\flat_\phi 1$ *eventually.*

Proof. Clearly, if $\operatorname{nwt}(A) = r$, then $\operatorname{dwt}(A^\phi) = r$ and so $\mathfrak{v}(A^\phi) = 1$ eventually. Suppose $\operatorname{nwt}(A) < r$. To show that $\mathfrak{v}(A^\phi) \prec^\flat_\phi 1$ eventually, we may replace A by A^{ϕ_0} for suitable active ϕ_0 and assume that $n := \operatorname{nwt}(A) = \operatorname{dwt}(A^\phi) = \operatorname{dwm}(A^\phi)$ for all active $\phi \preccurlyeq 1$. Thus $v(A^\phi) = v(A) + nv\phi$ for all active $\phi \preccurlyeq 1$ by [ADH, 11.1.11(i)]. Using (3.1.1) we therefore obtain for active $\phi \preccurlyeq 1$:

$$\mathfrak{v}(A^\phi) \asymp a_r\phi^r/a_n\phi^n = \mathfrak{v}(A)\phi^{r-n} \preccurlyeq \phi^{r-n} \preccurlyeq \phi.$$

Take $x \in K^\times$ with $x \not\asymp 1$ and $x' \asymp 1$; then $x \succ 1$, so $x^{-1} \asymp x^\dagger \prec 1$ is active. Hence for active $\phi \preccurlyeq x^{-1}$ we have $\phi \prec^\flat_\phi 1$ and thus $\mathfrak{v}(A^\phi) \prec^\flat_\phi 1$. $\square$

Corollary 3.1.22. *The following conditions on K are equivalent:*

(i) *K is λ-free;*

(ii) $\operatorname{nwt}(B) \leqslant 1$ *for all* $B \in K[\partial]$ (*so* $\mathfrak{v}(B^\phi) \prec^\flat_\phi 1$ *eventually*)*;*

(iii) $\operatorname{nwt}(B) \leqslant 1$ *for all* $B \in K[\partial]$ *of order* 2.

Proof. The implication (i) $\Rightarrow$ (ii) follows from [ADH, 13.7.10] and Lemma 3.1.21, and (ii) $\Rightarrow$ (iii) is clear. Suppose K is not λ-free. Take $\lambda \in K$ such that $\phi^\dagger + \lambda \prec \phi$ for all active ϕ ([ADH, 11.6.1]); set $B := (\partial + \lambda)\partial = \partial^2 + \lambda\partial$. Then for active ϕ we have $B^\phi = \phi^2\big(\delta^2 + (\phi^\dagger + \lambda)\phi^{-1}\delta\big)$, so $\operatorname{dwt}(B^\phi) = 2$. Thus (iii) $\Rightarrow$ (i). $\square$

Lemma 3.1.20 leads to an "eventual" version of Corollary 3.1.14:

Lemma 3.1.23. *Suppose K is λ-free and $B \in K[\partial]$ is such that* $\operatorname{order}(B) \leqslant r$ *and $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+1}A$, where $\mathfrak{v} := \mathfrak{v}(A) \prec^\flat 1$. Then $\mathscr{E}^{\mathrm{e}}(A+B) = \mathscr{E}^{\mathrm{e}}(A)$.*

Proof. By [ADH, 10.1.3, 11.7.18] and Corollary 1.7.10 we can pass to an extension to arrange that K is ω-free. Next, by [ADH, 11.7.23] and (0.7) we extend further to arrange that K is algebraically closed and newtonian, and thus d-valued by Lemma 1.2.9. Then $\mathscr{E}^{\mathrm{e}}(A) = v(\ker^{\neq} A)$ by Proposition 1.4.2, and A splits over K by (0.8). It remains to show that $\mathscr{E}^{\mathrm{e}}(A) \subseteq \mathscr{E}^{\mathrm{e}}(A+B)$: the reverse inclusion then follows by interchanging A and $A+B$, using $\mathfrak{v}(A) \sim \mathfrak{v}(A+B)$. Let $\gamma \in \mathscr{E}^{\mathrm{e}}(A)$. Take $\mathfrak{n} \in \ker^{\neq} A$ with $v\mathfrak{n} = \gamma$. Then $A \in K[\partial](\partial - \mathfrak{n}^\dagger)$ by [ADH, 5.1.21] and so $\mathfrak{n}^\dagger \preccurlyeq \mathfrak{v}^{-1}$, by [ADH, 5.1.22] and Corollary 3.1.6. Now $\mathscr{E}^{\mathrm{e}}(A) \subseteq \mathscr{E}(A)$, so $\gamma = v\mathfrak{n} \in \mathscr{E}(A+B)$ by Corollary 3.1.14. Let $\phi \preccurlyeq 1$ be active; it remains to show that then $\gamma \in \mathscr{E}\big((A+B)^\phi\big)$. By Lemma 3.1.20, $A^\phi \asymp_{\Delta(\mathfrak{v})} A$; also $B^\phi \preccurlyeq B$ by [ADH, 11.1.4]. Lemma 3.1.20 gives $\mathfrak{v} \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}(A^\phi)$, hence $B^\phi \prec_{\Delta(\mathfrak{v})} \mathfrak{v}(A^\phi)^{r+1}A^\phi$. Thus with K^ϕ, A^ϕ, B^ϕ in the role of K, A, B, the above argument leading to $\gamma \in \mathscr{E}(A+B)$ gives $\gamma \in \mathscr{E}(A^\phi + B^\phi) = \mathscr{E}\big((A+B)^\phi\big)$. $\square$

For $r = 1$ we can weaken the hypothesis of λ-freeness:

Corollary 3.1.24. *Suppose K has asymptotic integration, $r = 1$, and $B \in K[\partial]$ of order $\leqslant 1$ satisfies $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^2 A$, where $\mathfrak{v} := \mathfrak{v}(A) \prec^\flat 1$. Then $\mathscr{E}^{\mathrm{e}}(A+B) = \mathscr{E}^{\mathrm{e}}(A)$.*

Proof. Using Lemma 1.2.10 we replace K by an immediate extension to arrange $\mathrm{I}(K) \subseteq K^\dagger$. Then $\mathscr{E}^{\mathrm{e}}(A) = v(\ker^{\neq} A)$ by Lemma 1.4.9. Now argue as in the proof of Lemma 3.1.23. $\square$

In the next proposition and its corollary K is d*-valued with algebraically closed constant field C and divisible group $K^\dagger$ of logarithmic derivatives.* We choose a complement Λ of the $\mathbb{Q}$-linear subspace $K^\dagger$ of K. Then we have the set $\mathscr{E}^{\mathrm{u}}(A)$ of ultimate exceptional values of A with respect to Λ. The following stability result will be crucial in Section 4.4:

Proposition 3.1.25. *Suppose K is $\upomega$-free, $\operatorname{I}(K) \subseteq K^\dagger$, and $B \in K[\partial]$ of order $\leqslant r$ satisfies $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+1}A$, where $\mathfrak{v} := \mathfrak{v}(A) \prec^\flat 1$. Then $\mathscr{E}^{\mathrm{u}}(A+B) = \mathscr{E}^{\mathrm{u}}(A)$.*

Proof. Let Ω be the differential fraction field of the universal exponential extension $\mathrm{U} = K\big[\mathrm{e}(\Lambda)\big]$ of K from Section 2.2. Equip Ω with a spectral extension of the valuation of K; see Section 2.5. Apply Lemma 3.1.23 to Ω in place of K to get $\mathscr{E}^{\mathrm{e}}_\Omega(A+B) = \mathscr{E}^{\mathrm{e}}_\Omega(A)$. Hence $\mathscr{E}^{\mathrm{u}}(A+B) = \mathscr{E}^{\mathrm{u}}(A)$ by (2.5.3). $\square$

In a similar manner we obtain an analogue of Corollary 3.1.24:

Corollary 3.1.26. *Suppose K has asymptotic integration, $\operatorname{I}(K) \subseteq K^\dagger$, $r = 1$, and $B \in K[\partial]$ satisfies* $\operatorname{order}(B) \leqslant 1$ *and $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^2 A$, where $\mathfrak{v} := \mathfrak{v}(A) \prec^\flat 1$. Then $\mathscr{E}^{\mathrm{u}}(A+B) = \mathscr{E}^{\mathrm{u}}(A)$.*

Proof. Let Ω be as in the proof of Proposition 3.1.25. Then Ω is ungrounded by Lemma 2.5.3, hence $|\mathscr{E}^{\mathrm{e}}_\Omega(A)| \leqslant 1$ and $v(\ker^{\neq}_\Omega A) \subseteq \mathscr{E}^{\mathrm{e}}_\Omega(A)$ by [ADH, p. 481]. But $\dim_C \ker_\Omega A = 1$, so $v(\ker^{\neq}_\Omega A) = \mathscr{E}^{\mathrm{e}}_\Omega(A)$. The proof of Lemma 3.1.23 with Ω in place of K now gives $\mathscr{E}^{\mathrm{e}}_\Omega(A+B) = \mathscr{E}^{\mathrm{e}}_\Omega(A)$, so $\mathscr{E}^{\mathrm{u}}(A+B) = \mathscr{E}^{\mathrm{u}}(A)$ by (2.5.3). $\square$

In the "real" case we have the following variant of Proposition 3.1.25:

Proposition 3.1.27. *Suppose $K = H[\mathrm{i}]$, $\mathrm{i}^2 = -1$, where H is a real closed H-field with asymptotic integration such that $H^\dagger = H$ and $\operatorname{I}(H)\mathrm{i} \subseteq K^\dagger$. Let $B \in K[\partial]$ of order $\leqslant r$ be such that $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+1}A$ with $\mathfrak{v} := \mathfrak{v}(A) \prec^\flat 1$. Let Λ be a complement of the subspace $K^\dagger$ of the $\mathbb{Q}$-linear space K. Then $\mathscr{E}^{\mathrm{u}}(A+B) = \mathscr{E}^{\mathrm{u}}(A)$, where the ultimate exceptional values are with respect to Λ.*

Proof. Take an H-closed extension F of H with $C_F = C_H$ as in Corollary 2.5.23. Then the algebraically closed d-valued H-asymptotic extension $L := F[\mathrm{i}]$ of K is $\upomega$-free, $C_L = C$, $\operatorname{I}(L) \subseteq L^\dagger$, and $L^\dagger \cap K = K^\dagger$. Take a complement $\Lambda_L \supseteq \Lambda$ of the subspace $L^\dagger$ of the $\mathbb{Q}$-linear space L. Let $\mathrm{U}_L = L\big[\mathrm{e}(\Lambda_L)\big]$ be the universal exponential extension of L from Section 2.2; it has the universal exponential extension $\mathrm{U} := K\big[\mathrm{e}(\Lambda)\big]$ of K as a differential subring. Let Ω, Ω_L be the differential fraction fields of U, U_L, respectively, and equip Ω_L with a spectral extension of the valuation of L; then the restriction of this valuation to Ω is a spectral extension of the valuation of K (see remarks preceding Lemma 2.5.16). Lemma 3.1.23 applied to Ω_L in place of K yields $\mathscr{E}^{\mathrm{e}}_{\Omega_L}(A+B) = \mathscr{E}^{\mathrm{e}}_{\Omega_L}(A)$, hence $\mathscr{E}^{\mathrm{e}}_\Omega(A+B) = \mathscr{E}^{\mathrm{e}}_\Omega(A)$ by Lemma 2.5.16 and thus $\mathscr{E}^{\mathrm{u}}(A+B) = \mathscr{E}^{\mathrm{u}}(A)$. $\square$

The span of the linear part of a differential polynomial. In this subsection $P \in K\{Y\}^{\neq}$ has order r. Recall that the *linear part* of P is the differential operator

$$L_P \ := \ \sum_n \frac{\partial P}{\partial Y^{(n)}}(0)\,\partial^n \in K[\partial]$$

of order $\leqslant r$. We have $L_{P_{\times\mathfrak{m}}} = L_P\mathfrak{m}$ [ADH, p. 242]; hence items 3.1.9, 3.1.10 and 3.1.12 above yield information about the span of $L_{P_{\times\mathfrak{m}}}$ (provided $L_P \neq 0$). We now want to similarly investigate the span of the linear part

$$L_{P_{+a}} \ = \ \sum_n \frac{\partial P}{\partial Y^{(n)}}(a)\,\partial^n$$

of the additive conjugate P_{+a} of P by some $a \prec 1$. In the next two lemmas we assume $\operatorname{order}(L_P) = r$ (in particular, $L_P \neq 0$), $\mathfrak{v}(L_P) \prec 1$, and $a \prec 1$, we set

$$L := L_P, \quad L^+ := L_{P_{+a}}, \quad \mathfrak{v} := \mathfrak{v}(L),$$

and set $L_n := \frac{\partial P}{\partial Y^{(n)}}(0)$ and $L_n^+ := \frac{\partial P}{\partial Y^{(n)}}(a)$, so $L = \sum_n L_n \partial^n$, $L^+ = \sum_n L_n^+ \partial^n$. Recall from [ADH, 4.2] the decomposition of P into homogeneous parts: $P = \sum_d P_d$ where $P_d = \sum_{|\boldsymbol{i}|=d} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$; we set $P_{>1} := \sum_{d>1} P_d$.

Lemma 3.1.28. *Suppose $P_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v} P_1$ and $n \leqslant r$. Then*

(i) *$L_r^+ \sim_{\Delta(\mathfrak{v})} L_r$, and thus $\operatorname{order}(L^+) = \operatorname{order}(L) = r$;*
(ii) *if $L_n \asymp_{\Delta(\mathfrak{v})} L$, then $L_n^+ \sim_{\Delta(\mathfrak{v})} L_n$, and so $v(L_n^+) = v(L_n)$;*
(iii) *if $L_n \prec_{\Delta(\mathfrak{v})} L$, then $L_n^+ \prec_{\Delta(\mathfrak{v})} L$, and so $v(L_n^+) > v(L)$.*

In particular, $L^+ \sim_{\Delta(\mathfrak{v})} L$, $\operatorname{dwt} L^+ = \operatorname{dwt} L$, and $\mathfrak{v}(L^+) \sim_{\Delta(\mathfrak{v})} \mathfrak{v}$.

Proof. Take $Q, R \in K\{Y\}$ with $\deg_{Y^{(n)}} Q \leqslant 0$ and $R \in Y^{(n)} K\{Y\}$, such that

$$P \;=\; Q + (L_n + R)Y^{(n)}, \qquad \text{so} \qquad \frac{\partial P}{\partial Y^{(n)}} \;=\; \frac{\partial R}{\partial Y^{(n)}} Y^{(n)} + L_n + R.$$

Now $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v} P_1$, so $\frac{\partial P}{\partial Y^{(n)}} - L_n \prec_{\Delta(\mathfrak{v})} \mathfrak{v} P_1$. In $K[\partial]$ we thus have

$$L_n^+ - L_n \;=\; \frac{\partial P}{\partial Y^{(n)}}(a) - L_n \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v} L \;\asymp\; L_r.$$

So $L_n^+ - L_n \prec_{\Delta(\mathfrak{v})} L$ and (taking $r = n$) $L_r^+ - L_r \prec_{\Delta(\mathfrak{v})} L_r$. This yields (i)–(iii). □

Lemma 3.1.29. *Suppose $P_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{m+1} P_1$, and let $A, B \in K[\partial]$ be such that $L = A + B$, $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{m+1} L$. Then*

$$L^+ \;=\; A + B^+ \quad \text{where } B^+ \in K[\partial],\ B^+ \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{m+1} L^+.$$

In particular, $L - L^+ \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{m+1} L$.

Proof. Let $A_n, B_n \in K$ be such that $A = \sum_n A_n \partial^n$ and $B = \sum_n B_n \partial^n$, so $L_n = A_n + B_n$. Let any n (possibly $> r$) be given and take $Q, R \in K\{Y\}$ as in the proof of Lemma 3.1.28. Then $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{m+1} P_1$. Since $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{m+1} L$, this yields

$$\frac{\partial P}{\partial Y^{(n)}} - A_n \;=\; \frac{\partial R}{\partial Y^{(n)}} Y^{(n)} + B_n + R \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{m+1} P_1.$$

We have $L_n^+ = \frac{\partial P}{\partial Y^{(n)}}(a)$, so

$$L_n^+ - A_n \;=\; \frac{\partial P}{\partial Y^{(n)}}(a) - A_n \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{m+1} L.$$

By Lemma 3.1.28 we have $L^+ \sim_{\Delta(\mathfrak{v})} L$, hence $B^+ = L^+ - A \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{m+1} L^+$. □

3.2. Holes and Slots

Throughout this section K is an H-asymptotic field with small derivation and with rational asymptotic integration. We set $\Gamma := v(K^\times)$. So K is pre-d-valued, $\Gamma \neq \{0\}$ has no least positive element, and $\Psi \cap \Gamma^> \neq \emptyset$. We let a, b, f, g range over K, and ϕ, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{v}$, $\mathfrak{w}$ (possibly decorated) over $K^\times$. As at the end of the previous section we shorten "active in K" to "active".

Holes. A **hole** in K is a triple $(P,\mathfrak{m},\widehat{a})$ where $P\in K\{Y\}\setminus K$ and $\widehat{a}$ is an element of $\widehat{K}\setminus K$, for some immediate asymptotic extension $\widehat{K}$ of K, such that $\widehat{a}\prec\mathfrak{m}$ and $P(\widehat{a})=0$. (The extension $\widehat{K}$ may vary with $\widehat{a}$.) The **order**, **degree**, and **complexity** of a hole $(P,\mathfrak{m},\widehat{a})$ in K are defined as the order, (total) degree, and complexity, respectively, of the differential polynomial P. A hole $(P,\mathfrak{m},\widehat{a})$ in K is called **minimal** if no hole in K has smaller complexity; then P is a minimal annihilator of $\widehat{a}$ over K.

If $(P,\mathfrak{m},\widehat{a})$ is a hole in K, then $\widehat{a}$ is a K-external zero of P, in the sense of Section 1.7. Conversely, every K-external zero $\widehat{a}$ of a differential polynomial $P\in K\{Y\}^{\neq}$ gives for every $\mathfrak{m}\succ\widehat{a}$ a hole $(P,\mathfrak{m},\widehat{a})$ in K. By Proposition 1.7.35 and Corollary 1.7.41:

Lemma 3.2.1. *Let $r\in\mathbb{N}^{\geqslant 1}$, and suppose K is $\uplambda$-free. Then*

$$K \text{ is } \upomega\text{-free and } r\text{-newtonian} \iff K \text{ has no hole of order} \leqslant r.$$

Thus for $\upomega$-free K, being newtonian is equivalent to having no holes. Recall that K being henselian is equivalent to K having no proper immediate algebraic valued field extension, and hence to K having no hole of order 0.

Minimal holes are like the "minimal counterexamples" in certain combinatorial settings, and we need to understand such holes in a rather detailed way for later use in inductive arguments. Below we also consider the more general notion of *Z-minimal hole*, which has an important role to play as well. We recall that $Z(K,\widehat{a})$ is the set of all $Q\in K\{Y\}^{\neq}$ that vanish at $(K,\widehat{a})$ as defined in [ADH, 11.4].

Lemma 3.2.2. *Let $(P,\mathfrak{m},\widehat{a})$ be a hole in K. Then $P\in Z(K,\widehat{a})$. If $(P,\mathfrak{m},\widehat{a})$ is minimal, then P is an element of minimal complexity of $Z(K,\widehat{a})$.*

Proof. Let a, $\mathfrak{v}$ with $\widehat{a}-a\prec\mathfrak{v}$. Since $\widehat{a}\notin K$ lies in an immediate extension of K we can take $\mathfrak{n}$ with $\mathfrak{n}\asymp\widehat{a}-a$. By [ADH, 11.2.1] we then have $\operatorname{ndeg}_{\prec\mathfrak{v}}P_{+a}\geqslant\operatorname{ndeg}P_{+a,\times\mathfrak{n}}\geqslant 1$. Hence $P\in Z(K,\widehat{a})$. Suppose P is not of minimal complexity in $Z(K,\widehat{a})$. Take $Q\in Z(K,\widehat{a})$ of minimal complexity. Then [ADH, 11.4.8] yields a K-external zero $\widehat{b}$ of Q, and any $\mathfrak{n}\succ\widehat{b}$ gives a hole $(Q,\mathfrak{n},\widehat{b})$ in K of smaller complexity than $(P,\mathfrak{m},\widehat{a})$. □

In connection with the next result, note that K being 0-newtonian just means that K is henselian as a valued field.

Corollary 3.2.3. *Suppose K is $\uplambda$-free and has a minimal hole of order $r\geqslant 1$. Then K is $(r-1)$-newtonian, and $\upomega$-free if $r\geqslant 2$.*

Proof. This is clear for $r=1$ (and doesn't need $\uplambda$-freeness), and for $r\geqslant 2$ follows from Lemma 3.2.1. □

Corollary 3.2.4. *Suppose K is $\upomega$-free and has a minimal hole of order $r\geqslant 2$. Assume also that C is algebraically closed and Γ is divisible. Then K is* d*-valued, r-linearly closed, and r-linearly newtonian.*

Proof. This follows from Lemma 1.2.9, Corollary 1.7.42, and Corollary 3.2.3. □

Here is a linear version of Lemma 3.2.1:

Lemma 3.2.5. *If K is $\uplambda$-free, then*

$$K \text{ is } 1\text{-linearly newtonian} \iff K \text{ has no hole of degree } 1 \text{ and order } 1.$$

If $r \in \mathbb{N}^{\geqslant 1}$ and K is ω-free, then

$$K \text{ is } r\text{-linearly newtonian} \iff K \text{ has no hole of degree } 1 \text{ and order} \leqslant r.$$

Proof. The first statement follows from Lemma 1.7.33, and the second statement from Lemma 1.7.34. □

Corollary 3.2.6. *If K is ω-free and has a minimal hole in K of order r and degree > 1, then K is r-linearly newtonian.*

Lemma 3.2.7. *Suppose K has a hole $(P, \mathfrak{m}, \widehat{a})$ of degree 1, and $L_P \in K[\partial]^{\neq}$ splits over K. Then K has a hole of complexity $(1, 1, 1)$.*

Proof. Let $(P, \mathfrak{m}, \widehat{a})$ as in the hypothesis have minimal order. Then $\operatorname{order} P \geqslant 1$, so $\operatorname{order} P = \operatorname{order} L_P$. Take $A, B \in K[\partial]$ such that $\operatorname{order} A = 1$ and $L_P = AB$. If $\operatorname{order} B = 0$, then $(P, \mathfrak{m}, \widehat{a})$ has complexity $(1, 1, 1)$. Assume $\operatorname{order} B \geqslant 1$. Then $B(\widehat{a}) \notin K$: otherwise, taking $Q \in K\{Y\}$ of degree 1 with $L_Q = B$ and $Q(0) = -B(\widehat{a})$ yields a hole $(Q, \mathfrak{m}, \widehat{a})$ in K where $\deg Q = 1$ and L_Q splits over K, and $(Q, \mathfrak{m}, \widehat{a})$ has smaller order than $(P, \mathfrak{m}, \widehat{a})$. Set $\widehat{b} := B(\widehat{a})$ and take $R \in K\{Y\}$ of degree 1 with $L_R = A$ and $R(0) = P(0)$. Then

$$R(\widehat{b}) \;=\; R(0) + L_R(\widehat{b}) \;=\; P(0) + L_P(\widehat{a}) \;=\; P(\widehat{a}) \;=\; 0,$$

hence for any $\mathfrak{n} \succ \widehat{b}$, $(R, \mathfrak{n}, \widehat{b})$ is a hole in K of complexity $(1, 1, 1)$. □

Corollary 3.2.8. *Suppose K is ω-free, C is algebraically closed, and Γ is divisible. Then every minimal hole in K of degree 1 has order 1. If in addition K is 1-linearly newtonian, then every minimal hole in K has degree > 1.*

Proof. The first statement follows from Corollary 3.2.4 and the preceding lemma. For the second statement, use the first and Lemma 3.2.5. □

Let $(P, \mathfrak{m}, \widehat{a})$ be a hole in K. We say $(P, \mathfrak{m}, \widehat{a})$ is **Z-minimal** if P has minimal complexity in $Z(K, \widehat{a})$. Thus if $(P, \mathfrak{m}, \widehat{a})$ is minimal, then it is Z-minimal by Lemma 3.2.2. If $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, then by [ADH, remarks following 11.4.3], the differential polynomial P is a minimal annihilator of $\widehat{a}$ over K. Note also that $\operatorname{ndeg} P_{\times \mathfrak{m}} \geqslant 1$ by [ADH, 11.2.1]. In more detail:

Lemma 3.2.9. *Let $(P, \mathfrak{m}, \widehat{a})$ be a hole in K. Then for all $\mathfrak{n}$ with $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$,*

$$1 \;\leqslant\; \operatorname{dmul} P_{\times \mathfrak{n}} \;\leqslant\; \operatorname{ddeg} P_{\times \mathfrak{n}} \;\leqslant\; \operatorname{ddeg} P_{\times \mathfrak{m}}.$$

In particular, $\operatorname{ddeg}_{\prec \mathfrak{m}} P \geqslant 1$.

Proof. Assume $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$. Then $\widehat{a} = \mathfrak{n}\widehat{b}$ with $\widehat{b} \prec 1$; put $Q := P_{\times \mathfrak{n}} \in K\{Y\}^{\neq}$. Then $Q(\widehat{b}) = 0$, hence $D_Q(0) = 0$ and so $\operatorname{dmul} Q = \operatorname{dmul} P_{\times \mathfrak{n}} \geqslant 1$. The rest follows from [ADH, 6.6.5(ii), 6.6.7, 6.6.9] and $\Gamma^{>}$ having no least element. □

In the next lemma, (λ_ρ), (ω_ρ) are pc-sequences in K as in [ADH, 11.5, 11.7].

Lemma 3.2.10. *Suppose K is λ-free and $\omega \in K$ is such that $\omega_\rho \rightsquigarrow \omega$ (so K is not ω-free). Then we have a hole $(P, \mathfrak{m}, \lambda)$ in K where $P = 2Y' + Y^2 + \omega$ and $\lambda_\rho \rightsquigarrow \lambda$, and each such hole in K is a Z-minimal hole in K.*

Proof. From [ADH, 11.7.13] we obtain λ in an immediate asymptotic extension of K such that $\lambda_\rho \rightsquigarrow \lambda$ and $P(\lambda) = 0$. Taking any $\mathfrak{m}$ with $\lambda \prec \mathfrak{m}$ then yields a hole $(P, \mathfrak{m}, \lambda)$ in K with $\lambda_\rho \rightsquigarrow \lambda$, and each such hole in K is a Z-minimal hole in K by [ADH, 11.4.13, 11.7.12]. □

Corollary 3.2.11. *If K is λ-free but not ω-free, then each minimal hole in K of positive order has complexity $(1,1,1)$ or complexity $(1,1,2)$. If K is a Liouville closed H-field and not ω-free, then $(P,\mathfrak{m},\lambda)$ is a minimal hole of complexity $(1,1,2)$, where ω, P, λ, $\mathfrak{m}$ are as in Lemma 3.2.10.*

Here the second part uses Corollary 1.7.29 and Lemma 3.2.5.

Slots. In some arguments the notion of a hole in K turns out to be too stringent. Therefore we introduce a more flexible version of it:

Definition 3.2.12. A **slot** in K is a triple $(P,\mathfrak{m},\widehat{a})$ where $P \in K\{Y\} \setminus K$ and $\widehat{a}$ is an element of $\widehat{K} \setminus K$, for some immediate asymptotic extension $\widehat{K}$ of K, such that $\widehat{a} \prec \mathfrak{m}$ and $P \in Z(K,\widehat{a})$. The **order**, **degree**, and **complexity** of such a slot in K are defined to be the order, degree, and complexity of the differential polynomial P, respectively. A slot in K of degree 1 is also called a **linear** slot in K. A slot $(P,\mathfrak{m},\widehat{a})$ in K is Z**-minimal** if P is of minimal complexity among elements of $Z(K,\widehat{a})$.

Thus by Lemma 3.2.2, holes in K are slots in K, and a hole in K is Z-minimal iff it is Z-minimal as a slot in K. From [ADH, 11.4.13] we obtain:

Corollary 3.2.13. *Let $(P,\mathfrak{m},\widehat{a})$ be a Z-minimal slot in K and (a_ρ) be a divergent pc-sequence in K such that $a_\rho \rightsquigarrow \widehat{a}$. Then P is a minimal differential polynomial of (a_ρ) over K.*

We say that slots $(P,\mathfrak{m},\widehat{a})$ and $(Q,\mathfrak{n},\widehat{b})$ in K are **equivalent** if $P=Q$, $\mathfrak{m}=\mathfrak{n}$, and $v(\widehat{a}-a)=v(\widehat{b}-a)$ for all a; note that then $Z(K,\widehat{a})=Z(K,\widehat{b})$, so $(P,\mathfrak{m},\widehat{a})$ is Z-minimal iff $(P,\mathfrak{m},\widehat{b})$ is Z-minimal. Clearly this is an equivalence relation on the class of slots in K. The following lemma often allows us to pass from a Z-minimal slot to a Z-minimal hole:

Lemma 3.2.14. *Let $(P,\mathfrak{m},\widehat{a})$ be a Z-minimal slot in K. Then $(P,\mathfrak{m},\widehat{a})$ is equivalent to a Z-minimal hole in K.*

Proof. By [ADH, 11.4.8] we obtain $\widehat{b}$ in an immediate asymptotic extension of K with $P(\widehat{b})=0$ and $v(\widehat{a}-a)=v(\widehat{b}-a)$ for all a. In particular $\widehat{b}\notin K$, $\widehat{b}\prec\mathfrak{m}$, so $(P,\mathfrak{m},\widehat{b})$ is a hole in K equivalent to $(P,\mathfrak{m},\widehat{a})$. $\square$

By [ADH, 11.4.8] the extension below containing $\widehat{b}$ is not required to be immediate:

Corollary 3.2.15. *If $(P,\mathfrak{m},\widehat{a})$ is a Z-minimal hole in K and $\widehat{b}$ in an asymptotic extension of K satisfies $P(\widehat{b})=0$ and $v(\widehat{a}-a)=v(\widehat{b}-a)$ for all a, then there is an isomorphism $K\langle\widehat{a}\rangle \to K\langle\widehat{b}\rangle$ of valued differential fields over K sending $\widehat{a}$ to $\widehat{b}$.*

In particular, equivalent Z-minimal holes $(P,\mathfrak{m},\widehat{a})$, $(P,\mathfrak{m},\widehat{b})$ in K yield an isomorphism $K\langle\widehat{a}\rangle \to K\langle\widehat{b}\rangle$ of valued differential fields over K sending $\widehat{a}$ to $\widehat{b}$.

From Lemmas 3.2.1 and 3.2.14 we obtain:

Corollary 3.2.16. *Let $r\in\mathbb{N}^{\geqslant 1}$, and suppose K is ω-free. Then*

$$K \text{ is } r\text{-newtonian} \iff K \text{ has no slot of order} \leqslant r.$$

Let $(P,\mathfrak{m},\widehat{a})$ be a slot in K. Then $(bP,\mathfrak{m},\widehat{a})$ for $b\neq 0$ is a slot in K of the same complexity as $(P,\mathfrak{m},\widehat{a})$, and if $(P,\mathfrak{m},\widehat{a})$ is Z-minimal, then so is $(bP,\mathfrak{m},\widehat{a})$; likewise with "hole in K" in place of "slot in K". For active ϕ we have the **compositional conjugate** $(P^{\phi},\mathfrak{m},\widehat{a})$ by ϕ of $(P,\mathfrak{m},\widehat{a})$: it is a slot in K^{ϕ} of the same complexity as $(P,\mathfrak{m},\widehat{a})$, it is Z-minimal if $(P,\mathfrak{m},\widehat{a})$ is, and it is a hole (minimal hole) in K^{ϕ} if $(P,\mathfrak{m},\widehat{a})$ is a hole (minimal hole, respectively) in K. If the slots $(P,\mathfrak{m},\widehat{a})$, $(Q,\mathfrak{n},\widehat{b})$ in K are equivalent, then so are $(bP,\mathfrak{m},\widehat{a})$, $(bQ,\mathfrak{n},\widehat{b})$ for $b\neq 0$, as well as the slots $(P^{\phi},\mathfrak{m},\widehat{a})$, $(Q^{\phi},\mathfrak{n},\widehat{b})$ in K^{ϕ} for active ϕ.

The following conventions are in force in the rest of this section:

We let r range over natural numbers $\geqslant 1$ and let $(P,\mathfrak{m},\widehat{a})$ denote a slot in K of order r, so $P\notin K[Y]$ has order r. We set $w:=\operatorname{wt}(P)$, so $w\geqslant r\geqslant 1$.

Thus $\operatorname{wt}(P_{+a})=\operatorname{wt}(P_{\times\mathfrak{n}})=\operatorname{wt}(P^{\phi})=w$.

Refinements and multiplicative conjugates. For a, $\mathfrak{n}$ such that $\widehat{a}-a\prec\mathfrak{n}\preccurlyeq\mathfrak{m}$ we obtain a slot $(P_{+a},\mathfrak{n},\widehat{a}-a)$ in K of the same complexity as $(P,\mathfrak{m},\widehat{a})$ [ADH, 4.3, 11.4]. Slots of this form are said to **refine** $(P,\mathfrak{m},\widehat{a})$ and are called **refinements** of $(P,\mathfrak{m},\widehat{a})$. A refinement of a refinement of $(P,\mathfrak{m},\widehat{a})$ is itself a refinement of $(P,\mathfrak{m},\widehat{a})$. If $(P,\mathfrak{m},\widehat{a})$ is Z-minimal, then so is any refinement of $(P,\mathfrak{m},\widehat{a})$. If $(P,\mathfrak{m},\widehat{a})$ is a hole in K, then so is each of its refinements, and likewise with "minimal hole" in place of "hole". For active ϕ, $(P_{+a},\mathfrak{n},\widehat{a}-a)$ refines $(P,\mathfrak{m},\widehat{a})$ iff $(P^{\phi}_{+a},\mathfrak{n},\widehat{a}-a)$ refines $(P^{\phi},\mathfrak{m},\widehat{a})$. If $(P,\mathfrak{m},\widehat{a})$, $(P,\mathfrak{m},\widehat{b})$ are equivalent slots in K and $(P_{+a},\mathfrak{n},\widehat{a}-a)$ refines $(P,\mathfrak{m},\widehat{a})$, then $(P_{+a},\mathfrak{n},\widehat{b}-a)$ refines $(P,\mathfrak{m},\widehat{b})$, and the slots $(P_{+a},\mathfrak{n},\widehat{a}-a)$, $(P_{+a},\mathfrak{n},\widehat{b}-a)$ in K are equivalent. Conversely, if $(P,\mathfrak{m},\widehat{a})$ and $(P,\mathfrak{m},\widehat{b})$ are slots in K with equivalent refinements, then $(P,\mathfrak{m},\widehat{a})$ and $(P,\mathfrak{m},\widehat{b})$ are equivalent.

Lemma 3.2.17. *Let $(P_{+a},\mathfrak{n},\widehat{a}-a)$ be a slot in K. Then $(P_{+a},\mathfrak{n},\widehat{a}-a)$ refines $(P,\mathfrak{m},\widehat{a})$, or $(P,\mathfrak{m},\widehat{a})$ refines $(P_{+a},\mathfrak{n},\widehat{a}-a)$.*

Proof. If $\mathfrak{n}\preccurlyeq\mathfrak{m}$, then $\widehat{a}-a\prec\mathfrak{n}\preccurlyeq\mathfrak{m}$, so $(P_{+a},\mathfrak{n},\widehat{a}-a)$ refines $(P,\mathfrak{m},\widehat{a})$, whereas if $\mathfrak{m}\prec\mathfrak{n}$, then $(\widehat{a}-a)-(-a)=\widehat{a}\prec\mathfrak{m}\preccurlyeq\mathfrak{n}$, so

$$(P,\mathfrak{m},\widehat{a})=\big((P_{+a})_{+(-a)},\mathfrak{m},(\widehat{a}-a)-(-a)\big)$$

refines $(P_{+a},\mathfrak{n},\widehat{a}-a)$. ☐

Lemma 3.2.18. *Let $Q\in K\{Y\}^{\neq}$ be such that $Q\notin Z(K,\widehat{a})$. Then there is a refinement $(P_{+a},\mathfrak{n},\widehat{a}-a)$ of $(P,\mathfrak{m},\widehat{a})$ such that $\operatorname{ndeg} Q_{+a,\times\mathfrak{n}}=0$ and $\widehat{a}-a\prec\mathfrak{n}\prec\widehat{a}$.*

Proof. Take b, $\mathfrak{v}$ such that $\widehat{a}-b\prec\mathfrak{v}$ and $\operatorname{ndeg}_{\prec\mathfrak{v}} Q_{+b}=0$. We shall find an a such that $\operatorname{ndeg}_{\prec\mathfrak{v}} Q_{+a}=0$, $\widehat{a}-a\preccurlyeq\widehat{a}$, and $\widehat{a}-a\prec\mathfrak{v}$: if $\widehat{a}-b\preccurlyeq\widehat{a}$, we take $a:=b$; if $\widehat{a}-b\succ\widehat{a}$, then $-b\sim\widehat{a}-b$ and so $\operatorname{ndeg}_{\prec\mathfrak{v}} Q=\operatorname{ndeg}_{\prec\mathfrak{v}} Q_{+b}=0$ by [ADH, 11.2.7], hence $a:=0$ works. We next arrange $\widehat{a}-a\prec\widehat{a}$: if $\widehat{a}-a\asymp\widehat{a}$, take a_1 with $\widehat{a}-a_1\prec\widehat{a}$, so $a-a_1\prec\mathfrak{v}$, hence $\operatorname{ndeg}_{\prec\mathfrak{v}} Q_{+a_1}=\operatorname{ndeg}_{\prec\mathfrak{v}} Q_{+a}=0$, and thus a can be replaced by a_1. Since $\Gamma^{>}$ has no least element, we can choose $\mathfrak{n}$ with $\widehat{a}-a\prec\mathfrak{n}\prec\widehat{a},\mathfrak{v}$, and then $(P_{+a},\mathfrak{n},\widehat{a}-a)$ refines $(P,\mathfrak{m},\widehat{a})$ as desired. ☐

If $(P_{+a},\mathfrak{m},\widehat{a}-a)$ refines $(P,\mathfrak{m},\widehat{a})$, then $D_{P_{+a,\times\mathfrak{m}}}=D_{P_{\times\mathfrak{m},+(a/\mathfrak{m})}}=D_{P_{\times\mathfrak{m}}}$ by [ADH, 6.6.5(iii)], and thus

$$\operatorname{ddeg} P_{+a,\times\mathfrak{m}} \;=\; \operatorname{ddeg} P_{\times\mathfrak{m}}, \qquad \operatorname{dmul} P_{+a,\times\mathfrak{m}} \;=\; \operatorname{dmul} P_{\times\mathfrak{m}}.$$

In combination with Lemma 3.2.9 this has some useful consequences:

Corollary 3.2.19. *Suppose $(P,\mathfrak{m},\widehat{a})$ is a hole in K such that $\operatorname{ddeg} P_{\times\mathfrak{m}} = 1$. Then $\operatorname{ddeg}_{\prec\mathfrak{m}} P = 1$, and for all $\mathfrak{n}$ with $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$, $(P,\mathfrak{n},\widehat{a})$ refines $(P,\mathfrak{m},\widehat{a})$ with $\operatorname{ddeg} P_{\times\mathfrak{n}} = \operatorname{dmul} P_{\times\mathfrak{n}} = 1$.*

Corollary 3.2.20. *Suppose $(P_{+a},\mathfrak{n},\widehat{a}-a)$ refines the hole $(P,\mathfrak{m},\widehat{a})$ in K. Then*

$$\operatorname{ddeg} P_{\times\mathfrak{m}} \ = \ 1 \implies \operatorname{ddeg} P_{+a,\times\mathfrak{n}} \ = \ \operatorname{dmul} P_{+a,\times\mathfrak{n}} \ = \ 1.$$

Proof. Use

$$1 \leqslant \operatorname{dmul} P_{+a,\times\mathfrak{n}} \leqslant \operatorname{ddeg} P_{+a,\times\mathfrak{n}} \leqslant \operatorname{ddeg} P_{+a,\times\mathfrak{m}} = \operatorname{ddeg} P_{\times\mathfrak{m}},$$

where the first inequality follows from Lemma 3.2.9 applied to $(P_{+a},\mathfrak{n},\widehat{a}-a)$. ☐

If $(P_{+a},\mathfrak{m},\widehat{a}-a)$ refines $(P,\mathfrak{m},\widehat{a})$, then in analogy with ddeg and dmul,

$$\operatorname{ndeg} P_{+a,\times\mathfrak{m}} \ = \ \operatorname{ndeg} P_{\times\mathfrak{m}}, \qquad \operatorname{nmul} P_{+a,\times\mathfrak{m}} \ = \ \operatorname{nmul} P_{\times\mathfrak{m}}.$$

(Use compositional conjugation by active ϕ.) Lemma 3.2.9 goes through for slots, provided we use ndeg and nmul instead of ddeg and dmul:

Lemma 3.2.21. *Suppose $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$. Then*

$$1 \ \leqslant \ \operatorname{nmul} P_{\times\mathfrak{n}} \ \leqslant \ \operatorname{ndeg} P_{\times\mathfrak{n}} \ \leqslant \ \operatorname{ndeg} P_{\times\mathfrak{m}}.$$

Proof. By [ADH, 11.2.3(iii), 11.2.5] it is enough to show $\operatorname{nmul} P_{\times\mathfrak{n}} \geqslant 1$. Replacing $(P,\mathfrak{m},\widehat{a})$ by its refinement $(P,\mathfrak{n},\widehat{a})$ we arrange $\mathfrak{m}=\mathfrak{n}$. Now $\Gamma^{>}$ has no smallest element, so by definition of $Z(K,\widehat{a})$ and [ADH, p. 483] we have

$$1 \ \leqslant \ \operatorname{ndeg}_{\prec\mathfrak{m}} P \ = \ \max\big\{\operatorname{nmul} P_{\times\mathfrak{v}} : \mathfrak{v} \prec \mathfrak{m}\big\}.$$

Thus by [ADH, 11.2.5] we can take $\mathfrak{v}$ with $\widehat{a} \prec \mathfrak{v} \prec \mathfrak{m}$ with $\operatorname{nmul} P_{\times\mathfrak{v}} \geqslant 1$, and hence $\operatorname{nmul} P_{\times\mathfrak{m}} \geqslant 1$, again by [ADH, 11.2.5]. ☐

Lemma 3.2.21 yields results analogous to Corollaries 3.2.19 and 3.2.20 above:

Corollary 3.2.22. *If $\operatorname{ndeg} P_{\times\mathfrak{m}} = 1$, then for all $\mathfrak{n}$ with $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$, $(P,\mathfrak{n},\widehat{a})$ refines $(P,\mathfrak{m},\widehat{a})$ and $\operatorname{ndeg} P_{\times\mathfrak{n}} = \operatorname{nmul} P_{\times\mathfrak{n}} = 1$.*

Corollary 3.2.23. *If $(P_{+a},\mathfrak{n},\widehat{a}-a)$ refines $(P,\mathfrak{m},\widehat{a})$, then*

$$\operatorname{ndeg} P_{\times\mathfrak{m}} \ = \ 1 \implies \operatorname{ndeg} P_{+a,\times\mathfrak{n}} \ = \ \operatorname{nmul} P_{+a,\times\mathfrak{n}} \ = \ 1.$$

Any triple $(P_{\times\mathfrak{n}},\mathfrak{m}/\mathfrak{n},\widehat{a}/\mathfrak{n})$ is also a slot in K, with the same complexity as $(P,\mathfrak{m},\widehat{a})$; it is called the **multiplicative conjugate** of $(P,\mathfrak{m},\widehat{a})$ by $\mathfrak{n}$. If $(P,\mathfrak{m},\widehat{a})$ is Z-minimal, then so is any multiplicative conjugate. If $(P,\mathfrak{m},\widehat{a})$ is a hole in K, then so is any multiplicative conjugate; likewise with "minimal hole" in place of "hole". If two slots in K are equivalent, then so are their multiplicative conjugates by $\mathfrak{n}$.

Refinements and multiplicative conjugates interact in the following way: Suppose the slot $(P_{+a},\mathfrak{n},\widehat{a}-a)$ refines $(P,\mathfrak{m},\widehat{a})$. Multiplicative conjugation $(P_{+a},\mathfrak{n},\widehat{a}-a)$ in K by $\mathfrak{v}$ then results in the slot $(P_{+a,\times\mathfrak{v}},\mathfrak{n}/\mathfrak{v},(\widehat{a}-a)/\mathfrak{v})$ in K. On the other hand, first taking the multiplicative conjugate $(P_{\times\mathfrak{v}},\mathfrak{m}/\mathfrak{v},\widehat{a}/\mathfrak{v})$ of $(P,\mathfrak{m},\widehat{a})$ by $\mathfrak{v}$ and then refining to $(P_{\times\mathfrak{v},+a/\mathfrak{v}},\mathfrak{n}/\mathfrak{v},\widehat{a}/\mathfrak{v}-a/\mathfrak{v})$ results in the same slot in K, thanks to the identity $P_{+a,\times\mathfrak{v}} = P_{\times\mathfrak{v},+a/\mathfrak{v}}$.

Quasilinear slots. Note that $\operatorname{ndeg} P_{\times\mathfrak{m}} \geqslant 1$ by Lemma 3.2.21. We call $(P, \mathfrak{m}, \widehat{a})$ **quasilinear** if $P_{\times\mathfrak{m}}$ is quasilinear, that is, $\operatorname{ndeg} P_{\times\mathfrak{m}} = 1$. If $(P, \mathfrak{m}, \widehat{a})$ is quasilinear, then so is any slot in K equivalent to $(P, \mathfrak{m}, \widehat{a})$, any multiplicative conjugate of $(P, \mathfrak{m}, \widehat{a})$, as well as any refinement of $(P, \mathfrak{m}, \widehat{a})$, by Corollary 3.2.23. If $(P, \mathfrak{m}, \widehat{a})$ is linear, then it is quasilinear by Lemma 3.2.21.

Let (a_ρ) be a divergent pc-sequence in K with $a_\rho \rightsquigarrow \widehat{a}$ and for each index ρ, let $\mathfrak{m}_\rho \in K^\times$ be such that $\mathfrak{m}_\rho \asymp \widehat{a} - a_\rho$. Take an index ρ_0 such that $\mathfrak{m}_\sigma \prec \mathfrak{m}_\rho \prec \mathfrak{m}$ for all $\sigma > \rho \geqslant \rho_0$, cf. [ADH, 2.2].

Lemma 3.2.24. *Let $\sigma \geqslant \rho \geqslant \rho_0$. Then*

(i) $(P_{+a_{\rho+1}}, \mathfrak{m}_\rho, \widehat{a} - a_{\rho+1})$ *is a refinement of* $(P, \mathfrak{m}, \widehat{a})$*;*
(ii) *if* $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ *is a refinement of* $(P, \mathfrak{m}, \widehat{a})$*, then* $\mathfrak{m}_\rho \preccurlyeq \mathfrak{n}$ *for all sufficiently large ρ, and for such ρ,* $(P_{+a_{\rho+1}}, \mathfrak{m}_\rho, \widehat{a} - a_{\rho+1})$ *refines* $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$*;*
(iii) $(P_{+a_{\sigma+1}}, \mathfrak{m}_\sigma, \widehat{a} - a_{\sigma+1})$ *refines* $(P_{+a_{\rho+1}}, \mathfrak{m}_\rho, \widehat{a} - a_{\rho+1})$*.*

Proof. Part (i) follows from $\widehat{a} - a_{\rho+1} \asymp \mathfrak{m}_{\rho+1} \prec \mathfrak{m}_\rho \preccurlyeq \mathfrak{m}$. For (ii) let $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ be a refinement of $(P, \mathfrak{m}, \widehat{a})$. Since $\widehat{a} - a \prec \mathfrak{n}$, we have $\mathfrak{m}_\rho \preccurlyeq \mathfrak{n}$ for all sufficiently large ρ. For such ρ, with $b := a_{\rho+1} - a$ we have

$$(P_{+a_{\rho+1}}, \mathfrak{m}_\rho, \widehat{a} - a_{\rho+1}) \;=\; \big((P_{+a})_{+b}, \mathfrak{m}_\rho, (\widehat{a} - a) - b\big)$$

and

$$(\widehat{a} - a) - b \;=\; \widehat{a} - a_{\rho+1} \;\asymp\; \mathfrak{m}_{\rho+1} \;\prec\; \mathfrak{m}_\rho \;\preccurlyeq\; \mathfrak{n}.$$

Hence $(P_{+a_{\rho+1}}, \mathfrak{m}_\rho, \widehat{a} - a_{\rho+1})$ refines $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$. Part (iii) follows from (i) and (ii). $\square$

Let $\boldsymbol{a} = c_K(a_\rho)$ be the cut defined by (a_ρ) in K and $\operatorname{ndeg}_{\boldsymbol{a}} P$ be the Newton degree of P in $\boldsymbol{a}$ as introduced in [ADH, 11.2]. Then $\operatorname{ndeg}_{\boldsymbol{a}} P$ is the eventual value of $\operatorname{ndeg} P_{+a_\rho, \times\mathfrak{m}_\rho}$. Increasing ρ_0 we arrange that additionally for all $\rho \geqslant \rho_0$ we have $\operatorname{ndeg} P_{+a_\rho, \times\mathfrak{m}_\rho} = \operatorname{ndeg}_{\boldsymbol{a}} P$.

Corollary 3.2.25. $(P, \mathfrak{m}, \widehat{a})$ *has a quasilinear refinement iff* $\operatorname{ndeg}_{\boldsymbol{a}} P = 1$*.*

Proof. By Lemma 3.2.21 and [ADH, 11.2.8] we have

$$1 \leqslant \operatorname{ndeg} P_{+a_{\rho+1}, \times\mathfrak{m}_\rho} = \operatorname{ndeg} P_{+a_\rho, \times\mathfrak{m}_\rho}. \tag{3.2.1}$$

Thus if $\operatorname{ndeg}_{\boldsymbol{a}} P = 1$, then for $\rho \geqslant \rho_0$, the refinement $(P_{+a_{\rho+1}}, \mathfrak{m}_\rho, \widehat{a} - a_{\rho+1})$ of $(P, \mathfrak{m}, \widehat{a})$ is quasilinear. Conversely, if $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is a quasilinear refinement of $(P, \mathfrak{m}, \widehat{a})$, then Lemma 3.2.24(ii) yields a $\rho \geqslant \rho_0$ such that $\mathfrak{m}_\rho \preccurlyeq \mathfrak{n}$, and then $(P_{+a_{\rho+1}}, \mathfrak{m}_\rho, \widehat{a} - a_{\rho+1})$ in K refines $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ and hence is also quasilinear, so $\operatorname{ndeg}_{\boldsymbol{a}} P = \operatorname{ndeg} P_{+a_\rho, \times\mathfrak{m}_\rho} = 1$ by (3.2.1). $\square$

Lemma 3.2.26. *Assume K is* d*-valued and ω-free, and Γ is divisible. Then every Z-minimal slot in K of positive order has a quasilinear refinement.*

Proof. Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal. Take a divergent pc-sequence (a_ρ) in K such that $a_\rho \rightsquigarrow \widehat{a}$. Then P is a minimal differential polynomial of (a_ρ) over K, by Corollary 3.2.13. Hence $\operatorname{ndeg}_{\boldsymbol{a}} P = 1$ by [ADH, 14.5.1], where $\boldsymbol{a} := c_K(a_\rho)$. Now Corollary 3.2.25 gives a quasilinear refinement of $(P, \mathfrak{m}, \widehat{a})$. $\square$

Remark. Suppose K is a real closed H-field that is λ-free but not ω-free. (For example, the real closure of the H-field $\mathbb{R}\langle\omega\rangle$ from [ADH, 13.9.1] satisfies these conditions, by [ADH, 11.6.8, 11.7.23, 13.9.1].) Take $(P, \mathfrak{m}, \lambda)$ as in Lemma 3.2.10.

Then by Corollary 3.2.25 and [ADH, 11.7.9], $(P, \mathfrak{m}, \lambda)$ has no quasilinear refinement. Thus Lemma 3.2.26 fails if "ω-free" is replaced by "λ-free".

Lemma 3.2.27. *Let L be an r-newtonian H-asymptotic extension of K such that $\Gamma^<$ is cofinal in $\Gamma_L^<$, and suppose $(P, \mathfrak{m}, \widehat{a})$ is quasilinear. Then $P(\widehat{b}) = 0$ and $\widehat{b} \prec \mathfrak{m}$ for some $\widehat{b} \in L$.*

Proof. Lemma 3.2.21 and $\operatorname{ndeg} P_{\times\mathfrak{m}} = 1$ gives $\mathfrak{n} \prec \mathfrak{m}$ with $\operatorname{ndeg}_{\times\mathfrak{n}} P = 1$. By [ADH, p. 480], $\operatorname{ndeg} P_{\times\mathfrak{n}}$ does not change in passing from K to L. As L is r-newtonian this yields $\widehat{b} \preccurlyeq \mathfrak{n}$ in L with $P(\widehat{b}) = 0$. □

In the next two corollaries we assume that K is d-valued and ω-free, and that L is a newtonian H-asymptotic extension of K.

Corollary 3.2.28. *If $(P, \mathfrak{m}, \widehat{a})$ is quasilinear, then $P(\widehat{b}) = 0$, $\widehat{b} \prec \mathfrak{m}$ for some $\widehat{b} \in L$.*

Proof. By [21, Theorem B], K has a newtonization K^* inside L. Such K^* is d-algebraic over K by (0.7), so $\Gamma^<$ is cofinal in $\Gamma_{K^*}^<$ by Theorem 1.3.1. Thus we can apply Lemma 3.2.27 to K^* in the role of L. □

Here is a variant of Lemma 3.2.14:

Corollary 3.2.29. *Suppose Γ is divisible and $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal. Then there exists $\widehat{b} \in L$ such that $K\langle\widehat{b}\rangle$ is an immediate extension of K and $(P, \mathfrak{m}, \widehat{b})$ is a hole in K equivalent to $(P, \mathfrak{m}, \widehat{a})$. (Thus if $(P, \mathfrak{m}, \widehat{a})$ is also a hole in K, then there is an embedding $K\langle\widehat{a}\rangle \to L$ of valued differential fields over K.)*

Proof. By Lemma 3.2.26 we may refine $(P, \mathfrak{m}, \widehat{a})$ to arrange that $(P, \mathfrak{m}, \widehat{a})$ is quasilinear. Then [ADH, 11.4.8] gives $\widehat{b}$ in an immediate H-asymptotic extension of K with $P(\widehat{b}) = 0$ and $v(\widehat{a} - a) = v(\widehat{b} - a)$ for all a. So $(P, \mathfrak{m}, \widehat{b})$ is a hole in K equivalent to $(P, \mathfrak{m}, \widehat{a})$. The immediate d-algebraic extension $K\langle\widehat{b}\rangle$ of K is ω-free by Theorem 1.3.1. Then (0.7) gives a newtonian d-algebraic immediate extension M of $K\langle\widehat{b}\rangle$ and thus of K. Then M is a newtonization of K by [ADH, 14.5.4] and thus embeds over K into L. The rest follows from Corollary 3.2.15. □

Remark. Lemma 3.2.26 and Corollary 3.2.29 go through with the hypothesis "Γ is divisible" replaced by "K is henselian". The proofs are the same, using [21, 3.3] in place of [ADH, 14.5.1] in the proof of Lemma 3.2.26, and [21, 3.5] in place of [ADH, 14.5.4] in the proof of Corollary 3.2.29.

For $r = 1$ we can weaken the hypothesis of ω-freeness in Corollary 3.2.29:

Corollary 3.2.30. *Suppose K is λ-free and Γ is divisible, and $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal of order $r = 1$ with a quasilinear refinement. Let L be a newtonian H-asymptotic extension of K. Then there exists $\widehat{b} \in L$ such that $K\langle\widehat{b}\rangle$ is an immediate extension of K and $(P, \mathfrak{m}, \widehat{b})$ is a hole in K equivalent to $(P, \mathfrak{m}, \widehat{a})$. (So if $(P, \mathfrak{m}, \widehat{a})$ is also a hole in K, then we have an embedding $K\langle\widehat{a}\rangle \to L$ of valued differential fields over K.)*

Proof. Take a divergent pc-sequence (a_ρ) in K with $a_\rho \rightsquigarrow \widehat{a}$. Then $\operatorname{ndeg}_{\boldsymbol{a}} P = 1$ for $\boldsymbol{a} := c_K(a_\rho)$, by Corollary 3.2.25, and P is a minimal differential polynomial of (a_ρ) over K, by [ADH, 11.4.13]. The equality $\operatorname{ndeg}_{\boldsymbol{a}} P = 1$ remains valid when passing from K, $\boldsymbol{a}$ to L, $c_L(a_\rho)$, respectively, by Lemma 1.7.8. Hence [ADH, 14.1.10] yields $\widehat{b} \in L$ such that $P(\widehat{b}) = 0$ and $a_\rho \rightsquigarrow \widehat{b}$, so $v(\widehat{a} - a) = v(\widehat{b} - a)$ for all a. Then $K\langle\widehat{b}\rangle$ is an immediate extension of K by [ADH, 9.7.6], so $(P, \mathfrak{m}, \widehat{b})$ is a hole in K equivalent to $(P, \mathfrak{m}, \widehat{a})$. For the rest use Corollary 3.2.15. $\square$

The linear part of a slot. We define the **linear part** of $(P, \mathfrak{m}, \widehat{a})$ to be the linear part $L_{P_{\times\mathfrak{m}}} \in K[\partial]$ of $P_{\times\mathfrak{m}}$. By [ADH, p. 242] and (0.1) we have

$$L_{P_{\times\mathfrak{m}}} \;=\; L_P\,\mathfrak{m} \;=\; \sum_{n=0}^{r} \frac{\partial P_{\times\mathfrak{m}}}{\partial Y^{(n)}}(0)\,\partial^n \;=\; \mathfrak{m}S_P(0)\partial^r + \text{lower order terms in } \partial.$$

The slot $(P, \mathfrak{m}, \widehat{a})$ has the same linear part as each of its multiplicative conjugates. The linear part of a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ is given by

$$\begin{aligned} L_{P_{+a,\times\mathfrak{n}}} \;&=\; L_{P_{+a}}\mathfrak{n} \;=\; \sum_{m=0}^{r} \left(\sum_{n=m}^{r} \binom{n}{m} \mathfrak{n}^{(n-m)} \frac{\partial P}{\partial Y^{(n)}}(a) \right) \partial^m \\ &=\; \mathfrak{n}\,S_P(a)\,\partial^r + \text{lower order terms in } \partial. \end{aligned}$$

(See [ADH, (5.1.1)].) By [ADH, 5.7.5] we have $(P^\phi)_d = (P_d)^\phi$ for $d \in \mathbb{N}$; in particular $L_{P^\phi} = (L_P)^\phi$ and so $\operatorname{order}(L_{P^\phi}) = \operatorname{order}(L_P)$. A particularly favorable situation occurs when L_P splits over a given differential field extension E of K (which includes requiring $L_P \neq 0$). Typically, E is an algebraic closure of K. In any case, L_P splits over E iff $L_{P_{\times\mathfrak{n}}}$ splits over E, iff L_{P^ϕ} splits over E^ϕ. Thus:

Lemma 3.2.31. *Suppose* $\deg P = 1$ *and* L_P *splits over* E*. Then the linear part of any refinement of* $(P, \mathfrak{m}, \widehat{a})$ *and any multiplicative conjugate of* $(P, \mathfrak{m}, \widehat{a})$ *also splits over* E*, and any compositional conjugate of* $(P, \mathfrak{m}, \widehat{a})$ *by an active* ϕ *splits over* E^ϕ*.*

Let $\boldsymbol{i} = (i_0, \dots, i_r)$ range over $\mathbb{N}^{1+r}$. As in [ADH, 4.2] we set

$$P_{(\boldsymbol{i})} \;:=\; \frac{P^{(\boldsymbol{i})}}{\boldsymbol{i}!} \qquad \text{where } P^{(\boldsymbol{i})} \;:=\; \frac{\partial^{|\boldsymbol{i}|} P}{\partial^{i_0} Y \cdots \partial^{i_r} Y^{(r)}}.$$

If $|\boldsymbol{i}| = i_0 + \cdots + i_r \geqslant 1$, then $\mathrm{c}(P_{(\boldsymbol{i})}) < \mathrm{c}(P)$. Note that for $\boldsymbol{i} = (0, \dots, 0, 1)$ we have $P_{(\boldsymbol{i})} = S_P \neq 0$, since $\operatorname{order} P = r$. We now aim for Corollary 3.2.34.

Lemma 3.2.32. *Suppose that* $(P, \mathfrak{m}, \widehat{a})$ *is* Z*-minimal. Then* $(P, \mathfrak{m}, \widehat{a})$ *has a refinement* $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ *such that for all* $\boldsymbol{i}$ *with* $|\boldsymbol{i}| \geqslant 1$ *and* $P_{(\boldsymbol{i})} \neq 0$*,*

$$\operatorname{ndeg}\,(P_{(\boldsymbol{i})})_{+a,\times\mathfrak{n}} \;=\; 0.$$

Proof. Let $\boldsymbol{i}$ range over the (finitely many) elements of $\mathbb{N}^{1+r}$ satisfying $|\boldsymbol{i}| \geqslant 1$ and $P_{(\boldsymbol{i})} \neq 0$. Each $P_{(\boldsymbol{i})}$ has smaller complexity than P, so $P_{(\boldsymbol{i})} \notin Z(K, \widehat{a})$. Then $Q := \prod_{\boldsymbol{i}} P_{(\boldsymbol{i})} \notin Z(K, \widehat{a})$ by [ADH, 11.4.4], so Lemma 3.2.18 gives a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ with $\operatorname{ndeg} Q_{+a,\times\mathfrak{n}} = 0$. Then $\operatorname{ndeg}\,(P_{(\boldsymbol{i})})_{+a,\times\mathfrak{n}} = 0$ for all $\boldsymbol{i}$, by [ADH, remarks before 11.2.6]. $\square$

From [ADH, (4.3.3)] we recall that $(P_{(\boldsymbol{i})})_{+a} = (P_{+a})_{(\boldsymbol{i})}$. Also recall that $(P_{+a})_{\boldsymbol{i}} = P_{(\boldsymbol{i})}(a)$ by Taylor expansion. In particular, if $P_{(\boldsymbol{i})} = 0$, then $(P_{+a})_{\boldsymbol{i}} = 0$.

Lemma 3.2.33. *Suppose* $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ *refines* $(P, \mathfrak{m}, \widehat{a})$ *and* $\boldsymbol{i}$ *is such that* $|\boldsymbol{i}| \geqslant 1$*,* $P_{(\boldsymbol{i})} \neq 0$*, and* $\operatorname{ndeg}\,(P_{(\boldsymbol{i})})_{\times\mathfrak{m}} = 0$*. Then*

$$\operatorname{ndeg}\,(P_{(\boldsymbol{i})})_{+a,\times\mathfrak{n}} \;=\; 0, \qquad (P_{+a})_{\boldsymbol{i}} \;\sim\; P_{\boldsymbol{i}}.$$

Proof. Using [ADH, 11.2.4, 11.2.3(iii), 11.2.5] we get

$$\operatorname{ndeg}(P_{(\boldsymbol{i})})_{+a,\times\mathfrak{n}} \;=\; \operatorname{ndeg}(P_{(\boldsymbol{i})})_{+\widehat{a},\times\mathfrak{n}} \;\leqslant\; \operatorname{ndeg}(P_{(\boldsymbol{i})})_{+\widehat{a},\times\mathfrak{m}} \;=\; \operatorname{ndeg}(P_{(\boldsymbol{i})})_{\times\mathfrak{m}} \;=\; 0,$$

so $\operatorname{ndeg}(P_{(\boldsymbol{i})})_{+a,\times\mathfrak{n}} = 0$. Thus $P_{(\boldsymbol{i})} \notin Z(K,\widehat{a})$, hence $(P_{+a})_{\boldsymbol{i}} = P_{(\boldsymbol{i})}(a) \sim P_{(\boldsymbol{i})}(\widehat{a})$ by [ADH, 11.4.3]; applying this to $a = 0$, $\mathfrak{n} = \mathfrak{m}$ yields $P_{\boldsymbol{i}} = P_{(\boldsymbol{i})}(0) \sim P_{(\boldsymbol{i})}(\widehat{a})$. $\square$

Combining Lemmas 3.2.32 and 3.2.33 gives:

Corollary 3.2.34. *Every Z-minimal slot in K of order r has a refinement $(P,\mathfrak{m},\widehat{a})$ such that for all refinements $(P_{+a},\mathfrak{n},\widehat{a}-a)$ of $(P,\mathfrak{m},\widehat{a})$ and all $\boldsymbol{i}$ with $|\boldsymbol{i}| \geqslant 1$ and $P_{(\boldsymbol{i})} \neq 0$ we have $(P_{+a})_{\boldsymbol{i}} \sim P_{\boldsymbol{i}}$ (and thus* $\operatorname{order} L_{P_{+a}} = \operatorname{order} L_P = r$*).*

Here the condition "of order r" may seem irrelevant, but is forced on us because refinements preserve order and by our convention that P has order r.

Special slots. The slot $(P,\mathfrak{m},\widehat{a})$ in K is said to be **special** if $\widehat{a}/\mathfrak{m}$ is special over K in the sense of [ADH, p. 167]: some nontrivial convex subgroup Δ of Γ is cofinal in $v\big(\frac{\widehat{a}}{\mathfrak{m}}-K\big)$. If $(P,\mathfrak{m},\widehat{a})$ is special, then so are $(bP,\mathfrak{m},\widehat{a})$ for $b \neq 0$, any multiplicative conjugate of $(P,\mathfrak{m},\widehat{a})$, any compositional conjugate of $(P,\mathfrak{m},\widehat{a})$, and any slot in K equivalent to $(P,\mathfrak{m},\widehat{a})$. Also, by Lemma 1.5.1:

Lemma 3.2.35. *If $(P,\mathfrak{m},\widehat{a})$ is special, then so is any refinement.*

Here is our main source of special slots:

Lemma 3.2.36. *Let K be r-linearly newtonian, and $\boldsymbol{\omega}$-free if $r > 1$. Suppose $(P,\mathfrak{m},\widehat{a})$ is quasilinear, and Z-minimal or a hole in K. Then $(P,\mathfrak{m},\widehat{a})$ is special.*

Proof. Use Lemma 3.2.14 to arrange $(P,\mathfrak{m},\widehat{a})$ is a hole in K. Next arrange $\mathfrak{m} = 1$ by replacing $(P,\mathfrak{m},\widehat{a})$ with $(P_{\times\mathfrak{m}},1,\widehat{a}/\mathfrak{m})$. So $\operatorname{ndeg} P = 1$, hence $\widehat{a}$ is special over K by Proposition 1.5.12 (if $r > 1$) and 1.5.18 (if $r = 1$). $\square$

Next an approximation result that will be needed in [6]:

Lemma 3.2.37. *Suppose $\mathfrak{m} = 1$, $(P,1,\widehat{a})$ is special and Z-minimal, and $\widehat{a} - a \preccurlyeq \mathfrak{n} \prec 1$ for some a. Then $\widehat{a} - b \prec \mathfrak{n}^{r+1}$ for some b, and $P(b) \prec \mathfrak{n}P$ for any such b.*

Proof. Using Lemma 3.2.14 we arrange $P(\widehat{a}) = 0$. The differential polynomial $Q := \sum_{|\boldsymbol{i}|\geqslant 1} P_{(\boldsymbol{i})}(\widehat{a})Y^{\boldsymbol{i}} \in \widehat{K}\{Y\}$ has order $\leqslant r$ and $\operatorname{mul}(Q) \geqslant 1$, and Taylor expansion yields, for all a:

$$P(a) \;=\; P(\widehat{a}) + \sum_{|\boldsymbol{i}|\geqslant 1} P_{(\boldsymbol{i})}(\widehat{a})(a-\widehat{a})^{\boldsymbol{i}} \;=\; Q(a-\widehat{a}).$$

Since $\widehat{a}$ is special over K, we have b with $\widehat{a} - b \prec \mathfrak{n}^{r+1}$, and then by Lemma 1.1.10 we have $Q(b-\widehat{a}) \prec \mathfrak{n}Q \preccurlyeq \mathfrak{n}P$. $\square$

3.3. The First Normalization Theorems

Throughout this section K is an H-asymptotic field with small derivation and with rational asymptotic integration. We set $\Gamma := v(K^\times)$. The notational conventions introduced in the last section remain in force: a, b, f, g range over K; ϕ, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{v}$, $\mathfrak{w}$ over $K^\times$. As at the end of Section 3.1 we shall frequently use for $\mathfrak{v} \prec 1$ the coarsening of v by the convex subgroup $\Delta(\mathfrak{v}) = \big\{\gamma \in \Gamma : \gamma = o(v\mathfrak{v})\big\}$ of Γ.

We fix a slot $(P, \mathfrak{m}, \widehat{a})$ in K of order $r \geqslant 1$, and set $w := \operatorname{wt}(P)$ (so $w \geqslant r \geqslant 1$). In the next subsections we introduce various conditions on $(P, \mathfrak{m}, \widehat{a})$. These conditions will be shown to be related as follows:

$$\begin{tikzcd} \text{strictly normal} \arrow[r, Rightarrow] & \text{normal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{steep} \\ & \text{quasilinear} & \text{deep} \arrow[l, Rightarrow] \arrow[u, Rightarrow] \end{tikzcd}$$

Thus "deep + strictly normal" yields the rest. The main results of this section are Theorem 3.3.33 and its variants 3.3.34, 3.3.36, and 3.3.48.

Steep and deep slots. In this subsection, if $\operatorname{order}(L_{P_{\times\mathfrak{m}}}) = r$, then we set

$$\mathfrak{v} \ := \ \mathfrak{v}(L_{P_{\times\mathfrak{m}}}).$$

The slot $(P, \mathfrak{m}, \widehat{a})$ in K is said to be **steep** if $\operatorname{order}(L_{P_{\times\mathfrak{m}}}) = r$ and $\mathfrak{v} \prec^\flat 1$. Thus

$$(P, \mathfrak{m}, \widehat{a}) \text{ is steep} \iff (P_{\times\mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{a}/\mathfrak{n}) \text{ is steep} \iff (bP, \mathfrak{m}, \widehat{a}) \text{ is steep}$$

for $b \neq 0$. If $(P, \mathfrak{m}, \widehat{a})$ is steep, then so is any slot in K equivalent to $(P, \mathfrak{m}, \widehat{a})$. If $(P, \mathfrak{m}, \widehat{a})$ is steep, then so is any slot $(P^\phi, \mathfrak{m}, \widehat{a})$ in K^ϕ for active $\phi \preccurlyeq 1$, by Lemma 3.1.20, and thus $\operatorname{nwt}(L_{P_{\times\mathfrak{m}}}) < r$. Below we tacitly use that if $(P, \mathfrak{m}, \widehat{a})$ is steep, then

$$\mathfrak{n} \asymp_{\Delta(\mathfrak{v})} \mathfrak{v} \implies [\mathfrak{n}] = [\mathfrak{v}], \qquad \mathfrak{n} \prec 1,\ [\mathfrak{n}] = [\mathfrak{v}] \implies \mathfrak{n} \prec^\flat 1.$$

Note also that if $(P, \mathfrak{m}, \widehat{a})$ is steep, then $\mathfrak{v}^\dagger \asymp_{\Delta(\mathfrak{v})} 1$ by [ADH, 9.2.10(iv)].

Lemma 3.3.1. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is steep, $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$ and $[\mathfrak{n}/\mathfrak{m}] \leqslant [\mathfrak{v}]$. Then*

$$\operatorname{order}(L_{P_{\times\mathfrak{n}}}) \ = \ r, \qquad \mathfrak{v}(L_{P_{\times\mathfrak{n}}}) \ \asymp_{\Delta(\mathfrak{v})} \ \mathfrak{v},$$

so $(P, \mathfrak{n}, \widehat{a})$ is a steep refinement of $(P, \mathfrak{m}, \widehat{a})$.

Proof. Replace $(P, \mathfrak{m}, \widehat{a})$, $\mathfrak{n}$ by $(P_{\times\mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$, $\mathfrak{n}/\mathfrak{m}$, respectively, to arrange $\mathfrak{m} = 1$. Set $L := L_P$ and $\widetilde{L} := L_{P_{\times\mathfrak{n}}}$. Then $\widetilde{L} = L\mathfrak{n} \asymp_{\Delta(\mathfrak{v})} \mathfrak{n}L$ by [ADH, 6.1.3]. Hence

$$\widetilde{L}_r \ = \ \mathfrak{n}L_r \ \asymp \ \mathfrak{n}\mathfrak{v}L \ \asymp_{\Delta(\mathfrak{v})} \ \mathfrak{v}\widetilde{L}.$$

Since $\mathfrak{v}(\widetilde{L})\widetilde{L} \asymp \widetilde{L}_r$, this gives $\mathfrak{v}(\widetilde{L})\widetilde{L} \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}\widetilde{L}$, and thus $\mathfrak{v}(\widetilde{L}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. □

If $(P, \mathfrak{m}, \widehat{a})$ is steep and linear, then

$$L_{P_{+a,\times\mathfrak{m}}} = L_{P_{\times\mathfrak{m},+(a/\mathfrak{m})}} = L_{P_{\times\mathfrak{m}}},$$

so any refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ is also steep and linear.

Lemma 3.3.2. *Suppose* $\operatorname{order} L_{P_{\times\mathfrak{m}}} = r$*. Then $(P, \mathfrak{m}, \widehat{a})$ has a refinement $(P, \mathfrak{n}, \widehat{a})$ such that* $\operatorname{nwt} L_{P_{\times\mathfrak{n}}} = 0$*, and $(P^\phi, \mathfrak{n}, \widehat{a})$ is steep, eventually.*

Proof. Replacing $(P, \mathfrak{m}, \widehat{a})$ by $(P_{\times\mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$ we arrange $\mathfrak{m} = 1$. Take $\mathfrak{n}_1$ with $\widehat{a} \prec \mathfrak{n}_1 \prec 1$. Then $\operatorname{order}(P_1)_{\times\mathfrak{n}_1} = \operatorname{order} P_1 = \operatorname{order} L_P = r$, and thus $(P_1)_{\times\mathfrak{n}_1} \neq 0$. So [ADH, 11.3.6] applied to $(P_1)_{\times\mathfrak{n}_1}$ in place of P yields an $\mathfrak{n}$ with $\mathfrak{n}_1 \prec \mathfrak{n} \prec 1$ and $\operatorname{nwt}(P_1)_{\times\mathfrak{n}} = 0$, so $\operatorname{nwt} L_{P_{\times\mathfrak{n}}} = 0$. Hence by Lemma 3.1.21, $(P^\phi, \mathfrak{n}, \widehat{a})$ is steep, eventually. □

Recall that the separant $S_P = \partial P/\partial Y^{(r)}$ of P has lower complexity than P. Below we sometimes use the identity $S_{P^{\phi}_{\times\mathfrak{m}}} = \phi^r (S_{P_{\times\mathfrak{m}}})^{\phi}$ from (0.1).

The slot $(P, \mathfrak{m}, \widehat{a})$ in K is said to be **deep** if it is steep and for all active $\phi \preccurlyeq 1$,

(D1) $\operatorname{ddeg} S_{P^{\phi}_{\times\mathfrak{m}}} = 0$ (hence $\operatorname{ndeg} S_{P_{\times\mathfrak{m}}} = 0$), and

(D2) $\operatorname{ddeg} P^{\phi}_{\times\mathfrak{m}} = 1$ (hence $\operatorname{ndeg} P_{\times\mathfrak{m}} = 1$).

If $\deg P = 1$, then (D1) is automatic, for all active $\phi \preccurlyeq 1$. If $(P, \mathfrak{m}, \widehat{a})$ is deep, then so are $(P_{\times\mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{a}/\mathfrak{n})$ and $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$, as well as every slot in K equivalent to $(P, \mathfrak{m}, \widehat{a})$ and the slot $(P^{\phi}, \mathfrak{m}, \widehat{a})$ in K^{ϕ} for active $\phi \preccurlyeq 1$. Every deep slot in K is quasilinear, by (D2). If $\deg P = 1$, then $(P, \mathfrak{m}, \widehat{a})$ is quasilinear iff $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is deep for some active $\phi \preccurlyeq 1$. Moreover, if $(P, \mathfrak{m}, \widehat{a})$ is a deep hole in K, then $\operatorname{dmul} P^{\phi}_{\times\mathfrak{m}} = 1$ for all active $\phi \preccurlyeq 1$, by (D2) and Lemma 3.2.9.

Example 3.3.3. Suppose $P = Y' + gY - u$ where $g, u \in K$ and $\mathfrak{m} = 1$, $r = 1$. Set $L := L_P = \partial + g$ and $\mathfrak{v} := \mathfrak{v}(L)$. Then $\mathfrak{v} = 1$ if $g \preccurlyeq 1$, and $\mathfrak{v} = 1/g$ if $g \succ 1$. Thus

$$(P, 1, \widehat{a}) \text{ is steep} \quad\Longleftrightarrow\quad g \succ^{\flat} 1 \quad\Longleftrightarrow\quad g \succ 1 \text{ and } g^{\dagger} \succcurlyeq 1.$$

Note that $(P, 1, \widehat{a})$ is steep iff L is steep as defined in Section 1.4. Also,

$$(P, 1, \widehat{a}) \text{ is deep} \quad\Longleftrightarrow\quad (P, 1, \widehat{a}) \text{ is steep and } g \succcurlyeq u.$$

Hence if $u = 0$, then $(P, 1, \widehat{a})$ is deep iff it is steep.

Lemma 3.3.4. *For steep $(P, \mathfrak{m}, \widehat{a})$, the following are equivalent:*

(i) $(P^{\phi}, \mathfrak{m}, \widehat{a})$ *is deep, eventually;*

(ii) $\operatorname{ndeg} S_{P_{\times\mathfrak{m}}} = 0$ *and* $\operatorname{ndeg} P_{\times\mathfrak{m}} = 1$.

Note that if $\operatorname{ddeg} S_{P_{\times\mathfrak{m}}} = 0$ or $\operatorname{ndeg} S_{P_{\times\mathfrak{m}}} = 0$, then $S_{P_{\times\mathfrak{m}}}(0) \neq 0$, so $\operatorname{order} L_{P_{\times\mathfrak{m}}} = r$.

Lemma 3.3.5. *Suppose $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ refines the hole $(P, \mathfrak{m}, \widehat{a})$ in K. Then:*

(i) $\operatorname{ddeg} S_{P_{\times\mathfrak{m}}} = 0 \implies \operatorname{ddeg} S_{P_{+a,\times\mathfrak{n}}} = 0$*;*

(ii) $\operatorname{ddeg} P_{\times\mathfrak{m}} = 1 \implies \operatorname{ddeg} P_{+a,\times\mathfrak{n}} = 1$*;*

(iii) $\operatorname{ndeg} S_{P_{\times\mathfrak{m}}} = 0 \implies S_P(a) \sim S_P(0)$.

Thus if $(P, \mathfrak{m}, \widehat{a})$ is deep and $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is steep, then $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is deep.

Proof. Suppose $\operatorname{ddeg} S_{P_{\times\mathfrak{m}}} = 0$. Then $\operatorname{ddeg} S_{P_{+a,\times\mathfrak{n}}} = 0$ follows from

$$\operatorname{ddeg} S_{P_{+a,\times\mathfrak{n}}} = \operatorname{ddeg} (S_P)_{+a,\times\mathfrak{n}} \quad\text{and}\quad \operatorname{ddeg}(S_P)_{\times\mathfrak{m}} = \operatorname{ddeg} S_{P_{\times\mathfrak{m}}}$$

(consequences of (0.1)), and

$$\operatorname{ddeg} (S_P)_{+a,\times\mathfrak{n}} = \operatorname{ddeg} (S_P)_{+\widehat{a},\times\mathfrak{n}} \leqslant \operatorname{ddeg} (S_P)_{+\widehat{a},\times\mathfrak{m}} = \operatorname{ddeg} (S_P)_{\times\mathfrak{m}}$$

which holds by [ADH, 6.6.7]. This proves (i). Corollary 3.2.20 yields (ii), and (iii) is contained in Lemma 3.2.33. □

Lemmas 3.2.14 and 3.3.5 give:

Corollary 3.3.6. *If $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal and deep, then each steep refinement of $(P, \mathfrak{m}, \widehat{a})$ is deep.*

Here is another sufficient condition on refinements of deep holes to remain deep:

Lemma 3.3.7. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is a deep hole in K, and $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ refines $(P, \mathfrak{m}, \widehat{a})$ with $[\mathfrak{n}/\mathfrak{m}] \leqslant [\mathfrak{v}]$. Then $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ is deep with $\mathfrak{v}(L_{P_{+a,\times\mathfrak{n}}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$.*

Proof. From $(P, \mathfrak{m}, \widehat{a})$ we pass to the hole $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ and then to $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$. We first show that $\operatorname{order} L_{P_{+a,\times\mathfrak{m}}} = r$ and $\mathfrak{v}(L_{P_{+a,\times\mathfrak{m}}}) \sim \mathfrak{v}$, from which it follows that $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ is steep, hence deep by Lemma 3.3.5. By Corollary 3.2.20,

$$\operatorname{ddeg} P_{+a,\times\mathfrak{m}} \;=\; \operatorname{dmul} P_{+a,\times\mathfrak{m}} \;=\; 1,$$

so $(P_{+a,\times\mathfrak{m}})_1 \sim P_{+a,\times\mathfrak{m}}$. Also

$$(P_{\times\mathfrak{m}})_1 \sim P_{\times\mathfrak{m}} \sim P_{\times\mathfrak{m},+(a/\mathfrak{m})} = P_{+a,\times\mathfrak{m}},$$

by [ADH, 4.5.1(i)], and thus $(P_{+a,\times\mathfrak{m}})_1 \sim (P_{\times\mathfrak{m}})_1$. By (0.1) and Lemma 3.3.5(iii),

$$S_{P_{+a,\times\mathfrak{m}}}(0) \;=\; \mathfrak{m} S_P(a) \;\sim\; \mathfrak{m} S_P(0) \;=\; S_{P_{\times\mathfrak{m}}}(0),$$

so $S_{P_{+a,\times\mathfrak{m}}}(0) \sim S_{P_{\times\mathfrak{m}}}(0)$. This gives $\mathfrak{v}(L_{P_{+a,\times\mathfrak{m}}}) \sim \mathfrak{v}$ as promised.

Next, Lemma 3.3.1 applied to $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ in the role of $(P, \mathfrak{m}, \widehat{a})$ gives that $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ is steep with $\mathfrak{v}(L_{P_{+a,\times\mathfrak{n}}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. Now Lemma 3.3.5 applied to $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ and $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ in the role of $(P, \mathfrak{m}, \widehat{a})$ and $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$, respectively, gives that $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ is deep. $\square$

Lemmas 3.2.14 and 3.3.7 give a version for Z-minimal slots:

Corollary 3.3.8. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal and deep, and $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ refines $(P, \mathfrak{m}, \widehat{a})$ with $[\mathfrak{n}/\mathfrak{m}] \leqslant [\mathfrak{v}]$, where $\mathfrak{v} := \mathfrak{v}(L_{P_{\times\mathfrak{m}}})$. Then $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ is deep with $\mathfrak{v}(L_{P_{+a,\times\mathfrak{n}}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$.*

Next we turn to the task of turning Z-minimal slots into deep ones.

Lemma 3.3.9. *Every quasilinear Z-minimal slot in K of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that:*

(i) $\operatorname{ndeg}(P_{(\boldsymbol{i})})_{\times\mathfrak{m}} = 0$ *for all $\boldsymbol{i}$ with $|\boldsymbol{i}| \geqslant 1$ and $P_{(\boldsymbol{i})} \neq 0$;*
(ii) $\operatorname{ndeg} P_{\times\mathfrak{m}} = \operatorname{nmul} P_{\times\mathfrak{m}} = 1$, *and*
(iii) $\operatorname{nwt} L_{P_{\times\mathfrak{m}}} = 0$.

Proof. By Corollary 3.2.22, any quasilinear $(P, \mathfrak{m}, \widehat{a})$ satisfies (ii). Any refinement of a quasilinear $(P, \mathfrak{m}, \widehat{a})$ remains quasilinear, by Corollary 3.2.23. By Lemma 3.2.32 and a subsequent remark any quasilinear Z-minimal slot in K of order r can be refined to a quasilinear $(P, \mathfrak{m}, \widehat{a})$ that satisfies (i), and by Lemma 3.2.33, any further refinement of such $(P, \mathfrak{m}, \widehat{a})$ continues to satisfy (i). Thus to prove the lemma, assume we are given a quasilinear $(P, \mathfrak{m}, \widehat{a})$ satisfying (i); it is enough to show that then $(P, \mathfrak{m}, \widehat{a})$ has a refinement $(P, \mathfrak{n}, \widehat{a})$ satisfying (iii) with $\mathfrak{n}$ instead of $\mathfrak{m}$ (and thus also (i) and (ii) with $\mathfrak{n}$ instead of $\mathfrak{m}$).

Take $\widetilde{\mathfrak{m}}$ with $\widehat{a} \prec \widetilde{\mathfrak{m}} \prec \mathfrak{m}$. Then $(P_{\times\widetilde{\mathfrak{m}}})_1 \neq 0$ by (ii), so [ADH, 11.3.6] applied to $(P_1)_{\times\widetilde{\mathfrak{m}}}$ in place of P yields an $\mathfrak{n}$ with $\widetilde{\mathfrak{m}} \prec \mathfrak{n} \prec \mathfrak{m}$ and $\operatorname{nwt}(P_1)_{\times\mathfrak{n}} = 0$. Hence the refinement $(P, \mathfrak{n}, \widehat{a})$ of $(P, \mathfrak{m}, \widehat{a})$ satisfies (iii) with $\mathfrak{n}$ instead of $\mathfrak{m}$. $\square$

Corollary 3.3.10. *Every quasilinear Z-minimal slot in K of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $\operatorname{nwt} L_{P_{\times\mathfrak{m}}} = 0$, and $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is deep, eventually.*

Proof. Given a quasilinear Z-minimal slot in K of order r, we take a refinement $(P, \mathfrak{m}, \widehat{a})$ as in Lemma 3.3.9. Then $\operatorname{ndeg} S_{P_{\times\mathfrak{m}}} = 0$ by (i) of that lemma, so $\operatorname{order} L_{P_{\times\mathfrak{m}}} = r$ by the remark that precedes Lemma 3.3.5. Then (iii) of Lemma 3.3.9 and Lemma 3.1.21 give that $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is steep, eventually. Using now $\operatorname{ndeg} S_{P_{\times\mathfrak{m}}} = 0$ and (ii) of Lemma 3.3.9 we obtain from Lemma 3.3.4 that $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is deep, eventually. $\square$

Lemma 3.2.26 and the previous lemma and its corollary now yield:

Lemma 3.3.11. *Suppose K is* d*-valued and* ω*-free, and* Γ *is divisible. Then every Z-minimal slot in K of order r has a refinement $(P,\mathfrak{m},\widehat{a})$ satisfying* (i)–(iii) *in Lemma 3.3.9.*

Corollary 3.3.12. *Suppose K is* d*-valued and* ω*-free, and* Γ *is divisible. Then every Z-minimal slot in K of order r has a quasilinear refinement $(P,\mathfrak{m},\widehat{a})$ such that* $\operatorname{nwt} L_{P_{\times\mathfrak{m}}}=0$*, and $(P^{\phi},\mathfrak{m},\widehat{a})$ is deep, eventually.*

Approximating Z-minimal slots. In this subsection we set, as before,

$$\mathfrak{v} \ := \ \mathfrak{v}(L_{P_{\times\mathfrak{m}}}),$$

provided $L_{P_{\times\mathfrak{m}}}$ has order r. The next lemma is a key approximation result.

Lemma 3.3.13. *Suppose $(P,\mathfrak{m},\widehat{a})$ is Z-minimal and steep, and*

$$\operatorname{ddeg} P_{\times\mathfrak{m}} \ = \ \operatorname{ndeg} P_{\times\mathfrak{m}} \ = \ 1, \qquad \operatorname{ddeg} S_{P_{\times\mathfrak{m}}} \ = \ 0.$$

Then there exists an a such that $\widehat{a}-a \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$.

Proof. We can arrange $\mathfrak{m}=1$ and $P \asymp 1$. Then $\operatorname{ddeg} P=1$ gives $P_1 \asymp 1$, so $S_P(0)\asymp\mathfrak{v}$. Take $Q,R_1,\dots,R_n\in K\{Y\}$ $(n\geqslant 1)$ of order $<r$ such that

$$P \ = \ Q+R_1Y^{(r)}+\dots+R_n(Y^{(r)})^n, \qquad S_P \ = \ R_1+\dots+nR_n(Y^{(r)})^{n-1}.$$

Then $R_1(0)=S_P(0)\asymp\mathfrak{v}$. As $\operatorname{ddeg} S_P=0$, this gives $S_P\sim R_1(0)$, hence

$$R \ := \ P-Q \ \sim \ R_1(0)Y^{(r)} \ \asymp \ \mathfrak{v} \ \prec_{\Delta(\mathfrak{v})} \ 1 \ \asymp \ P,$$

so $P\sim_{\Delta(\mathfrak{v})} Q$. Thus $Q\neq 0$, and $Q\notin Z(K,\widehat{a})$ because $\operatorname{order} Q<r$. Now Lemma 3.2.18 gives a refinement $(P_{+a},\mathfrak{n},\widehat{a}-a)$ of $(P,1,\widehat{a})$ such that $\operatorname{ndeg} Q_{+a,\times\mathfrak{n}}=0$ and $\mathfrak{n}\prec 1$. We claim that then $\widehat{a}-a\prec_{\Delta(\mathfrak{v})} 1$. (Establishing this claim finishes the proof.) Suppose the claim is false. Then $\widehat{a}-a\asymp_{\Delta(\mathfrak{v})} 1$, so $\mathfrak{n}\asymp_{\Delta(\mathfrak{v})} 1$, hence $Q_{+a,\times\mathfrak{n}}\asymp_{\Delta(\mathfrak{v})} Q_{+a}\asymp Q$ by [ADH, 4.5.1]. Likewise, $R_{+a,\times\mathfrak{n}}\asymp_{\Delta(\mathfrak{v})} R$. Using $P_{+a,\times\mathfrak{n}}=Q_{+a,\times\mathfrak{n}}+R_{+a,\times\mathfrak{n}}$ gives $Q_{+a,\times\mathfrak{n}}\sim_{\Delta(\mathfrak{v})} P_{+a,\times\mathfrak{n}}$, so $Q_{+a,\times\mathfrak{n}}\sim^{\flat} P_{+a,\times\mathfrak{n}}$. Then $\operatorname{ndeg} Q_{+a,\times\mathfrak{n}} = \operatorname{ndeg} P_{+a,\times\mathfrak{n}} = 1$ by Lemma 1.7.2 and Corollary 3.2.23, a contradiction. □

Lemmas 3.2.9 and 3.3.13, and a remark following the definition of *deep* give:

Corollary 3.3.14. *If $(P,\mathfrak{m},\widehat{a})$ is Z-minimal, steep, and linear, then there exists an a such that $\widehat{a}-a\prec_{\Delta(\mathfrak{v})}\mathfrak{m}$.*

Corollary 3.3.15. *Suppose $(P,\mathfrak{m},\widehat{a})$ is Z-minimal, deep, and special. Then for all $n\geqslant 1$ there is an a with $\widehat{a}-a\prec\mathfrak{v}^n\mathfrak{m}$.*

Proof. We arrange $\mathfrak{m}=1$ in the usual way. Let Δ be the convex subgroup of Γ that is cofinal in $v(\widehat{a}-K)$. Lemma 3.3.13 gives an element $\gamma\in v(\widehat{a}-K)$ with $\gamma\geqslant\delta/m$ for some $m\geqslant 1$. Hence $v(\widehat{a}-K)$ contains for every $n\geqslant 1$ an element $>n\delta$. □

Combining Lemma 3.2.36 with Corollary 3.3.15 yields:

Corollary 3.3.16. *If K is r-linearly newtonian,* ω*-free if $r>1$, and $(P,\mathfrak{m},\widehat{a})$ is Z-minimal and deep, then for all $n\geqslant 1$ there is an a such that $\widehat{a}-a\prec\mathfrak{v}^n\mathfrak{m}$.*

Normal slots. We say that our slot $(P, \mathfrak{m}, \widehat{a})$ in K, with linear part L, is **normal** if order $L = r$ and, with $\mathfrak{v} := \mathfrak{v}(L)$ and $w := \operatorname{wt}(P)$,

(N1) $\mathfrak{v} \prec^\flat 1$;
(N2) $(P_{\times\mathfrak{m}})_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$.

Note that then $\mathfrak{v} \prec 1$, $\operatorname{dwt}(L) < r$, $(P, \mathfrak{m}, \widehat{a})$ is steep, and

$$P_{\times\mathfrak{m}} \sim_{\Delta(\mathfrak{v})} P(0) + (P_{\times\mathfrak{m}})_1 \qquad (\text{so } \operatorname{ddeg} P_{\times\mathfrak{m}} \leqslant 1). \tag{3.3.1}$$

If order $L = r$, $\mathfrak{v} := \mathfrak{v}(L)$, and L is monic, then $(P_{\times\mathfrak{m}})_1 \asymp \mathfrak{v}^{-1}$, so that (N2) is then equivalent to: $(P_{\times\mathfrak{m}})_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^w$. If $\deg P = 1$, then order $L = r$ and (N2) automatically holds, hence $(P, \mathfrak{m}, \widehat{a})$ is normal iff it is steep. Thus by Lemma 3.1.21:

Lemma 3.3.17. *If* $\deg P = 1$ *and* $\operatorname{nwt}(L) < r$, *then* $(P^\phi, \mathfrak{m}, \widehat{a})$ *is normal, eventually.*

If $(P, \mathfrak{m}, \widehat{a})$ is normal, then so are $(P_{\times\mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{a}/\mathfrak{n})$ and $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$. In particular, $(P, \mathfrak{m}, \widehat{a})$ is normal iff $(P_{\times\mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$ is normal. If $(P, \mathfrak{m}, \widehat{a})$ is normal, then so is any equivalent slot. Hence by (3.3.1) and Lemmas 3.2.9 and 3.2.14:

Lemma 3.3.18. *If* $(P, \mathfrak{m}, \widehat{a})$ *is normal, and* $(P, \mathfrak{m}, \widehat{a})$ *is* Z*-minimal or is a hole in* K*, then* $\operatorname{ddeg} P_{\times\mathfrak{m}} = \operatorname{dmul} P_{\times\mathfrak{m}} = 1$.

Example. Let $K \supseteq \mathbb{R}(\mathrm{e}^x)$ be an H-subfield of $\mathbb{T}$, $\mathfrak{m} = 1$, $r = 2$. If $P = D + R$ where

$$D \ = \ \mathrm{e}^{-x}\, Y'' - Y, \qquad R \ = \ f + \mathrm{e}^{-4x}\, Y^5 \quad (f \in K),$$

then $\mathfrak{v} = -\mathrm{e}^{-x} \prec^\flat 1$, $P_1 = D \sim -Y$, $w = 2$, and $P_{>1} = \mathrm{e}^{-4x}\, Y^5 \prec_{\Delta(\mathfrak{v})} \mathrm{e}^{-3x}\, P_1$, so $(P, 1, \widehat{a})$ is normal. However, if $P = D + S$ with D as above and $S = f + \mathrm{e}^{-3x}\, Y^5$ ($f \in K$), then $P_{>1} = \mathrm{e}^{-3x}\, Y^5 \succcurlyeq_{\Delta(\mathfrak{v})} \mathrm{e}^{-3x}\, P_1$, so $(P, 1, \widehat{a})$ is not normal.

Lemma 3.3.19. *Suppose* $\operatorname{order}(L) = r$ *and* $\mathfrak{v}$ *is such that* (N1) *and* (N2) *hold, and* $\mathfrak{v}(L) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. *Then* $(P, \mathfrak{m}, \widehat{a})$ *is normal.*

Proof. Put $\mathfrak{w} := \mathfrak{v}(L)$. Then $[\mathfrak{w}] = [\mathfrak{v}]$, and so $\mathfrak{v} \prec^\flat 1$ gives $\mathfrak{w} \prec^\flat 1$. Also,

$$(P_{\times\mathfrak{m}})_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1 \asymp_{\Delta(\mathfrak{v})} \mathfrak{w}^{w+1}(P_{\times\mathfrak{m}})_1.$$

Hence (N1), (N2) hold with $\mathfrak{w}$ in place of $\mathfrak{v}$. □

Lemma 3.3.20. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is normal and* $\phi \preccurlyeq 1$ *is active. Then the slot* $(P^\phi, \mathfrak{m}, \widehat{a})$ *in* K^ϕ *is normal.*

Proof. We arrange $\mathfrak{m} = 1$ and put $\mathfrak{v} := \mathfrak{v}(L)$, $\mathfrak{w} := \mathfrak{v}(L_{P^\phi})$. Now $L_{P^\phi} = L^\phi$, so $\mathfrak{v} \asymp_{\Delta(\mathfrak{v})} \mathfrak{w}$ and $\mathfrak{v} \prec^\flat_\phi 1$ by Lemma 3.1.20. By [ADH, 11.1.1], $[\phi] < [\mathfrak{v}]$, and (N2) we have

$$(P^\phi)_{>1} \ = \ (P_{>1})^\phi \ \asymp_{\Delta(\mathfrak{v})} \ P_{>1} \ \prec_{\Delta(\mathfrak{v})} \ \mathfrak{v}^{w+1} P_1 \ \asymp_{\Delta(\mathfrak{v})} \ \mathfrak{v}^{w+1} P_1^\phi,$$

which by Lemma 3.3.19 applied to $(P^\phi, 1, \widehat{a})$ in the role of $(P, \mathfrak{m}, \widehat{a})$ gives normality of $(P^\phi, 1, \widehat{a})$. □

Corollary 3.3.21. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is normal. Then* $(P, \mathfrak{m}, \widehat{a})$ *is quasilinear.*

Proof. Lemma 3.2.21 gives $\operatorname{ndeg} P_{\times\mathfrak{m}} \geqslant 1$. The parenthetical remark after (3.3.1) above and Lemma 3.3.20 gives $\operatorname{ndeg} P_{\times\mathfrak{m}} \leqslant 1$. □

Combining Lemmas 3.3.18 and 3.3.20 yields:

Corollary 3.3.22. *If $(P,\mathfrak{m},\widehat{a})$ is normal and linear, and $(P,\mathfrak{m},\widehat{a})$ is Z-minimal or a hole in K, then $(P,\mathfrak{m},\widehat{a})$ is deep.*

There are a few occasions later where we need to change the "monomial" $\mathfrak{m}$ in $(P,\mathfrak{m},\widehat{a})$ while preserving key properties of this slot. Here is what we need:

Lemma 3.3.23. *Let $u \in K$, $u \asymp 1$. Then $(P,u\mathfrak{m},\widehat{a})$ refines $(P,\mathfrak{m},\widehat{a})$, and if $(P_{+a},\mathfrak{n},\widehat{a}-a)$ refines $(P,\mathfrak{m},\widehat{a})$, then so does $(P_{+a},u\mathfrak{n},\widehat{a}-a)$. If $(P,\mathfrak{m},\widehat{a})$ is quasilinear, respectively deep, respectively normal, then so is $(P,u\mathfrak{m},\widehat{a})$.*

Proof. The refinement claims are clearly true, and quasilinearity is preserved since $\operatorname{ndeg} P_{\times u\mathfrak{m}} = \operatorname{ndeg} P_{\times\mathfrak{m}}$ by [ADH, 11.2.3(iii)]. "Steep" is preserved by Lemma 3.3.1, and hence "deep" is preserved using (0.1) and [ADH, 6.6.5(ii)]. Normality is preserved because steepness is,

$$(P_{\times u\mathfrak{m}})_d \;=\; (P_d)_{\times u\mathfrak{m}} \;\asymp\; (P_d)_{\times\mathfrak{m}} \;=\; (P_{\times\mathfrak{m}})_d \quad \text{for all } d\in\mathbb{N}$$

by [ADH, 4.3, 4.5.1(ii)], and $\mathfrak{v}(L_{P_{\times u\mathfrak{m}}}) \asymp \mathfrak{v}(L_{P_{\times\mathfrak{m}}})$ by Lemma 3.1.2. □

Here is a useful invariance property of normal slots:

Lemma 3.3.24. *Suppose $(P,\mathfrak{m},\widehat{a})$ is normal and $a \prec \mathfrak{m}$. Then L_P and $L_{P_{+a}}$ have order r. If in addition K is $\uplambda$-free or $r=1$, then $\mathscr{E}^{\mathrm{e}}(L_P) = \mathscr{E}^{\mathrm{e}}(L_{P_{+a}})$.*

Proof. $L_{P_{\times\mathfrak{m}}} = L_P\mathfrak{m}$ (so L_P has order r), and $L_{P_{+a,\times\mathfrak{m}}} = L_{P_{\times\mathfrak{m},+a/\mathfrak{m}}} = L_{P_{+a}}\mathfrak{m}$. The slot $(P_{\times\mathfrak{m}},1,\widehat{a}/\mathfrak{m})$ in K is normal and $a/\mathfrak{m} \prec 1$. Thus we can apply Lemma 3.1.28(i) to $\widehat{K}$, $P_{\times\mathfrak{m}}$, $a/\mathfrak{m}$ in place of K, P, a to give order $L_{P_{+a}} = r$. Next, applying likewise Lemma 3.1.29 with $L := L_{P_{\times\mathfrak{m}}}$, $\mathfrak{v} := \mathfrak{v}(L_{P_{\times\mathfrak{m}}})$, $m=r$, $B=0$, gives

$$L_P\mathfrak{m} - L_{P_{+a}}\mathfrak{m} \;=\; L_{P_{\times\mathfrak{m}}} - L_{P_{\times\mathfrak{m},+a/\mathfrak{m}}} \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{r+1}L_P\mathfrak{m}.$$

Hence, if K is $\uplambda$-free, then $\mathscr{E}^{\mathrm{e}}(L_P\mathfrak{m}) = \mathscr{E}^{\mathrm{e}}(L_{P_{+a}}\mathfrak{m})$ by Lemma 3.1.23, so

$$\mathscr{E}^{\mathrm{e}}(L_P) \;=\; \mathscr{E}^{\mathrm{e}}(L_P\mathfrak{m}) + v(\mathfrak{m}) \;=\; \mathscr{E}^{\mathrm{e}}(L_{P_{+a}}\mathfrak{m}) + v(\mathfrak{m}) \;=\; \mathscr{E}^{\mathrm{e}}(L_{P_{+a}}).$$

If $r=1$ we obtain the same equality from Corollary 3.1.24. □

Normality under refinements. In this subsection we study how normality behaves under more general refinements. This is not needed to prove the main result of this section, Theorem 3.3.33, but is included to obtain useful variants of it.

Proposition 3.3.25. *Suppose $(P,\mathfrak{m},\widehat{a})$ is normal. Let a refinement $(P_{+a},\mathfrak{m},\widehat{a}-a)$ of $(P,\mathfrak{m},\widehat{a})$ be given. Then this refinement is also normal.*

Proof. By the remarks following the definition of "multiplicative conjugate" in Section 3.2 and after replacing the slots $(P,\mathfrak{m},\widehat{a})$ and $(P_{+a},\mathfrak{m},\widehat{a}-a)$ in K by $(P_{\times\mathfrak{m}},1,\widehat{a}/\mathfrak{m})$ and $\big(P_{\times\mathfrak{m},+a/\mathfrak{m}},1,(\widehat{a}-a)/\mathfrak{m}\big)$, respectively, we arrange that $\mathfrak{m}=1$. Let $\mathfrak{v} := \mathfrak{v}(L_P)$. By Lemma 3.1.28 we have $\operatorname{order}(L_{P_{+a}}) = r$, $\mathfrak{v}(L_{P_{+a}}) \sim_{\Delta(\mathfrak{v})} \mathfrak{v}$, and $(P_{+a})_1 \sim_{\Delta(\mathfrak{v})} P_1$. Using [ADH, 4.5.1(i)] we have for $d>1$ with $P_d \neq 0$,

$$(P_{+a})_d \;=\; \big((P_{\geqslant d})_{+a}\big)_d \;\preccurlyeq\; (P_{\geqslant d})_{+a} \;\sim\; P_{\geqslant d} \;\preccurlyeq\; P_{>1},$$

and using (N2), this yields

$$(P_{+a})_{>1} \;\preccurlyeq\; P_{>1} \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{w+1}P_1 \;\asymp\; \mathfrak{v}^{w+1}(P_{+a})_1.$$

Hence (N2) holds with $\mathfrak{m}=1$ and with P replaced by P_{+a}. Thus $(P_{+a},1,\widehat{a}-a)$ is normal, by Lemma 3.3.19. □

Proposition 3.3.26. *Suppose $(P,\mathfrak{m},\widehat{a})$ is a normal hole in K, $\widehat{a}\prec\mathfrak{n}\preccurlyeq\mathfrak{m}$, and $[\mathfrak{n}/\mathfrak{m}]\leqslant\big[\mathfrak{v}(L_{P_{\times\mathfrak{m}}})\big]$. Then the refinement $(P,\mathfrak{n},\widehat{a})$ of $(P,\mathfrak{m},\widehat{a})$ is also normal.*

Proof. As in the proof of Lemma 3.3.1 we arrange $\mathfrak{m}=1$ and set $L:=L_P$, $\mathfrak{v}:=\mathfrak{v}(L)$, and $\widetilde{L}:=L_{P_{\times\mathfrak{n}}}$, to obtain $[\mathfrak{n}]\leqslant[\mathfrak{v}]$ and $\mathfrak{v}(\widetilde{L})\asymp_{\Delta(\mathfrak{v})}\mathfrak{v}$. Recall from [ADH, 4.3] that $(P_{\times\mathfrak{n}})_d=(P_d)_{\times\mathfrak{n}}$ for $d\in\mathbb{N}$. For such d we have by [ADH, 6.1.3],

$$(P_d)_{\times\mathfrak{n}}\ \asymp_{\Delta(\mathfrak{v})}\ \mathfrak{n}^dP_d\ \preccurlyeq\ \mathfrak{n}^dP_{\geqslant d}.$$

In particular, $(P_{\times\mathfrak{n}})_1\asymp_{\Delta(\mathfrak{v})}\mathfrak{n}P_1$. By (N2) we also have, for $d>1$:

$$P_{\geqslant d}\ \preccurlyeq\ P_{>1}\ \prec_{\Delta(\mathfrak{v})}\ \mathfrak{v}^{w+1}P_1.$$

By Lemma 3.3.18 we have $P\sim P_1$. For $d>1$ we have by [ADH, 6.1.3],

$$\mathfrak{n}^dP\ \asymp\ \mathfrak{n}^dP_1\ \asymp_{\Delta(\mathfrak{v})}\ \mathfrak{n}^{d-1}(P_1)_{\times\mathfrak{n}}\ \preccurlyeq\ (P_1)_{\times\mathfrak{n}}\ =\ (P_{\times\mathfrak{n}})_1\ \preccurlyeq\ P_{\times\mathfrak{n}}$$

and thus

$$(P_{\times\mathfrak{n}})_d\ =\ (P_d)_{\times\mathfrak{n}}\ \preccurlyeq_{\Delta(\mathfrak{v})}\ \mathfrak{n}^dP_{\geqslant d}\ \prec_{\Delta(\mathfrak{v})}\ \mathfrak{v}^{w+1}\mathfrak{n}^dP_1\ \preccurlyeq_{\Delta(\mathfrak{v})}\ \mathfrak{v}^{w+1}(P_{\times\mathfrak{n}})_1.$$

Hence (N2) holds with $\mathfrak{m}$ replaced by $\mathfrak{n}$. Thus $(P,\mathfrak{n},\widehat{a})$ is normal, using $\mathfrak{v}(\widetilde{L})\asymp_{\Delta(\mathfrak{v})}\mathfrak{v}$ and Lemmas 3.3.1 and 3.3.19. ☐

From Lemma 3.2.14 and Proposition 3.3.26 we obtain:

Corollary 3.3.27. *Suppose $(P,\mathfrak{m},\widehat{a})$ is normal and Z-minimal, $\widehat{a}\prec\mathfrak{n}\preccurlyeq\mathfrak{m}$, and $[\mathfrak{n}/\mathfrak{m}]\leqslant\big[\mathfrak{v}(L_{P_{\times\mathfrak{m}}})\big]$. Then the refinement $(P,\mathfrak{n},\widehat{a})$ of $(P,\mathfrak{m},\widehat{a})$ is also normal.*

In the rest of this subsection $\mathfrak{m}=1$, $\widehat{a}\prec\mathfrak{n}\prec1$, $\operatorname{order}(L_P)=r$, and $[\mathfrak{v}]<[\mathfrak{n}]$ where $\mathfrak{v}:=\mathfrak{v}(L_P)$. So $(P,\mathfrak{n},\widehat{a})$ refines $(P,1,\widehat{a})$, $L_{P_{\times\mathfrak{n}}}=L_P\mathfrak{n}$, and $\operatorname{order}L_{P_{\times\mathfrak{n}}}=r$.

Lemma 3.3.28. *Suppose $(P,1,\widehat{a})$ is steep, $\mathfrak{v}(L_{P_{\times\mathfrak{n}}})\preccurlyeq\mathfrak{v}$, and $P_{>1}\preccurlyeq P_1$. Then $(P,\mathfrak{n},\widehat{a})$ is normal.*

Proof. Put $\mathfrak{w}:=\mathfrak{v}(L_{P_{\times\mathfrak{n}}})$. Then $[\mathfrak{w}]<[\mathfrak{n}]$ by Corollary 3.1.10, and $\mathfrak{w}\preccurlyeq\mathfrak{v}\prec^\flat1$ gives $\mathfrak{w}\prec^\flat1$. It remains to show that $(P_{\times\mathfrak{n}})_{>1}\prec_{\Delta(\mathfrak{w})}\mathfrak{w}^{w+1}(P_{\times\mathfrak{n}})_1$. Using $[\mathfrak{n}]>[\mathfrak{w}]$ it is enough that $(P_{\times\mathfrak{n}})_{>1}\prec_\Delta\mathfrak{w}^{w+1}(P_{\times\mathfrak{n}})_1$, where $\Delta:=\Delta(\mathfrak{n})$. Since $\mathfrak{w}\asymp_\Delta1$, it is even enough that $(P_{\times\mathfrak{n}})_{>1}\prec_\Delta(P_{\times\mathfrak{n}})_1$, to be derived below. Let $d>1$. Then by [ADH, 6.1.3] and $P_d\preccurlyeq P_{>1}\preccurlyeq P_1$ we have

$$(P_{\times\mathfrak{n}})_d\ =\ (P_d)_{\times\mathfrak{n}}\ \asymp_\Delta\ P_d\mathfrak{n}^d\ \preccurlyeq\ P_1\,\mathfrak{n}^d.$$

In view of $\mathfrak{n}\prec_\Delta1$ and $d>1$ we have

$$P_1\,\mathfrak{n}^d\ \prec_\Delta\ P_1\,\mathfrak{n}\ \asymp_\Delta\ (P_1)_{\times\mathfrak{n}}\ =\ (P_{\times\mathfrak{n}})_1,$$

using again [ADH, 6.1.3]. Thus $(P_{\times\mathfrak{n}})_d\prec_\Delta(P_{\times\mathfrak{n}})_1$, as promised. ☐

Corollary 3.3.29. *If $(P,1,\widehat{a})$ is normal and $\mathfrak{v}(L_{P_{\times\mathfrak{n}}})\preccurlyeq\mathfrak{v}$, then $(P,\mathfrak{n},\widehat{a})$ is normal.*

In the next lemma and its corollary K is d-valued and for every $q\in\mathbb{Q}^>$ there is given an element $\mathfrak{n}^q$ of $K^\times$ such that $(\mathfrak{n}^q)^\dagger=q\mathfrak{n}^\dagger$; the remark before Lemma 3.1.15 gives $v(\mathfrak{n}^q)=qv(\mathfrak{n})$ for $q\in\mathbb{Q}^>$. Hence for $0<q\leqslant1$ in $\mathbb{Q}$ we have $\widehat{a}\prec\mathfrak{n}\preccurlyeq\mathfrak{n}^q\prec1$, so $(P,\mathfrak{n}^q,\widehat{a})$ refines $(P,1,\widehat{a})$.

Lemma 3.3.30. *Suppose $(P,1,\widehat{a})$ is steep and $P_{>1}\preccurlyeq P_1$. Then $(P,\mathfrak{n}^q,\widehat{a})$ is normal, for all but finitely many $q\in\mathbb{Q}$ with $0<q\leqslant1$.*

Proof. We have $\mathfrak{n}^\dagger \succcurlyeq 1$ by $\mathfrak{n} \prec \mathfrak{v} \prec 1$ and $\mathfrak{v}^\dagger \succcurlyeq 1$. Lemma 3.1.16 gives $\mathfrak{v}(L_{P_{\times \mathfrak{n}^q}}) \preccurlyeq \mathfrak{v}$ for all but finitely many $q \in \mathbb{Q}^>$. Suppose $\mathfrak{v}(L_{P_{\times \mathfrak{n}^q}}) \preccurlyeq \mathfrak{v}$, $0 < q \leqslant 1$ in $\mathbb{Q}$. Then $(P, \mathfrak{n}^q, \widehat{a})$ is normal by Lemma 3.3.28 applied with $\mathfrak{n}^q$ instead of $\mathfrak{n}$. $\square$

Corollary 3.3.31. *If $(P, 1, \widehat{a})$ is normal, then $(P, \mathfrak{n}^q, \widehat{a})$ is normal for all but finitely many $q \in \mathbb{Q}$ with $0 < q \leqslant 1$.*

Normalizing. If in this subsection $\operatorname{order}(L_{P_{\times \mathfrak{m}}}) = r$, then $\mathfrak{v} := \mathfrak{v}(L_{P_{\times \mathfrak{m}}})$. Towards proving that normality can always be achieved we first show:

Lemma 3.3.32. *Suppose Γ is divisible, $(P, \mathfrak{m}, \widehat{a})$ is a deep hole in K, and $\widehat{a} - a \prec \mathfrak{v}^{w+2}\mathfrak{m}$ for some a. Then $(P, \mathfrak{m}, \widehat{a})$ has a refinement that is deep and normal.*

Proof. Replacing $(P, \mathfrak{m}, \widehat{a})$ by $(P_{\times \mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$ and renaming we arrange $\mathfrak{m} = 1$. Take a such that $\widehat{a} - a \prec \mathfrak{v}^{w+2}$. For $e := w + \frac{3}{2}$, let $\mathfrak{v}^e$ be an element of $K^\times$ with $v(\mathfrak{v}^e) = e\, v(\mathfrak{v})$. *Claim*: the refinement $(P_{+a}, \mathfrak{v}^e, \widehat{a} - a)$ of $(P, 1, \widehat{a})$ is deep and normal. By Lemma 3.3.7, $(P_{+a}, \mathfrak{v}^e, \widehat{a} - a)$ is deep, so we do have $\operatorname{order}(L_{P_{+a, \times \mathfrak{v}^e}}) = r$ and $\mathfrak{v}(L_{P_{+a, \times \mathfrak{v}^e}}) \prec^\flat 1$. Lemma 3.3.7 also yields $\mathfrak{v}(L_{P_{+a, \times \mathfrak{v}^e}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. Since $\operatorname{ddeg} P = \operatorname{dmul} P = 1$, we can use Corollary 3.2.20 for $\mathfrak{n} = \mathfrak{v}^e$ and for $\mathfrak{n} = 1$ to obtain

$$\operatorname{ddeg} P_{+a, \times \mathfrak{v}^e} \;=\; \operatorname{dmul} P_{+a, \times \mathfrak{v}^e} \;=\; \operatorname{ddeg} P_{+a} \;=\; \operatorname{dmul} P_{+a} = 1$$

and thus $(P_{+a, \times \mathfrak{v}^e})_1 \sim P_{+a, \times \mathfrak{v}^e}$; also $P_1 \sim P \sim P_{+a} \sim (P_{+a})_1$, where $P \sim P_{+a}$ follows from $a \prec 1$ and [ADH, 4.5.1(i)]. Now let $d > 1$. Then

$$\begin{aligned} (P_{+a, \times \mathfrak{v}^e})_d \;&\asymp_{\Delta(\mathfrak{v})}\; (\mathfrak{v}^e)^d (P_{+a})_d \;\preccurlyeq\; (\mathfrak{v}^e)^d P_{+a} \;\sim\; (\mathfrak{v}^e)^d (P_{+a})_1 \\ &\asymp_{\Delta(\mathfrak{v})}\; (\mathfrak{v}^e)^{d-1} (P_{+a, \times \mathfrak{v}^e})_1 \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{w+1} (P_{+a, \times \mathfrak{v}^e})_1, \end{aligned}$$

using [ADH, 6.1.3] for $\asymp_{\Delta(\mathfrak{v})}$. So $(P_{+a}, \mathfrak{v}^e, \widehat{a} - a)$ is normal by Lemma 3.3.19. $\square$

We can now finally show:

Theorem 3.3.33. *Suppose K is $\upomega$-free and r-linearly newtonian, and Γ is divisible. Then every Z-minimal slot in K of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep and normal, eventually.*

Proof. By Lemma 3.2.14 it is enough to show this for Z-minimal holes in K of order r. Given such hole in K, use Corollary 3.3.12 to refine it to a hole $(P, \mathfrak{m}, \widehat{a})$ such that $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep, eventually. Replacing $(P, \mathfrak{m}, \widehat{a})$ by $(P^\phi, \mathfrak{m}, \widehat{a})$ for a suitable active $\phi \preccurlyeq 1$ we arrange that $(P, \mathfrak{m}, \widehat{a})$ itself is deep. Then an appeal to Corollary 3.3.16 followed by an application of Lemma 3.3.32 yields a deep and normal refinement of $(P, \mathfrak{m}, \widehat{a})$. Now apply Lemma 3.3.20 to this refinement. $\square$

Next we indicate some variants of Theorem 3.3.33:

Corollary 3.3.34. *Suppose K is* d*-valued and $\upomega$-free, and Γ is divisible. Then every minimal hole in K of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep and normal, eventually.*

Proof. Given a minimal hole in K of order r, use Corollary 3.3.12 to refine it to a hole $(P, \mathfrak{m}, \widehat{a})$ in K such that $\operatorname{nwt} L_{P_{\times \mathfrak{m}}} = 0$ and $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep, eventually. If $\deg P = 1$, then $(P^\phi, \mathfrak{m}, \widehat{a})$ is normal, eventually, by Lemma 3.3.17. If $\deg P > 1$, then K is r-linearly newtonian by Corollary 3.2.6, so we can use Theorem 3.3.33. $\square$

For $r = 1$ we can follow the proof of Theorem 3.3.33, using Corollary 3.3.10 in place of Corollary 3.3.12, to obtain:

Corollary 3.3.35. *If K is 1-linearly newtonian and Γ is divisible, then every quasilinear Z-minimal slot in K of order 1 has a refinement $(P,\mathfrak{m},\widehat{a})$ such that $(P^{\phi},\mathfrak{m},\widehat{a})$ is deep and normal, eventually.*

Here is another variant of Theorem 3.3.33:

Proposition 3.3.36. *If K is* d*-valued and $\upomega$-free, and Γ is divisible, then every Z-minimal special slot in K of order r has a refinement $(P,\mathfrak{m},\widehat{a})$ such that $(P^{\phi},\mathfrak{m},\widehat{a})$ is deep and normal, eventually.*

To establish this proposition we follow the proof of Theorem 3.3.33, using Lemma 3.2.35 to preserve specialness in the initial refining. Corollary 3.3.15 takes over the role of Corollary 3.3.16 in that proof.

For linear slots in K we can weaken the hypotheses of Theorem 3.3.33:

Corollary 3.3.37. *Suppose $\deg P = 1$. Then $(P,\mathfrak{m},\widehat{a})$ has a refinement $(P,\mathfrak{n},\widehat{a})$ such that $(P^{\phi},\mathfrak{n},\widehat{a})$ is deep and normal, eventually. Moreover, if K is $\uplambda$-free and $r>1$, then $(P^{\phi},\mathfrak{m},\widehat{a})$ is deep and normal, eventually.*

Proof. By the remarks before Lemma 3.3.17, $(P,\mathfrak{m},\widehat{a})$ is normal iff it is steep. Moreover, if $(P,\mathfrak{m},\widehat{a})$ is normal, then it is quasilinear by Corollary 3.3.21, and hence $(P^{\phi},\mathfrak{m},\widehat{a})$ is deep and normal, eventually, by the remarks before Example 3.3.3 and Lemma 3.3.20. By Lemma 3.3.2, $(P,\mathfrak{m},\widehat{a})$ has a refinement $(P,\mathfrak{n},\widehat{a})$ such that $(P^{\phi},\mathfrak{n},\widehat{a})$ is steep, eventually. This yields the first part. The second part follows from Corollary 3.1.22 and Lemma 3.3.17. $\square$

Corollary 3.3.38. *Suppose K is $\uplambda$-free, Γ is divisible, and $(P,\mathfrak{m},\widehat{a})$ is a quasilinear minimal hole in K of order $r=1$. Then $(P,\mathfrak{m},\widehat{a})$ has a refinement $(Q,\mathfrak{n},\widehat{b})$ such that $(Q^{\phi},\mathfrak{n},\widehat{b})$ is deep and normal, eventually.*

Proof. The case $\deg P = 1$ is part of Corollary 3.3.37. If $\deg P>1$, then K is 1-linearly newtonian by Lemma 3.2.5, so we can use Corollary 3.3.35. $\square$

Improving normality. *In this subsection $L := L_{P_{\times\mathfrak{m}}}$.* Note that if $(P,\mathfrak{m},\widehat{a})$ is a normal hole in K, then $P_{\times\mathfrak{m}} \sim (P_{\times\mathfrak{m}})_1$ by Lemma 3.3.18. We call our slot $(P,\mathfrak{m},\widehat{a})$ in K **strictly normal** if it is normal, but with the condition (N2) replaced by the stronger condition

(N2s) $(P_{\times\mathfrak{m}})_{\neq 1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$.

Thus for normal $(P,\mathfrak{m},\widehat{a})$ and $\mathfrak{v}=\mathfrak{v}(L)$ we have:

$$(P,\mathfrak{m},\widehat{a}) \text{ is strictly normal} \iff P(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1.$$

So if $(P,\mathfrak{m},\widehat{a})$ is normal and $P(0)=0$, then $(P,\mathfrak{m},\widehat{a})$ is strictly normal. Note that if $(P,\mathfrak{m},\widehat{a})$ is strictly normal, then

$$P_{\times\mathfrak{m}} \sim_{\Delta(\mathfrak{v})} (P_{\times\mathfrak{m}})_1 \qquad (\text{and hence } \operatorname{ddeg} P_{\times\mathfrak{m}} = 1).$$

If $(P,\mathfrak{m},\widehat{a})$ is strictly normal, then so are $(P_{\times\mathfrak{n}},\mathfrak{m}/\mathfrak{n},\widehat{a}/\mathfrak{n})$ and $(bP,\mathfrak{m},\widehat{a})$ for $b\neq 0$. Thus $(P,\mathfrak{m},\widehat{a})$ is strictly normal iff $(P_{\times\mathfrak{m}},1,\widehat{a}/\mathfrak{m})$ is strictly normal. If $(P,\mathfrak{m},\widehat{a})$ is strictly normal, then so is every equivalent slot in K. The proof of Lemma 3.3.23 shows that if $(P,\mathfrak{m},\widehat{a})$ is strictly normal and $u\in K$, $u\asymp 1$, then $(P,u\mathfrak{m},\widehat{a})$ is also strictly normal. The analogue of Lemma 3.3.19 goes through, with $(P_{\times\mathfrak{m}})_{\neq 1}$ instead of $(P_{\times\mathfrak{m}})_{>1}$ in the proof:

Lemma 3.3.39. *Suppose* $\operatorname{order}(L) = r$ *and* $\mathfrak{v}$ *are such that* (N1) *and* (N2s) *hold, and* $\mathfrak{v}(L) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. *Then* $(P, \mathfrak{m}, \widehat{a})$ *is strictly normal.*

Lemma 3.3.20 goes likewise through with "strictly normal" instead of "normal":

Lemma 3.3.40. *If* $(P, \mathfrak{m}, \widehat{a})$ *is strictly normal and* $\phi \preccurlyeq 1$ *is active, then the slot* $(P^{\phi}, \mathfrak{m}, \widehat{a})$ *in* K^{ϕ} *is strictly normal.* (*Hence if* $(P, \mathfrak{m}, \widehat{a})$ *is strictly normal, then* $(P, \mathfrak{m}, \widehat{a})$ *is quasilinear, and if in addition* $(P, \mathfrak{m}, \widehat{a})$ *is linear, then it is deep.*)

As to Proposition 3.3.25, here is a weak version for strict normality:

Lemma 3.3.41. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is a strictly normal hole in* K *and* $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+w+1}\mathfrak{m}$ *where* $\mathfrak{v} := \mathfrak{v}(L)$. *Then its refinement* $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ *is also strictly normal.*

Proof. As in the proof of Proposition 3.3.25 we arrange $\mathfrak{m} = 1$. We can also assume $P_1 \asymp 1$. From $P = P(0) + P_1 + P_{>1}$ we get

$$P(a) \ = \ P(0) + P_1(a) + P_{>1}(a),$$

where $P(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}$ and $P_{>1}(a) \preccurlyeq P_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}$ by (N2s) and $a \prec 1$; we show that also $P_1(a) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}$. To see this note that

$$0 \ = \ P(\widehat{a}) \ = \ P(0) + P_1(\widehat{a}) + P_{>1}(\widehat{a}),$$

where as before $P(0), P_{>1}(\widehat{a}) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}$, so $P_1(\widehat{a}) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}$. Lemma 1.1.10 applied to $(\widehat{K}, \preccurlyeq_{\Delta(\mathfrak{v})}, P_1)$ in place of $(K, \preccurlyeq, P)$, with $m = w+1$, $y = a - \widehat{a}$, yields $P_1(a - \widehat{a}) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}$, hence

$$P_1(a) \ = \ P_1(a - \widehat{a}) + P_1(\widehat{a}) \ \prec_{\Delta(\mathfrak{v})} \ \mathfrak{v}^{w+1}$$

as claimed. It remains to use $\mathfrak{v}(L_{P_{+a}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$ and the normality of $(P_{+a}, 1, \widehat{a} - a)$ obtained from Proposition 3.3.25 and its proof. □

We also have a version of Lemma 3.3.41 for Z-minimal slots, obtained from that lemma via Lemma 3.2.14:

Lemma 3.3.42. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is* Z*-minimal and strictly normal. Set* $\mathfrak{v} := \mathfrak{v}(L)$, *and suppose* $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+w+1}\mathfrak{m}$. *Then the refinement* $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ *of* $(P, \mathfrak{m}, \widehat{a})$ *is strictly normal.*

Next two versions of Proposition 3.3.26:

Lemma 3.3.43. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is a strictly normal hole in* K, $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$, *and* $[\mathfrak{n}/\mathfrak{m}] < \big[\mathfrak{v}(L)\big]$. *Then the refinement* $(P, \mathfrak{n}, \widehat{a})$ *of* $(P, \mathfrak{m}, \widehat{a})$ *is strictly normal.*

Proof. As in the proof of Proposition 3.3.26 we arrange $\mathfrak{m} = 1$ and, setting $\mathfrak{v} := \mathfrak{v}(L)$, $\widetilde{L} := L_{P_{\times\mathfrak{n}}}$, show that $\operatorname{order}(\widetilde{L}) = r$, $\mathfrak{v}(\widetilde{L}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$, and that (N2) holds with $\mathfrak{m}$ replaced by $\mathfrak{n}$. Now $[\mathfrak{n}] < [\mathfrak{v}]$ yields $\mathfrak{n} \asymp_{\Delta(\mathfrak{v})} 1$; together with $(P_{\times\mathfrak{n}})_1 \asymp_{\Delta(\mathfrak{v})} \mathfrak{n}P_1$ this gives $P(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}P_1 \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{n}})_1$. Hence (N2s) holds with $\mathfrak{m}$ replaced by $\mathfrak{n}$. Lemma 3.3.39 now yields that $(P, \mathfrak{n}, \widehat{a})$ is strictly normal. □

Lemma 3.3.44. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is a strictly normal hole in* K *and* $\widehat{a} \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$ *where* $\mathfrak{v} := \mathfrak{v}(L)$. *Assume also that for all* $q \in \mathbb{Q}^{>}$ *there is given an element* $\mathfrak{v}^q$ *of* $K^{\times}$ *with* $v(\mathfrak{v}^q) = q\,v(\mathfrak{v})$. *Then for all sufficiently small* $q \in \mathbb{Q}^{>}$ *and* $\mathfrak{n}$ *with* $\mathfrak{n} \asymp \mathfrak{v}^q\mathfrak{m}$ *we have:* $\widehat{a} \prec \mathfrak{n}$ *and the refinement* $(P, \mathfrak{n}, \widehat{a})$ *of* $(P, \mathfrak{m}, \widehat{a})$ *is strictly normal.*

Proof. We arrange $\mathfrak{m} = 1$ as usual, and take $q_0 \in \mathbb{Q}^>$ with $\widehat{a} \prec \mathfrak{v}^{q_0}$ and $P(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1+q_0} P_1$. Let $q \in \mathbb{Q}$, $0 < q \leqslant q_0$, and suppose $\mathfrak{n} \asymp \mathfrak{v}^q$. Then $(P, \mathfrak{n}, \widehat{a})$ is a refinement of $(P, 1, \widehat{a})$, and the proof of Proposition 3.3.26 gives: $\widetilde{L} := L_{P_{\times\mathfrak{n}}}$ has order r with $\mathfrak{v}(\widetilde{L}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$, $\mathfrak{n}P_1 \asymp_{\Delta(\mathfrak{v})} (P_{\times\mathfrak{n}})_1$, and (N2) holds with $\mathfrak{m}$ replaced by $\mathfrak{n}$. Hence

$$P(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1+q_0} P_1 \preccurlyeq \mathfrak{v}^{w+1} \mathfrak{n} P_1 \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} (P_{\times\mathfrak{n}})_1.$$

As in the proof of the last lemma we conclude that $(P, \mathfrak{n}, \widehat{a})$ is strictly normal. $\square$

Remark 3.3.45. In Lemmas 3.3.43 and 3.3.44 we assumed that $(P, \mathfrak{m}, \widehat{a})$ is a strictly normal hole in K. By Lemma 3.2.14 these lemmas go through if this hypothesis is replaced by "$(P, \mathfrak{m}, \widehat{a})$ is a strictly normal Z-minimal slot in K".

We now turn to refining a given normal hole to a strictly normal hole. We only do this under additional hypotheses, tailored so that we may employ Lemma 3.1.17. Therefore we assume in the rest of this subsection: *K is* d*-valued and for all $\mathfrak{v}$ and $q \in \mathbb{Q}^>$ we are given an element $\mathfrak{v}^q$ of $K^\times$ with $(\mathfrak{v}^q)^\dagger = q\mathfrak{v}^\dagger$.* Note that then $v(\mathfrak{v}^q) = q\,v(\mathfrak{v})$ for such q. (In particular, Γ is divisible.) We also adopt the convention that if $\operatorname{order} L = r$, then $\mathfrak{v} := \mathfrak{v}(L)$.

Lemma 3.3.46. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is a normal hole in K and $\widehat{a} - a \preccurlyeq \mathfrak{v}^{w+2}\mathfrak{m}$. Then the refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ is strictly normal.*

Proof. As usual we arrange that $\mathfrak{m} = 1$. By Proposition 3.3.25, $(P_{+a}, 1, \widehat{a} - a)$ is normal; the proof of this proposition gives $\operatorname{order}(L_{P_{+a}}) = r$, $\mathfrak{v}(L_{P_{+a}}) \sim_{\Delta(\mathfrak{v})} \mathfrak{v}$, $(P_{+a})_1 \sim_{\Delta(\mathfrak{v})} P_1$, and (N2) holds with $\mathfrak{m} = 1$ and P replaced by P_{+a}. It remains to show that $P_{+a}(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{+a})_1$, equivalently, $P(a) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} P_1$.

Let $\widehat{L} := L_{P_{+\widehat{a}}} \in \widehat{K}[\partial]$ and $R := P_{>1} \in K\{Y\}$; note that $P_{(\boldsymbol{i})} = R_{(\boldsymbol{i})}$ for $|\boldsymbol{i}| > 1$ and $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} P_1$. Hence Taylor expansion and $P(\widehat{a}) = 0$ give

$$\begin{aligned} P(a) &= P(\widehat{a}) + \widehat{L}(a - \widehat{a}) + \sum_{|\boldsymbol{i}|>1} P_{(\boldsymbol{i})}(\widehat{a}) \cdot (a - \widehat{a})^{\boldsymbol{i}} \\ &= \widehat{L}(a - \widehat{a}) + \sum_{|\boldsymbol{i}|>1} R_{(\boldsymbol{i})}(\widehat{a}) \cdot (a - \widehat{a})^{\boldsymbol{i}} \end{aligned}$$

$$\text{where } R_{(\boldsymbol{i})}(\widehat{a}) \cdot (a - \widehat{a})^{\boldsymbol{i}} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} P_1 \text{ for } |\boldsymbol{i}| > 1,$$

so it is enough to show $\widehat{L}(a - \widehat{a}) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} P_1$. Lemma 3.1.28 applied to $(\widehat{K}, \widehat{a})$ in place of (K, a) gives $\operatorname{order} \widehat{L} = r$ and $\widehat{L} \sim_{\Delta(\mathfrak{v})} L$. Since $\widehat{K}$ is d-valued, Lemma 3.1.17 yields a $q \in \mathbb{Q}$ with $w + 1 < q \leqslant w + 2$ and a $\mathfrak{w}$ such that $\widehat{L}\mathfrak{v}^q \asymp \mathfrak{w}\,\mathfrak{v}^q\,\widehat{L}$ where $[\mathfrak{w}] \leqslant [\mathfrak{v}^\dagger]$ and hence $\mathfrak{w} \asymp_{\Delta(\mathfrak{v})} 1$ (see the remark before Lemma 3.3.1). With $\mathfrak{n} \asymp a - \widehat{a}$ we have $\mathfrak{n} \preccurlyeq \mathfrak{v}^{w+2} \preccurlyeq \mathfrak{v}^q \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}$ and therefore

$$\widehat{L}(a - \widehat{a}) \preccurlyeq \widehat{L}\mathfrak{n} \preccurlyeq \widehat{L}\mathfrak{v}^q \asymp \mathfrak{w}\,\mathfrak{v}^q\,\widehat{L} \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}^q\widehat{L} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}\widehat{L}.$$

Hence $\widehat{L}(a - \widehat{a}) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} P_1$ as required. $\square$

In particular, if $(P, \mathfrak{m}, \widehat{a})$ is a normal hole in K and $\widehat{a} \preccurlyeq \mathfrak{v}^{w+2}\mathfrak{m}$, then $(P, \mathfrak{m}, \widehat{a})$ is strictly normal.

Corollary 3.3.47. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, deep, and normal. If $(P, \mathfrak{m}, \widehat{a})$ is special, then $(P, \mathfrak{m}, \widehat{a})$ has a deep and strictly normal refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ where $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$ and $\mathfrak{v}(L_{P_{+a,\times\mathfrak{m}}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. (Note that if K is r-linearly newtonian, and $\boldsymbol{\omega}$-free if $r > 1$, then $(P, \mathfrak{m}, \widehat{a})$ is special by Lemma 3.2.36.)*

Proof. By Lemma 3.2.14 we arrange that $(P, \mathfrak{m}, \widehat{a})$ is a hole in K. If $(P, \mathfrak{m}, \widehat{a})$ is special, Corollary 3.3.15 gives an a such that $\widehat{a} - a \preccurlyeq \mathfrak{v}^{w+2}\mathfrak{m}$, and then the refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ is strictly normal by Lemma 3.3.46, and deep with $\mathfrak{v}(L_{P_{+a,\times\mathfrak{m}}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$ by Lemma 3.3.7. □

This leads to a useful variant of Theorem 3.3.33:

Corollary 3.3.48. *Suppose K is ω-free and r-linearly newtonian. Then every Z-minimal slot in K of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep and strictly normal, eventually.*

Proof. Let a Z-minimal slot in K of order r be given. Use Theorem 3.3.33 to refine it to a slot $(P, \mathfrak{m}, \widehat{a})$ in K with an active ϕ_0 such that the slot $(P^{\phi_0}, \mathfrak{m}, \widehat{a})$ in K^{ϕ_0} is deep and normal. Corollary 3.3.47 gives a deep and strictly normal refinement $(P^{\phi_0}_{+a}, \mathfrak{m}, \widehat{a} - a)$ of $(P^{\phi_0}, \mathfrak{m}, \widehat{a})$. By Lemma 3.3.40 the slot $(P^{\phi}_{+a}, \mathfrak{m}, \widehat{a} - a)$ in K^ϕ is deep and strictly normal, for all active $\phi \preccurlyeq \phi_0$ (in K). Thus $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ refines the original Z-minimal slot in K and has the desired property. □

Corollaries 3.2.6 and 3.3.48 have the following consequence:

Corollary 3.3.49. *Suppose K is ω-free. Then every minimal hole in K of order r and degree > 1 has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep and strictly normal, eventually.*

Corollary 3.3.47 also gives the following variant of Corollary 3.3.48, where the role of Theorem 3.3.33 in its proof is taken over by Proposition 3.3.36:

Corollary 3.3.50. *Suppose K is ω-free. Then every Z-minimal special slot in K of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^\phi, \mathfrak{m}, \widehat{a})$ is deep and strictly normal, eventually.*

3.4. Isolated Slots

In this short section we study the concept of isolation, which plays well together with normality. *Throughout this section K is an H-asymptotic field with small derivation and with rational asymptotic integration.* We let a, b range over K and ϕ, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{w}$ over $K^\times$. We also let $(P, \mathfrak{m}, \widehat{a})$ be a slot in K of order $r \geqslant 1$. Recall that $v(\widehat{a} - K)$ is a cut in Γ without largest element. Note that $v\big((\widehat{a} - a) - K\big) = v(\widehat{a} - K)$ and $v(\widehat{a}\mathfrak{n} - K) = v(\widehat{a} - K) + v\mathfrak{n}$.

Definition 3.4.1. We say that $(P, \mathfrak{m}, \widehat{a})$ is **isolated** if for all $a \prec \mathfrak{m}$,

$$\operatorname{order}(L_{P_{+a}}) = r \quad \text{and} \quad \mathscr{E}^{\mathrm{e}}(L_{P_{+a}}) \cap v(\widehat{a} - K) \ < \ v(\widehat{a} - a);$$

equivalently, for all $a \prec \mathfrak{m}$: $\operatorname{order}(L_{P_{+a}}) = r$ and whenever $\mathfrak{w} \preccurlyeq \widehat{a} - a$ is such that $v(\mathfrak{w}) \in \mathscr{E}^{\mathrm{e}}(L_{P_{+a}})$, then $\mathfrak{w} \prec \widehat{a} - b$ for all b.

In particular, if $(P, \mathfrak{m}, \widehat{a})$ is isolated, then $v(\widehat{a}) \notin \mathscr{E}^{\mathrm{e}}(L_P)$. If $(P, \mathfrak{m}, \widehat{a})$ is isolated, then so is every equivalent slot in K, as well as $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$ and the slot $(P^\phi, \mathfrak{m}, \widehat{a})$ in K^ϕ for active ϕ in K. Moreover:

Lemma 3.4.2. *If $(P, \mathfrak{m}, \widehat{a})$ is isolated, then so is any refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of it.*

Proof. For the case $\mathfrak{n} = \mathfrak{m}$, use $v\big((\widehat{a} - a) - K\big) = v(\widehat{a} - K)$. The case $a = 0$ is clear. The general case reduces to these two special cases. □

Lemma 3.4.3. *Suppose $(P,\mathfrak{m},\widehat{a})$ is isolated. Then the multiplicative conjugate $(P_{\times\mathfrak{n}},\mathfrak{m}/\mathfrak{n},\widehat{a}/\mathfrak{n})$ of $(P,\mathfrak{m},\widehat{a})$ by $\mathfrak{n}$ is isolated.*

Proof. Let $a \prec \mathfrak{m}/\mathfrak{n}$. Then $a\mathfrak{n} \prec \mathfrak{m}$, so $\operatorname{order}(L_{P_{\times\mathfrak{n},+a}}) = \operatorname{order}(L_{P_{+a\mathfrak{n},\times\mathfrak{n}}}) = \operatorname{order}(L_{P_{+a\mathfrak{n}}}) = r$. Suppose $\mathfrak{w} \preccurlyeq (\widehat{a}/\mathfrak{n})-a$ and $v(\mathfrak{w}) \in \mathscr{E}^{\mathrm{e}}\big(L_{P_{\times\mathfrak{n},+a}}\big)$. Now $L_{P_{\times\mathfrak{n},+a}} = L_{P_{+a\mathfrak{n},\times\mathfrak{n}}} = L_{P_{+a\mathfrak{n}}}\mathfrak{n}$ and thus $\mathfrak{w}\mathfrak{n} \preccurlyeq \widehat{a}-a\mathfrak{n}$, $v(\mathfrak{w}\mathfrak{n}) \in \mathscr{E}^{\mathrm{e}}\big(P_{+a\mathfrak{n}}\big)$. But $(P,\mathfrak{m},\widehat{a})$ is isolated, so $v(\mathfrak{w}\mathfrak{n}) > v(\widehat{a}-K)$ and hence $v(\mathfrak{w}) > v\big((\widehat{a}/\mathfrak{n})-K\big)$. Thus $(P_{\times\mathfrak{n}},\mathfrak{m}/\mathfrak{n},\widehat{a}/\mathfrak{n})$ is isolated. $\square$

Lemma 3.4.4. *Suppose K is $\boldsymbol{\lambda}$-free or $r=1$, and $(P,\mathfrak{m},\widehat{a})$ is normal. Then*

$$(P,\mathfrak{m},\widehat{a}) \text{ is isolated} \iff \mathscr{E}^{\mathrm{e}}(L_P)\cap v(\widehat{a}-K) \ \leqslant\ v\mathfrak{m}.$$

Proof. Use Lemma 3.3.24; for the direction $\Rightarrow$, use also that $\widehat{a}-a \prec \mathfrak{m}$ iff $a \prec \mathfrak{m}$. $\square$

Lemma 3.4.5. *Suppose* $\deg P = 1$. *Then*

$$(P,\mathfrak{m},\widehat{a}) \text{ is isolated} \iff \mathscr{E}^{\mathrm{e}}(L_P)\cap v(\widehat{a}-K) \ \leqslant\ v\mathfrak{m}.$$

Proof. Use that $\operatorname{order} L_P = r$ and $L_{P_{+a}} = L_P$ for all a. $\square$

Proposition 3.4.6. *Suppose K is $\boldsymbol{\lambda}$-free or $r=1$, and $(P,\mathfrak{m},\widehat{a})$ is normal. Then $(P,\mathfrak{m},\widehat{a})$ has an isolated refinement.*

Proof. Suppose $(P,\mathfrak{m},\widehat{a})$ is not already isolated. Then Lemma 3.4.4 gives γ with

$$\gamma \in \mathscr{E}^{\mathrm{e}}(L_P)\cap v(\widehat{a}-K), \quad \gamma > v\mathfrak{m}.$$

We have $|\mathscr{E}^{\mathrm{e}}(L_P)| \leqslant r$, by [ADH, p. 481] if $r=1$, and Corollary 1.7.11 and $\boldsymbol{\lambda}$-freeness of K if $r>1$. Hence we can take $\gamma := \max \mathscr{E}^{\mathrm{e}}(L_P)\cap v(\widehat{a}-K)$, and then $\gamma > v\mathfrak{m}$. Take a and $\mathfrak{n}$ with $v(\widehat{a}-a) > \gamma = v(\mathfrak{n})$; then $(P_{+a},\mathfrak{n},\widehat{a}-a)$ is a refinement of $(P,\mathfrak{m},\widehat{a})$ and $a \prec \mathfrak{m}$. Let $b \prec \mathfrak{n}$; then $a+b \prec \mathfrak{m}$, so by Lemma 3.3.24,

$$\operatorname{order}(L_{(P_{+a})_{+b}}) \ =\ r, \qquad \mathscr{E}^{\mathrm{e}}(L_{(P_{+a})_{+b}}) \ =\ \mathscr{E}^{\mathrm{e}}(L_P).$$

Also $v\big((\widehat{a}-a)-b\big) > \gamma$, hence

$$\mathscr{E}^{\mathrm{e}}\big(L_{(P_{+a})_{+b}}\big)\cap v\big((\widehat{a}-a)-K\big) \ =\ \mathscr{E}^{\mathrm{e}}(L_P)\cap v(\widehat{a}-K) \ \leqslant\ \gamma \ <\ v\big((\widehat{a}-a)-b\big).$$

Thus $(P_{+a},\mathfrak{n},\widehat{a}-a)$ is isolated. $\square$

Remark 3.4.7. Proposition 3.4.6 goes through if instead of assuming that $(P,\mathfrak{m},\widehat{a})$ is normal, we assume that $(P,\mathfrak{m},\widehat{a})$ is linear. (Same argument, using Lemma 3.4.5 in place of Lemma 3.4.4 and $L_{(P_{+a})_{+b}} = L_P$ in place of Lemma 3.3.24.)

Corollary 3.4.8. *Suppose $r=1$, and $(P,\mathfrak{m},\widehat{a})$ is normal or linear. If $\mathscr{E}^{\mathrm{e}}(L_P)=\emptyset$, then $(P,\mathfrak{m},\widehat{a})$ is isolated. If $\mathscr{E}^{\mathrm{e}}(L_P)\neq\emptyset$, so $\mathscr{E}^{\mathrm{e}}(L_P)=\{v\mathfrak{g}\}$ where $\mathfrak{g}\in K^{\times}$, then $(P,\mathfrak{m},\widehat{a})$ is isolated iff $\mathfrak{m}\preccurlyeq\mathfrak{g}$ or $\widehat{a}-K\succ\mathfrak{g}$.*

This follows immediately from Lemmas 3.4.4 and 3.4.5. The results in the rest of this subsection are the *raison d'être* of isolated holes:

Proposition 3.4.9. *Suppose K is $\boldsymbol{\omega}$-free and $(P,\mathfrak{m},\widehat{a})$ is an isolated hole in K which is normal or linear. Let $\widehat{b}$ in an immediate asymptotic extension of K satisfy $P(\widehat{b})=0$ and $\widehat{b}\prec\mathfrak{m}$. Then $v(\widehat{a}-a)=v(\widehat{b}-a)$ for all a, so $\widehat{b}\notin K$.*

Proof. Replacing $(P, \mathfrak{m}, \widehat{a})$, $\widehat{b}$ by $(P_{\times\mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$, $\widehat{b}/\mathfrak{m}$, we arrange $\mathfrak{m} = 1$. Let a be given; we show $v(\widehat{a} - a) = v(\widehat{b} - a)$. This is clear if $a \succcurlyeq 1$, so assume $a \prec 1$. Corollary 3.3.21 (if $(P, \mathfrak{m}, \widehat{a})$ is normal) and Lemma 3.2.21 (if $(P, \mathfrak{m}, \widehat{a})$ is linear) give $\operatorname{ndeg} P = 1$. Thus P is in newton position at a by Corollary 3.2.23. Moreover $v(\widehat{a} - a) \notin \mathscr{E}^{\mathrm{e}}(L_{P_{+a}})$, hence $v(\widehat{a} - a) = v^{\mathrm{e}}(P, a)$ by Lemma 1.7.15. Likewise, if $v(\widehat{b} - a) \notin \mathscr{E}^{\mathrm{e}}(L_{P_{+a}})$, then $v(\widehat{b} - a) = v^{\mathrm{e}}(P, a)$ by Lemma 1.7.15, so $v(\widehat{a} - a) = v(\widehat{b} - a)$.

Thus to finish the proof it is enough to show that $\mathscr{E}^{\mathrm{e}}(L_{P_{+a}}) \cap v(\widehat{b} - K) \leqslant 0$. Now $|\mathscr{E}^{\mathrm{e}}(L_{P_{+a}})| \leqslant r$ by Corollary 1.4.5, so we have $b \prec 1$ such that

$$\mathscr{E}^{\mathrm{e}}(L_{P_{+a}}) \cap v(\widehat{b} - K) \ < \ v(\widehat{b} - b),$$

in particular, $v(\widehat{b} - b) \notin \mathscr{E}^{\mathrm{e}}(L_{P_{+a}})$. If $(P, \mathfrak{m}, \widehat{a})$ is normal, then Lemma 3.3.24 gives

$$\mathscr{E}^{\mathrm{e}}(L_{P_{+a}}) \ = \ \mathscr{E}^{\mathrm{e}}(L_P) \ = \ \mathscr{E}^{\mathrm{e}}(L_{P_{+b}}),$$

so by the above with b instead of a we have $v(\widehat{a} - b) = v(\widehat{b} - b)$. If $(P, \mathfrak{m}, \widehat{a})$ is linear, then $L_{P_{+a}} = L_P = L_{P_{+b}}$, and we obtain likewise $v(\widehat{a} - b) = v(\widehat{b} - b)$. Hence

$$\mathscr{E}^{\mathrm{e}}(L_{P_{+a}}) \cap v(\widehat{b} - K) \ \subseteq \ \mathscr{E}^{\mathrm{e}}(L_{P_{+a}}) \cap \Gamma^{< v(\widehat{a} - b)} \ \subseteq \ \mathscr{E}^{\mathrm{e}}(L_P) \cap v(\widehat{a} - K) \ \leqslant \ 0.$$

using Lemmas 3.4.4 and 3.4.5 for the last step. $\square$

Combining Proposition 3.4.9 with Corollary 3.2.15 yields:

Corollary 3.4.10. *Let K, $(P, \mathfrak{m}, \widehat{a})$, $\widehat{b}$ be as in Proposition 3.4.9, and assume also that $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal. Then there is an isomorphism $K\langle\widehat{a}\rangle \to K\langle\widehat{b}\rangle$ of valued differential fields over K sending $\widehat{a}$ to $\widehat{b}$.*

Using the Normalization Theorem, we now obtain:

Corollary 3.4.11. *Suppose K is $\upomega$-free and Γ is divisible. Then every minimal hole in K of order r has an isolated refinement $(P, \mathfrak{m}, \widehat{a})$ such that for any $\widehat{b}$ in an immediate asymptotic extension of K with $P(\widehat{b}) = 0$ and $\widehat{b} \prec \mathfrak{m}$ there is an isomorphism $K\langle\widehat{a}\rangle \to K\langle\widehat{b}\rangle$ of valued differential fields over K sending $\widehat{a}$ to $\widehat{b}$.*

Proof. Given a minimal linear hole in K of order r, use Remark 3.4.7 to refine it to an isolated minimal linear hole $(P, \mathfrak{m}, \widehat{a})$ in K of order r, and use Corollary 3.4.10. Suppose we are given a minimal nonlinear hole in K of order r. Then K is r-linearly newtonian by Corollary 3.2.6. Then Theorem 3.3.33 yields a refinement $(Q, \mathfrak{w}, \widehat{d})$ of it and an active θ in K such that the minimal hole $(Q^{\theta}, \mathfrak{w}, \widehat{d})$ in K^{θ} is normal. Proposition 3.4.6 gives an isolated refinement $(Q^{\theta}_{+d}, \mathfrak{v}, \widehat{d} - d)$ of $(Q^{\theta}, \mathfrak{w}, \widehat{d})$. Suitably refining $(Q^{\theta}_{+d}, \mathfrak{v}, \widehat{d} - d)$ further followed by compositionally conjugating with a suitable active element of K^{θ} yields by Theorem 3.3.33 and Lemma 3.4.2 a refinement $(P, \mathfrak{m}, \widehat{a})$ of $(Q, \mathfrak{w}, \widehat{d})$ (and thus of the originally given hole) and an active ϕ in K such that $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is both normal and isolated. Then $(P, \mathfrak{m}, \widehat{a})$ is isolated, and we can apply Corollary 3.4.10 to K^{ϕ} and $(P^{\phi}, \mathfrak{m}, \widehat{a})$ in the role of K and $(P, \mathfrak{m}, \widehat{a})$. $\square$

For $r = 1$ we can replace "$\upomega$-free" in Proposition 3.4.9 and Corollary 3.4.10 by the weaker "$\uplambda$-free" (same proofs, using Lemma 1.7.20 instead of Lemma 1.7.15):

Proposition 3.4.12. *Suppose K is λ-free, $(P,\mathfrak{m},\widehat{a})$ is an isolated hole in K of order $r=1$, and suppose $(P,\mathfrak{m},\widehat{a})$ is normal or linear. Let $\widehat{b}$ in an immediate asymptotic extension of K satisfy $P(\widehat{b})=0$ and $\widehat{b}\prec\mathfrak{m}$. Then $v(\widehat{a}-a)=v(\widehat{b}-a)$ for all a. (Hence if $(P,\mathfrak{m},\widehat{a})$ is Z-minimal, then there is an isomorphism $K\langle\widehat{a}\rangle\to K\langle\widehat{b}\rangle$ of valued differential fields over K sending $\widehat{a}$ to $\widehat{b}$.)*

This leads to an analogue of Corollary 3.4.11:

Corollary 3.4.13. *Suppose K is λ-free and Γ is divisible. Then every quasilinear minimal hole in K of order $r=1$ has an isolated refinement $(P,\mathfrak{m},\widehat{a})$ such that for any $\widehat{b}$ in an immediate asymptotic extension of K with $P(\widehat{b})=0$ and $\widehat{b}\prec\mathfrak{m}$ there is an isomorphism $K\langle\widehat{a}\rangle\to K\langle\widehat{b}\rangle$ of valued differential fields over K sending $\widehat{a}$ to $\widehat{b}$.*

Proof. Suppose we are given a quasilinear minimal hole in K of order $r=1$. Then Corollary 3.3.38 yields a refinement $(Q,\mathfrak{w},\widehat{d})$ of it and an active θ in K such that the quasilinear minimal hole $(Q^{\theta},\mathfrak{w},\widehat{d})$ in K^{θ} of order 1 is normal. Proposition 3.4.6 gives an isolated refinement $(Q^{\theta}_{+d},\mathfrak{v},\widehat{d}-d)$ of $(Q^{\theta},\mathfrak{w},\widehat{d})$, and then Corollary 3.3.38 yields a refinement $(P,\mathfrak{m},\widehat{a})$ of $(Q,\mathfrak{w},\widehat{d})$ and an active ϕ in K such that $(P^{\phi},\mathfrak{m},\widehat{a})$ is normal and isolated. Now apply Proposition 3.4.12 with K^{ϕ} and $(P^{\phi},\mathfrak{m},\widehat{a})$ in the role of K and $(P,\mathfrak{m},\widehat{a})$. □

Next a variant of Lemma 3.2.1 for $r=1$ without assuming ω-freeness:

Corollary 3.4.14. *Suppose K is 1-newtonian and Γ is divisible. Then K has no quasilinear Z-minimal slot of order 1.*

Proof. By Proposition 1.7.28, K is λ-free. Towards a contradiction, let $(P,\mathfrak{m},\widehat{a})$ be a quasilinear Z-minimal slot in K of order 1. By Lemma 3.2.14 we arrange that $(P,\mathfrak{m},\widehat{a})$ is a hole in H. Using Corollary 3.3.35, Lemma 3.4.2 and the remark before it, and Proposition 3.4.6, we can refine further so that $(P^{\phi},\mathfrak{m},\widehat{a})$ is normal and isolated for some active ϕ in K. Then there is no $y\in K$ with $P(y)=0$ and $y\prec\mathfrak{m}$, by Proposition 3.4.12, contradicting Lemma 3.2.27 for $L=K$. □

Finally, for isolated linear holes, without additional hypotheses:

Lemma 3.4.15. *Suppose $(P,\mathfrak{m},\widehat{a})$ is an isolated linear hole in K, and $\widehat{a}-a\prec\mathfrak{m}$. Then $P(a)\neq 0$, and $\gamma=v(\widehat{a}-a)$ is the unique element of $\Gamma\setminus\mathscr{E}^{\mathrm{e}}(L_P)$ such that $v^{\mathrm{e}}_{L_P}(\gamma)=v\big(P(a)\big)$.*

Proof. By Lemma 3.4.5, $\gamma:=v(\widehat{a}-a)\in\Gamma\setminus\mathscr{E}^{\mathrm{e}}(L_P)$. Since $\deg P=1$,

$$L_P(\widehat{a}-a)\ =\ L_P(\widehat{a})-L_P(a)\ =\ -P(0)-L_P(a)\ =\ -P(a),$$

so $P(a)\neq 0$. By Lemma 1.4.6, $v^{\mathrm{e}}_{L_P}(\gamma)=v\big(L_P(\widehat{a}-a)\big)=v\big(P(a)\big)$. □

In [8] we shall prove a version of Proposition 3.4.9 without the hypothesis that $\widehat{b}$ lies in an immediate extension of K. In Section 4.4 below we consider, in a more restricted setting, a variant of isolated slots, with ultimate exceptional values taking over the role played by exceptional values in Definition 3.4.1.

In this section K is a d-valued field of H-type with small derivation and rational asymptotic integration. (Later on we will impose additional restrictions on K.) We also let $\widehat{K}$ be an immediate asymptotic extension of K. We focus here on slots of complexity $(1,1,1)$ in K. As a byproduct we obtain in Corollary 3.5.18 a partial generalization of Corollary 3.3.49 to minimal holes in K of arbitrary degree. First we establish in the next subsection a useful formal identity. We let j, k range over $\mathbb{N}$ (in addition to m, n, as usual).

An integration identity. Let R be a differential ring, and let f, g, h, range over R. We use $\int f = g + \int h$ as a suggestive way to express that $f = g' + h$, and likewise, $\int f = g - \int h$ means that $f = g' - h$. For example,

$$\int f'g \ = \ fg - \int fg' \qquad \text{(integration by parts)}.$$

Let $\mathrm{e}, \xi \in R^\times$ satisfy $\mathrm{e}^\dagger = \xi$. We wish to expand $\int \mathrm{e}$ by iterated integration by parts. Now for $g = \mathrm{e}$ we have $g' \in R^\times$ with $\frac{g}{g'} = \frac{1}{\xi}$, so in view of $\mathrm{e} = g' \frac{\mathrm{e}}{g'}$:

$$\int \mathrm{e} \ = \ \int g' \frac{\mathrm{e}}{g'} \ = \ \frac{\mathrm{e}}{\xi} - \int g\left(\frac{\mathrm{e}}{g'}\right)',$$

and

$$\left(\frac{\mathrm{e}}{g'}\right)' = \left(\frac{1}{\xi}\right)' = \frac{-\xi'}{\xi^2} = \frac{-\xi^\dagger}{\xi},$$

and thus

$$\int \mathrm{e} \ = \ \frac{\mathrm{e}}{\xi} + \int \frac{\xi^\dagger}{\xi}\,\mathrm{e}\,.$$

More generally, using the above identities for $g = \mathrm{e}$,

$$\begin{aligned}
\int f\,\mathrm{e} \ &= \ \int g' f \frac{\mathrm{e}}{g'} \ = \ \frac{f}{\xi}\,\mathrm{e} - \int g\left(f\frac{\mathrm{e}}{g'}\right)' \\
&= \ \frac{f}{\xi}\,\mathrm{e} - \int g\left(f'\frac{\mathrm{e}}{g'} + f\left(\frac{\mathrm{e}}{g'}\right)'\right) \ = \ \frac{f}{\xi}\,\mathrm{e} - \int\left(\frac{f'}{\xi}\,\mathrm{e} + fg\left(\frac{-\xi^\dagger}{\xi}\right)\right) \\
&= \ \frac{f}{\xi}\,\mathrm{e} - \int\left(\frac{f'}{\xi}\,\mathrm{e} + \frac{-f\xi^\dagger}{\xi}\,\mathrm{e}\right) \ = \ \frac{f}{\xi}\,\mathrm{e} + \int\left(\frac{\xi^\dagger f - f'}{\xi}\right)\mathrm{e}\,.
\end{aligned}$$

Replacing f by f/ξ^k gives the following variant of this identity:

$$\int \frac{f}{\xi^k}\,\mathrm{e} \ = \ \frac{f}{\xi^{k+1}}\,\mathrm{e} + \int \frac{(k+1)\xi^\dagger f - f'}{\xi^{k+1}}\,\mathrm{e}\,.$$

Induction on m using the last identity yields:

Lemma 3.5.1. *Set $\zeta := \xi^\dagger$. Then*

$$\int f\,\mathrm{e} \ = \ \sum_{j=0}^{m} P_j(\zeta, f)\frac{\mathrm{e}}{\xi^{j+1}} + \int P_{m+1}(\zeta, f)\frac{\mathrm{e}}{\xi^{m+1}},$$

where the $P_j \in \mathbb{Q}\{Z,V\} = \mathbb{Q}\{Z\}\{V\}$ are independent of R, e, ξ:

$$P_0 \ := \ V, \qquad P_{j+1} \ := \ (j+1)ZP_j - P_j'.$$

Thus $P_j = P_{j0}V + P_{j1}V' + \cdots + P_{jj}V^{(j)}$ with all $P_{jk} \in \mathbb{Q}\{Z\}$ and $P_{jj} = (-1)^j$.

For example,

$$P_0 = V, \quad P_1 = ZV - V', \quad P_2 = (2Z^2 - Z')V - 3ZV' + V''.$$

An asymptotic expansion. *In this subsection* $\xi \in K$ *and* $\xi \succ^\flat 1$; equivalently, $\xi \in K$ satisfies $\xi \succ 1$ and $\zeta := \xi^\dagger \succcurlyeq 1$. *We also assume that* $\xi \notin \mathrm{I}(K) + K^\dagger$. Since $\widehat{K}$ is d-valued of H-type with asymptotic integration, it has by [ADH, 10.2.7] an immediate asymptotic extension $\widehat{K}(\phi)$ with $\phi' = \xi$. Then the algebraic closure of $\widehat{K}(\phi)$ is still d-valued of H-type, by [ADH, 9.5], and so [ADH, 10.4.1] yields a d-valued H-asymptotic extension L of this algebraic closure with an element $\mathrm{e} \neq 0$ such that $\mathrm{e}^\dagger = \xi$. All we need about L below is that it is a d-valued H-asymptotic extension of $\widehat{K}$ with elements ϕ and e such that $\phi' = \xi$ and $\mathrm{e} \neq 0$, $\mathrm{e}^\dagger = \xi$. Note that then L has small derivation, and $\xi \succ^\flat 1$ in L. (The element ϕ will only play an auxiliary role later in this subsection.)

Lemma 3.5.2. $v(\mathrm{e}) \notin \Gamma$.

Proof. Suppose otherwise. Take $a \in K^\times$ with $a\,\mathrm{e} \asymp 1$. Then $a^\dagger + \xi = (a\,\mathrm{e})^\dagger \in \mathrm{I}(L) \cap K = \mathrm{I}(K)$ and thus $\xi \in \mathrm{I}(K) + K^\dagger$, a contradiction. $\square$

By Lemma 3.5.2 there is for each $g \in L$ at most one $\widehat{f} \in \widehat{K}$ with $\big(\widehat{f}\frac{\mathrm{e}}{\xi}\big)' = g$. Let $f \in K^\times$ be given with $f \preccurlyeq 1$, and suppose $\widehat{f} \in \widehat{K}$ satisfies $\big(\widehat{f}\frac{\mathrm{e}}{\xi}\big)' = f\,\mathrm{e}$. Our aim is to show that with P_j as in Lemma 3.5.1, the series $\sum_{j=0}^\infty P_j(\zeta, f)\frac{1}{\xi^j}$ is a kind of asymptotic expansion of $\widehat{f}$. The partial sums

$$f_m := \sum_{j=0}^m P_j(\zeta, f)\frac{1}{\xi^j}$$

of this series lie in K, with $f_0 = f$ and $f_n - f_m \prec \xi^{-m}$ for $m < n$, by Lemma 1.1.13. More precisely, we show:

Proposition 3.5.3. *We have* $\widehat{f} - f_m \prec \xi^{-m}$ *for all* m. (*Thus:* $f \asymp 1 \Rightarrow \widehat{f} \sim f$.)

Towards the proof, note that by Lemma 3.5.1 with $R = L$,

$$\widehat{f}\frac{\mathrm{e}}{\xi} \;=\; \sum_{j=0}^m P_j(\zeta, f)\frac{\mathrm{e}}{\xi^{j+1}} + I_m, \quad I_m \in L, \text{ and thus}$$

$$\widehat{f} \;=\; f_m + \frac{\xi}{\mathrm{e}} I_m \tag{3.5.1}$$

where $I_m \in \widehat{K}\,\mathrm{e}$ satisfies $I_m' = P_{m+1}(\zeta, f)\frac{\mathrm{e}}{\xi^{m+1}}$, a condition that determines I_m uniquely up to an additive constant from C_L. The proof of Proposition 3.5.3 now rests on the following lemmas:

Lemma 3.5.4. *In* L *we have* $(\mathrm{e}\,\xi^l)^{(k)} \sim \mathrm{e}\,\xi^{l+k}$, *for all* k *and all* $l \in \mathbb{Z}$.

This is Corollary 1.1.14 with our L in the role of K there, and taking e^ϕ there as our $\mathrm{e} \in L$; note that here our $\phi \in L$ with $\phi' = \xi$ is needed.

Lemma 3.5.5. *Suppose* $\mathrm{e} \succ \xi^{m+1}$. *Then* $\frac{\xi}{\mathrm{e}} I_m \prec \xi^{-m}$.

Proof. This amounts to $I_m \prec \frac{\mathrm{e}}{\xi^{m+1}}$. Suppose $I_m \succcurlyeq \frac{\mathrm{e}}{\xi^{m+1}} \succ 1$. Then we have $I_m' \succcurlyeq \big(\frac{\mathrm{e}}{\xi^{m+1}}\big)' \sim \frac{\mathrm{e}}{\xi^m}$ by Lemma 3.5.4, so $P_{m+1}(\zeta, f)\frac{\mathrm{e}}{\xi^{m+1}} \succcurlyeq \frac{\mathrm{e}}{\xi^m}$, and thus $P_{m+1}(\zeta, f) \succcurlyeq \xi$, contradicting Lemma 1.1.13. $\square$

Lemma 3.5.6. *Suppose* $\mathrm{e} \preccurlyeq \xi^m$. *Then* $I_m \prec 1$ *and* $\frac{\xi}{\mathrm{e}} I_m \prec \xi^{-m}$.

Proof. Lemma 1.1.13 gives

$$P_{m+1}(\zeta, f)\frac{\mathrm{e}}{\xi^{m+1}} \preccurlyeq \zeta^N \frac{\mathrm{e}}{\xi^{m+1}} \preccurlyeq \frac{\zeta^N}{\xi} \quad \text{for some } N \in \mathbb{N},$$

so $v(I'_m) > \Psi_L$, and thus $I_m \preccurlyeq 1$. If $I_m \asymp 1$, then $v(\frac{\xi}{\mathrm{e}} I_m) = v(\xi) - v(\mathrm{e}) \notin \Gamma$, contradicting $\frac{\xi}{\mathrm{e}} I_m = \widehat{f} - f_m \in \widehat{K}$, by (3.5.1). Thus $I_m \prec 1$. Now assume towards a contradiction that $\frac{\xi}{\mathrm{e}} I_m \succcurlyeq \xi^{-m}$. Then $\frac{\mathrm{e}}{\xi^{m+1}} \preccurlyeq I_m \prec 1$, so $I'_m \succcurlyeq \left(\frac{\mathrm{e}}{\xi^{m+1}}\right)' \sim \frac{\mathrm{e}}{\xi^m}$ by Lemma 3.5.4, and this yields a contradiction as in the proof of Lemma 3.5.5. □

Proof of Proposition 3.5.3. Let m be given. If $\mathrm{e} \succ \xi^{m+1}$, then $\widehat{f} - f_m = \frac{\xi}{\mathrm{e}} I_m \prec \xi^{-m}$ by Lemma 3.5.5. Suppose $\mathrm{e} \preccurlyeq \xi^{m+1}$. Then Lemma 3.5.6 (with $m+1$ instead of m) gives $\widehat{f} - f_{m+1} \prec \xi^{-(m+1)}$, hence $\widehat{f} - f_m = (\widehat{f} - f_{m+1}) + (f_{m+1} - f_m) \prec \xi^{-m}$. □

Application to linear differential equations of order 1. Proposition 3.5.3 yields information about the asymptotics of solutions (in $\widehat{K}$) of certain linear differential equations of order 1 over K:

Corollary 3.5.7. *Let* $f, \xi \in K$, $f \preccurlyeq 1$, $\xi \succ^\flat 1$, $\xi \notin \mathrm{I}(K) + K^\dagger$, *and suppose* $y \in \widehat{K}$ *satisfies* $y' + \xi y = f$. *Then there is for every* m *an element* $y_m \in K$ *with* $y - y_m \prec \xi^{-m}$. *Also,* $f \asymp 1 \Rightarrow y \sim f\xi^{-1}$.

Proof. Take L and $\mathrm{e} \in L$ as at the beginning of the previous subsection, and set $\widehat{f} := y\xi \in \widehat{K}$. Then for $A := \partial + \xi$ we have $A(\widehat{f}/\xi) = f$, so

$$\left(\widehat{f}\,\frac{\mathrm{e}}{\xi}\right)' \;=\; (\widehat{f}/\xi)'\,\mathrm{e} + (\widehat{f}/\xi)\xi\,\mathrm{e} \;=\; A(\widehat{f}/\xi)\,\mathrm{e} \;=\; f\,\mathrm{e},$$

hence $\widehat{f}$ is as in the previous subsection. Now apply Proposition 3.5.3. □

Corollary 3.5.8. *Let* $g \in K$, $u \in K^\times$ *be such that* $g \notin \mathrm{I}(K) + K^\dagger$ *and* $\xi := g + u^\dagger \succ^\flat 1$. *Suppose* $z \in \widehat{K}$ *satisfies* $z' + gz = u$. *Then* $z \sim u/\xi$, *and for every* m *there is a* $z_m \in K$ *such that* $z - z_m \prec u\xi^{-m}$.

Proof. Set $A := \partial + g$. Then $A_{\ltimes u} = \partial + \xi$, so $A(z) = u$ yields for $y := z/u$ that $y' + \xi y = 1$. Now observe that $\xi \notin \mathrm{I}(K) + K^\dagger$ and use the previous corollary. □

Slots of order and degree 1. In the rest of this section we use the material above to analyze slots of order and degree 1 in K. *Below K is henselian and $(P, \mathfrak{m}, \widehat{f})$ is a slot in K with* $\operatorname{order} P = \deg P = 1$ *and* $\widehat{f} \in \widehat{K} \setminus K$. *We let f range over K, $\mathfrak{n}$ over $K^\times$, and ϕ over active elements of K.* Thus

$$\begin{aligned} P \;&=\; a(Y' + gY - u) \quad \text{where } a \in K^\times,\ g, u \in K, \\ P_{\times\mathfrak{n}} \;&=\; a\mathfrak{n}\big(Y' + (g + \mathfrak{n}^\dagger)Y - \mathfrak{n}^{-1}u\big). \end{aligned}$$

Since K is henselian, $(P, \mathfrak{m}, \widehat{f})$ is Z-minimal and thus equivalent to a hole in K, by Lemma 3.2.14. Also, $\operatorname{nmul} P_{\times\mathfrak{m}} = \operatorname{ndeg} P_{\times\mathfrak{m}} = 1$ by Lemma 3.2.21. We have $L_P = a(\partial + g)$, so

$$g \in K^\dagger \iff \ker L_P \neq \{0\}, \qquad\qquad g \in \mathrm{I}(K) + K^\dagger \iff \mathscr{E}^{\mathrm{e}}(L_P) \neq \emptyset,$$

using for the second equivalence the remark on $\mathscr{E}^{\mathrm{e}}(A)$ preceding Lemma 1.4.9. If $(P, \mathfrak{m}, \widehat{f})$ is isolated, then $P(f) \neq 0$ for $\widehat{f} - f \prec \mathfrak{m}$ by Lemmas 3.2.14 and 3.4.15, so, taking $f = 0$, we have $u \neq 0$.

Lemma 3.5.9. *Suppose $\partial K = K$ and $\mathrm{I}(K) \subseteq K^\dagger$. Then $\mathscr{E}^{\mathrm{e}}(L_P) = \emptyset$, so $(P, \mathfrak{m}, \widehat{f})$ is isolated by Lemma 3.4.5.*

Proof. Passing to an equivalent hole in K, arrange that $(P, \mathfrak{m}, \widehat{f})$ is a hole in K. Since $\partial K = K$ and $\widehat{f} \in \widehat{K} \setminus K$, the remark following Lemma 1.7.21 yields $g \notin K^\dagger = \mathrm{I}(K) + K^\dagger$, therefore $\mathscr{E}^{\mathrm{e}}(L_P) = \emptyset$. $\square$

Set $\mathfrak{v} := \mathfrak{v}(L_{P_{\times \mathfrak{m}}})$; thus $\mathfrak{v} = 1$ if $g + \mathfrak{m}^\dagger \preccurlyeq 1$ and $\mathfrak{v} = 1/(g + \mathfrak{m}^\dagger)$ otherwise. Hence from Example 3.3.3 and the remarks before Lemma 3.3.17 we obtain:

$$(P, \mathfrak{m}, \widehat{f}) \text{ is normal} \iff (P, \mathfrak{m}, \widehat{f}) \text{ is steep} \iff \mathfrak{v} \prec^\flat 1,$$
$$(P, \mathfrak{m}, \widehat{f}) \text{ is deep} \iff \mathfrak{v} \prec^\flat 1 \text{ and } u \preccurlyeq \mathfrak{m}/\mathfrak{v}.$$

We have $P(0) = -au$, and if $\mathfrak{v} \prec 1$, then $(P_{\times \mathfrak{m}})_1 \sim (a\mathfrak{m}/\mathfrak{v})Y$. Thus

$$(P, \mathfrak{m}, \widehat{f}) \text{ is strictly normal} \iff \mathfrak{v} \prec^\flat 1 \text{ and } u \prec_{\Delta(\mathfrak{v})} \mathfrak{m}\mathfrak{v}.$$

We say that $(P, \mathfrak{m}, \widehat{f})$ is **balanced** if $(P, \mathfrak{m}, \widehat{f})$ is steep and $P(0) \preccurlyeq S_{P_{\times \mathfrak{m}}}(0)$, equivalently, $(P, \mathfrak{m}, \widehat{f})$ is steep and $u \preccurlyeq \mathfrak{m}$. Thus

$$(P, \mathfrak{m}, \widehat{f}) \text{ is strictly normal} \implies (P, \mathfrak{m}, \widehat{f}) \text{ is balanced} \implies (P, \mathfrak{m}, \widehat{f}) \text{ is deep},$$

and with $b \in K^\times$,

$$(P, \mathfrak{m}, \widehat{f}) \text{ is balanced} \iff (P_{\times \mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{f}/\mathfrak{n}) \text{ is balanced} \iff (bP, \mathfrak{m}, \widehat{f}) \text{ is balanced}.$$

If $(P, \mathfrak{m}, \widehat{f})$ is balanced, then so is any slot in K equivalent to $(P, \mathfrak{m}, \widehat{f})$. Moreover, if $(P, \mathfrak{m}, \widehat{f})$ is a hole in K, then $P(0) = -L_P(\widehat{f})$, so $(P, \mathfrak{m}, \widehat{f})$ is balanced iff it is steep and $L_P(\widehat{f}) \preccurlyeq S_{P_{\times \mathfrak{m}}}(0)$. By Corollary 3.3.14, if $(P, \mathfrak{m}, \widehat{f})$ is steep, then $\widehat{f} - f \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$ for some f. For balanced $(P, \mathfrak{m}, \widehat{f})$ we have a variant of this fact:

Lemma 3.5.10. *Suppose $(P, \mathfrak{m}, \widehat{f})$ is balanced and $g \notin \mathrm{I}(K) + K^\dagger$. Then there is for all n an f such that $\widehat{f} - f \prec \mathfrak{v}^n \mathfrak{m}$.*

Proof. Replacing $(P, \mathfrak{m}, \widehat{f})$ by an equivalent hole in K, we arrange that $(P, \mathfrak{m}, \widehat{f})$ is a hole in K, and replacing $(P, \mathfrak{m}, \widehat{f})$ by $(P_{\times \mathfrak{m}}, 1, \widehat{f}/\mathfrak{m})$, that $\mathfrak{m} = 1$. Then $\widehat{f}' + g\widehat{f} = u$ with $g = 1/\mathfrak{v} \succ^\flat 1$, $g \notin \mathrm{I}(K) + K^\dagger$, and $u \preccurlyeq 1$. Hence the lemma follows from Corollary 3.5.7. $\square$

In the next corollary we assume that the subgroup $K^\dagger$ of K is divisible. (Since K is henselian and d-valued, this holds if the groups $C^\times$ and Γ are divisible.)

Corollary 3.5.11. *Suppose $(P, \mathfrak{m}, \widehat{f})$ is balanced and $g \notin \mathrm{I}(K) + K^\dagger$. Then $(P, \mathfrak{m}, \widehat{f})$ has a strictly normal refinement $(P_{+f}, \mathfrak{m}, \widehat{f} - f)$.*

Proof. First arrange that $(P, \mathfrak{m}, \widehat{f})$ is a hole in K. The previous lemma yields an f such that $\widehat{f} - f \preccurlyeq \mathfrak{v}^3 \mathfrak{m}$. Then $(P_{+f}, \mathfrak{m}, \widehat{f} - f)$ is a strictly normal refinement of $(P, \mathfrak{m}, \widehat{f})$, by Lemma 3.3.46 (where the latter uses divisibility of $K^\dagger$). $\square$

Lemma 3.5.12. *Suppose $(P, \mathfrak{m}, \widehat{f})$ is balanced with $v\widehat{f} \notin \mathscr{E}^{\mathrm{e}}(L_P)$ and $\widehat{f} - f \preccurlyeq \widehat{f}$. Then the refinement $(P_{+f}, \mathfrak{m}, \widehat{f} - f)$ of $(P, \mathfrak{m}, \widehat{f})$ is balanced.*

Proof. By Lemma 3.2.14 we arrange $(P, \mathfrak{m}, \widehat{f})$ is a hole. Replacing $(P, \mathfrak{m}, \widehat{f})$ and f by $(P_{\times\mathfrak{m}}, 1, \widehat{f}/\mathfrak{m})$ and $f/\mathfrak{m}$ we arrange next that $\mathfrak{m} = 1$. By the remark preceding Lemma 3.3.2, $(P_{+f}, 1, \widehat{f} - f)$ is steep. Take ϕ such that $v\widehat{f} \notin \mathscr{E}\big((L_P)^{\phi}\big)$, and set $\widehat{g} := \widehat{f} - f$, so $0 \neq \widehat{g} \preccurlyeq \widehat{f}$. Recall from [ADH, 5.7.5] that $L_{P^{\phi}} = (L_P)^{\phi}$ and hence $L_{P^{\phi}}(\widehat{f}) = L_P(\widehat{f})$ and $L_P(\widehat{g}) = L_{P^{\phi}}(\widehat{g})$. Thus

$$L_{P_{+f}}(\widehat{g}) \;=\; L_P(\widehat{g}) \;\preccurlyeq\; L_{P^{\phi}}\widehat{g} \;\preccurlyeq\; L_{P^{\phi}}\widehat{f} \;\asymp\; L_{P^{\phi}}(\widehat{f}) \;= L_P(\widehat{f}) \;\preccurlyeq\; S_P(0) \;=\; S_{P_{+f}}(0),$$

using [ADH, 4.5.1(iii)] to get the second $\preccurlyeq$ and $v\widehat{f} \notin \mathscr{E}(L_{P^{\phi}})$ to get $\asymp$; the last $\preccurlyeq$ uses $(P, 1, \widehat{f})$ being a hole. Therefore $(P_{+f}, 1, \widehat{g})$ is balanced. $\square$

Combining Lemmas 3.4.2 and 3.5.12 yields:

Corollary 3.5.13. *If $(P, \mathfrak{m}, \widehat{f})$ is balanced and isolated, and $\widehat{f} - f \preccurlyeq \widehat{f}$, then the refinement $(P_{+f}, \mathfrak{m}, \widehat{f} - f)$ of $(P, \mathfrak{m}, \widehat{f})$ is also balanced and isolated.*

We call $(P, \mathfrak{m}, \widehat{f})$ **proper** if the differential polynomial P is proper as defined in Section 1.7 (that is, $u \neq 0$ and $g + u^{\dagger} \succ^{\flat} 1$). If $(P, \mathfrak{m}, \widehat{f})$ is proper, then so are $(bP, \mathfrak{m}, \widehat{f})$ for $b \neq 0$ and $(P_{\times\mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{f}/\mathfrak{n})$, as well as each refinement $(P, \mathfrak{n}, \widehat{f})$ of $(P, \mathfrak{m}, \widehat{f})$ and each slot in K equivalent to $(P, \mathfrak{m}, \widehat{f})$. By Lemma 1.7.23, if $(P, \mathfrak{m}, \widehat{f})$ is proper, then so is $(P^{\phi}, \mathfrak{m}, \widehat{f})$ for $\phi \preccurlyeq 1$.

Lemma 3.5.14. *Suppose $(P, \mathfrak{m}, \widehat{f})$ is proper and $\mathfrak{m} \asymp u$; then $(P, \mathfrak{m}, \widehat{f})$ is balanced.*

Proof. Replacing $(P, \mathfrak{m}, \widehat{f})$ by $(P_{\times\mathfrak{m}}, 1, \widehat{f}/\mathfrak{m})$, we arrange $\mathfrak{m} = 1$. Then $u \asymp 1$ and thus $(P, 1, \widehat{f})$ is balanced. $\square$

Proposition 3.5.15. *Suppose $(P, \mathfrak{m}, \widehat{f})$ is proper and $v\widehat{f} \notin \mathscr{E}^{\mathrm{e}}(L_P)$. Then $(P, \mathfrak{m}, \widehat{f})$ has a balanced refinement.*

Proof. We arrange $\mathfrak{m} = 1$ as usual. By Lemmas 1.7.26 and 3.2.14 we have

$$\widehat{f} \;\sim\; u/(g + u^{\dagger}) \prec^{\flat} u.$$

Hence if $u \preccurlyeq 1$, then $(P, u, \widehat{f})$ refines $(P, 1, \widehat{f})$, and so $(P, u, \widehat{f})$ is balanced by Lemma 3.5.14. Assume now that $u \succ 1$. Then $1 \prec u \prec g$ by Lemma 1.7.25 and $\operatorname{nmul} P = 1$, and hence $u^{\dagger} \preccurlyeq g^{\dagger} \prec g$. So $g \sim g + u^{\dagger} \succ^{\flat} 1$, hence $(P, 1, \widehat{f})$ is steep, and $\widehat{f} \sim u/g$. Set $f := u/g \prec 1$; then $(P_{+f}, 1, \widehat{f} - f)$ is a steep refinement of $(P, 1, \widehat{f})$. Moreover

$$P_{+f}(0) = P(f) = af' \prec a = S_{P_{+f}}(0),$$

hence $(P_{+f}, 1, \widehat{f} - f)$ is balanced. $\square$

Corollary 3.5.16. *Suppose K is λ-free. Then there exists $\phi \preccurlyeq 1$ and a refinement $(P_{+f}, \mathfrak{n}, \widehat{f} - f)$ of $(P, \mathfrak{m}, \widehat{f})$ such that $(P^{\phi}_{+f}, \mathfrak{n}, \widehat{f} - f)$ is balanced.*

Proof. Using Remark 3.4.7 we can replace $(P, \mathfrak{m}, \widehat{f})$ by a refinement to arrange that $(P, \mathfrak{m}, \widehat{f})$ is isolated. Then $u \neq 0$ by the remark before Lemma 3.5.9, so by Lemma 1.7.24 , P^{ϕ} is proper, eventually. Now apply Proposition 3.5.15 to a proper (and isolated) $(P^{\phi}, \mathfrak{m}, \widehat{f})$ with $\phi \preccurlyeq 1$. $\square$

Corollary 3.5.17. *Suppose K is λ-free, $\partial K = K$, $\mathrm{I}(K) \subseteq K^\dagger$, and $K^\dagger$ is divisible. Then $(P, \mathfrak{m}, \widehat{f})$ has a refinement $(P_{+f}, \mathfrak{n}, \widehat{f} - f)$ such that $(P^\phi_{+f}, \mathfrak{n}, \widehat{f} - f)$ is strictly normal for some $\phi \preccurlyeq 1$.*

Proof. Corollary 3.5.16 yields a refinement $(P_{+f_1}, \mathfrak{n}_1, \widehat{f} - f_1)$ of $(P, \mathfrak{m}, \widehat{f})$ and a $\phi \preccurlyeq 1$ such that $(P^\phi_{+f_1}, \mathfrak{n}_1, \widehat{f} - f_1)$ is balanced. By Lemma 3.5.9 with K^ϕ in the role of K and $(P^\phi_{+f_1}, \mathfrak{n}_1, \widehat{f} - f_1)$ in the role of $(P, \mathfrak{m}, \widehat{f})$ we can apply Corollary 3.5.11 to $(P^\phi_{+f_1}, \mathfrak{n}_1, \widehat{f} - f_1)$ to give a strictly normal refinement $(P^\phi_{f_1+f_2}, \mathfrak{n}, \widehat{f} - f_1 - f_2)$ of it. Thus for $f := f_1 + f_2$ the refinement $(P_{+f}, \mathfrak{n}, \widehat{f} - f)$ of $(P, \mathfrak{m}, \widehat{f})$ has the property that $(P^\phi_{+f}, \mathfrak{n}, \widehat{f} - f)$ is strictly normal. $\square$

Combining this corollary with Corollaries 3.2.8, 3.3.49, and Lemma 3.3.40 yields:

Corollary 3.5.18. *If K is ω-free and algebraically closed with $\partial K = K$ and $\mathrm{I}(K) \subseteq K^\dagger$, then every minimal hole in K of order $\geqslant 1$ has a refinement $(Q, \mathfrak{n}, \widehat{g})$ such that $(Q^\phi, \mathfrak{n}, \widehat{g})$ is deep and strictly normal, eventually.*

Remark. Suppose K is λ-free, with $\partial K = K$, $\mathrm{I}(K) \subseteq K^\dagger$, and $K^\dagger$ is divisible. By Corollary 3.3.37 every linear slot in K of order $r \geqslant 1$ has a refinement $(Q, \mathfrak{n}, \widehat{g})$ such that $(Q^\phi, \mathfrak{n}, \widehat{g})$ is deep and normal, eventually. We don't know whether every linear minimal hole in K of order $r \geqslant 1$ has a refinement $(Q, \mathfrak{n}, \widehat{g})$ such that $(Q^\phi, \mathfrak{n}, \widehat{g})$ is deep and strictly normal, eventually. (For $r = 1$ this holds by Corollary 3.5.17.)

Part 4. Holes in H-Fields

Here we focus on holes in the algebraic closure K of a Liouville closed H-field H with small derivation. After the preliminary Sections 4.1 and 4.2 we come in Sections 4.3–4.5 to the technical heart of Part 4. Section 4.3 shows that every minimal hole in K gives rise to a Z-minimal slot $(Q, \mathfrak{n}, \widehat{b})$ in H such that the slot $(Q^{\phi}, \mathfrak{n}, \widehat{b})$ in H^{ϕ} is eventually *split-normal*, meaning *normal with its linear part "asymptotically" splitting over* K^{ϕ}; see Definition 4.3.3 for the precise definition, and Theorem 4.3.9 for the detailed statement of the main result of this section. When H is a Hardy field as in [6], this asymptotic splitting will allow us to define a contractive operator on a space of real-valued functions; this operator then has a fixed point whose germ y satisfies $Q(y) = 0$, $y \prec \mathfrak{n}$. A main difficulty then lies in guaranteeing that such germs y have similar asymptotic properties as $\widehat{b}$. Sections 4.4 and 4.5 prepare the ground for dealing with this: In Section 4.4 we strengthen the concept of isolated slot to *ultimate* slot (in H, or in K). This relies on the ultimate exceptional values of linear differential operators over K introduced in Part 2. In Section 4.5 we single out among split-normal slots those that are *repulsive-normal*, culminating in the proof of Theorem 4.5.28: an analogue of Theorem 4.3.9 producing from a minimal hole in K and for small enough active $\phi > 0$ in H a deep repulsive-normal ultimate slot in H^{ϕ}. This is further improved in Theorem 4.5.43.

4.1. Some Valuation-Theoretic Lemmas

The present section contains preliminaries for the next section on approximating splittings of linear differential operators; these facts in turn are used in Section 4.3 on split-normality. We shall often deal with real closed fields with extra structure, denoted usually by H, since the results in this section about such H will be applied to H-fields (and to Hardy fields in [6]). We begin by summarizing some purely valuation-theoretic facts.

Completion and specialization of real closed valued fields. Let H be a real closed valued field whose valuation ring $\mathcal{O}$ is convex in H (with respect to the unique ordering on H making H an ordered field). Using [ADH, 3.5.15] we equip the algebraic closure $K = H[\mathrm{i}]$ ($\mathrm{i}^2 = -1$) of H with its unique valuation ring lying over $\mathcal{O}$, which is $\mathcal{O} + \mathcal{O}\mathrm{i}$. We set $\Gamma := v(H^{\times})$, so $\Gamma_K = \Gamma$.

Lemma 4.1.1. *The completion H^{c} of the valued field H is real closed, its valuation ring is convex in H^{c}, and there is a unique valued field embedding $H^{\mathrm{c}} \to K^{\mathrm{c}}$ over H. Identifying H^{c} with its image under this embedding we have $H^{\mathrm{c}}[\mathrm{i}] = K^{\mathrm{c}}$.*

Proof. For the first two claims, see [ADH, 3.5.20]. By [ADH, 3.2.20] we have a unique valued field embedding $H^{\mathrm{c}} \to K^{\mathrm{c}}$ over H, and viewing H^{c} as a valued subfield of K^{c} via this embedding we have $K^{\mathrm{c}} = H^{\mathrm{c}}K = H^{\mathrm{c}}[\mathrm{i}]$ by [ADH, 3.2.29]. □

We identify H^{c} with its image in K^{c} as in the previous lemma. Fix a convex subgroup Δ of Γ. Let $\dot{\mathcal{O}}$ be the valuation ring of the coarsening of H by Δ, with maximal ideal $\dot{o}$. Then by [ADH, 3.5.11 and subsequent remarks] $\dot{\mathcal{O}}$ and $\dot{o}$ are convex in H, the specialization $\dot{H} = \dot{\mathcal{O}}/\dot{o}$ of H by Δ is naturally an ordered and valued field, and the valuation ring of $\dot{H}$ is convex in $\dot{H}$. Moreover, $\dot{H}$ is even real closed by [ADH, 3.5.16]. Likewise, the coarsening of K by Δ has valuation ring $\dot{\mathcal{O}}_K$ with maximal ideal $\dot{o}_K$ and valued residue field $\dot{K}$. Thus $\dot{\mathcal{O}}_K$ lies over $\dot{\mathcal{O}}$

by [ADH, 3.4, subsection *Coarsening and valued field extensions*], so $(K, \dot{\mathcal{O}}_K)$ is a valued field extension of $(H, \dot{\mathcal{O}})$. In addition:

Lemma 4.1.2. *$\dot{K}$ is a valued field extension of $\dot{H}$ and an algebraic closure of $\dot{H}$.*

Proof. The second part follows by general valuation theory from K being an algebraic closure of H. In fact, with the image of $\mathrm{i} \in \mathcal{O}_K \subseteq \dot{\mathcal{O}}_K$ in $\dot{K}$ denoted by the same symbol, we have $\dot{K} = \dot{H}[\mathrm{i}]$. $\square$

Next, let $\widehat{H}$ be an immediate valued field extension of H. We equip $\widehat{H}$ with the unique field ordering making it an ordered field extension of H in which $\mathcal{O}_{\widehat{H}}$ is convex; see [ADH, 3.5.12]. Choose i in a field extension of $\widehat{H}$ with $\mathrm{i}^2 = -1$. Equip $\widehat{H}[\mathrm{i}]$ with the unique valuation ring of $\widehat{H}[\mathrm{i}]$ that lies over $\mathcal{O}_{\widehat{H}}$, namely $\mathcal{O}_{\widehat{H}} + \mathcal{O}_{\widehat{H}}\mathrm{i}$ [ADH, 3.5.15]. Let $\widehat{a} = \widehat{b} + \widehat{c}\mathrm{i} \in \widehat{H}[\mathrm{i}] \setminus H[\mathrm{i}]$ with $\widehat{b}, \widehat{c} \in \widehat{H}$, and let b, c range over H. Then

$$v\big(\widehat{a} - (b + c\mathrm{i})\big) \ = \ \min\big\{v(\widehat{b} - b), v(\widehat{c} - c)\big\}$$

and thus $v\big(\widehat{a} - H[\mathrm{i}]\big) \subseteq v(\widehat{b} - H)$ and $v\big(\widehat{a} - H[\mathrm{i}]\big) \subseteq v(\widehat{c} - H)$.

Lemma 4.1.3. *We have $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$ or $v(\widehat{c} - H) \subseteq v(\widehat{b} - H)$. Moreover, the following are equivalent:*

(i) *$v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$;*

(ii) *for all b there is a c with $v\big(\widehat{a} - (b + c\mathrm{i})\big) = v(\widehat{b} - b)$;*

(iii) *$v\big(\widehat{a} - H[\mathrm{i}]\big) = v(\widehat{b} - H)$.*

Proof. For the first assertion, use that $v(\widehat{b} - H), v(\widehat{c} - H) \subseteq \Gamma_\infty$ are downward closed. Suppose $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$, and let b be given. If $\widehat{c} \in H$, then for $c := \widehat{c}$ we have $v\big(\widehat{a} - (b + c\mathrm{i})\big) = v(\widehat{b} - b)$. Suppose $\widehat{c} \notin H$. Then $v(\widehat{c} - H) \subseteq \Gamma$ does not have a largest element and $v(\widehat{b} - b) \in v(\widehat{c} - H)$, so we have c with $v(\widehat{b} - b) < v(\widehat{c} - c)$; thus

$$v\big(\widehat{a} - (b + c\mathrm{i})\big) = \min\big\{v(\widehat{b} - b), v(\widehat{c} - c)\big\} = v(\widehat{b} - b).$$

This shows (i) $\Rightarrow$ (ii). Moreover, (ii) $\Rightarrow$ (iii) follows from $v\big(\widehat{a} - H[\mathrm{i}]\big) \subseteq v(\widehat{b} - H)$, and (iii) $\Rightarrow$ (i) from $v\big(\widehat{a} - H[\mathrm{i}]\big) \subseteq v(\widehat{c} - H)$. $\square$

So if $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$, then: $\widehat{a}$ is special over $H[\mathrm{i}]$ $\iff$ $\widehat{b}$ is special over H.

To apply Lemma 4.1.3 to H-fields we assume in the next lemma more generally that H is equipped with a derivation making it a d-valued field and that $\widehat{H}$ is equipped with a derivation ∂ making it an asymptotic field extension of H; then $\widehat{H}$ is also d-valued with the same constant field as H [ADH, 9.1.2].

Lemma 4.1.4. *Suppose H is closed under integration. Then we have:*

$$v(\widehat{b} - H) \subseteq v(\widehat{c} - H) \implies v(\partial\widehat{b} - H) \subseteq v(\partial\widehat{c} - H).$$

Proof. Assume $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$. Let $b \in H$, and take $g \in H$ with $g' = b$; adding a suitable constant to g we arrange $\widehat{b} - g \not\asymp 1$. Next, take $h \in H$ with $\widehat{b} - g \asymp \widehat{c} - h$. Then

$$\partial\widehat{b} - b \ = \ \partial(\widehat{b} - g) \ \asymp \ \partial(\widehat{c} - h) \ = \ \partial\widehat{c} - h',$$

so $v(\partial\widehat{b} - b) \in v(\partial\widehat{c} - H)$. $\square$

Embedding into the completion. In this subsection K is an asymptotic field, $\Gamma := v(K^\times) \neq \{0\}$, and L is an asymptotic field extension of K such that Γ is cofinal in Γ_L.

Lemma 4.1.5. *Let $a \in L$ and let (a_ρ) be a c-sequence in K with $a_\rho \to a$ in L. Then for each n, $(a_\rho^{(n)})$ is a c-sequence in K with $a_\rho^{(n)} \to a^{(n)}$ in L.*

Proof. By induction on n it suffices to treat the case $n = 1$. Let $\gamma \in \Gamma_L$; we need to show the existence of an index σ such that $v(a' - a'_\rho) > \gamma$ for all $\rho > \sigma$. By [ADH, 9.2.6] we have $f \in L^\times$ with $f \prec 1$ and $v(f') \geqslant \gamma$. Take σ such that $v(a - a_\rho) > vf$ for all $\rho > \sigma$. Then $v(a' - a'_\rho) > v(f') \geqslant \gamma$ for $\rho > \sigma$. $\square$

Let K^{c} be the completion of the valued differential field K; then K^{c} is asymptotic by [ADH, 9.1.6]. Lemma 4.1.5 and [ADH, 3.2.13 and 3.2.15] give:

Corollary 4.1.6. *Let $(a_i)_{i\in I}$ be a family of elements of L such that a_i is the limit in L of a c-sequence in K, for each $i \in I$. Then there is a unique embedding $K\langle (a_i)_{i\in I}\rangle \to K^{\mathrm{c}}$ of valued differential fields over K.*

Next suppose that H is a real closed asymptotic field whose valuation ring $\mathcal{O}$ is convex in H with $\mathcal{O} \neq H$, the asymptotic extension $\widehat{H}$ of H is immediate, and i is an element of an asymptotic extension of $\widehat{H}$ with $i^2 = -1$. Then $i \notin \widehat{H}$, and we identify H^{c} with a valued subfield of $H[i]^{\mathrm{c}}$ as in Lemma 4.1.1, so that $H^{\mathrm{c}}[i] = H[i]^{\mathrm{c}}$ as in that lemma. Using also Lemma 4.1.5 we see that H^{c} is actually a valued *differential* subfield of the asymptotic field $H[i]^{\mathrm{c}}$, and so $H^{\mathrm{c}}[i] = H[i]^{\mathrm{c}}$ also as *asymptotic* fields. Thus by Corollary 4.1.6 applied to $K := H$ and $L := \widehat{H}$:

Corollary 4.1.7. *Let $a \in \widehat{H}[i]$ be the limit in $\widehat{H}[i]$ of a c-sequence in $H[i]$. Then $\operatorname{Re} a$, $\operatorname{Im} a$ are limits in $\widehat{H}$ of c-sequences in H, hence there is a unique embedding $H[i]\langle \operatorname{Re} a, \operatorname{Im} a\rangle \to H^{\mathrm{c}}[i]$ of valued differential fields over $H[i]$.*

4.2. Approximating Linear Differential Operators

In this section K is a valued differential field with small derivation, $\Gamma := v(K^\times)$. For later use we prove here Corollaries 4.2.6 and 4.2.9 and consider *strong splitting*. Much of this section rests on the following basic estimate for linear differential operators which split over K:

Lemma 4.2.1. *Let $b_1, \dots, b_r \in K$ and n be given. Then there exists $\gamma_0 \in \Gamma^{\geqslant}$ such that for all $b_1^\bullet, \dots, b_r^\bullet \in K$ and $\gamma \in \Gamma$ with $\gamma > \gamma_0$ and $v(b_i - b_i^\bullet) \geqslant (n+r)\gamma$ for $i = 1, \dots, r$, we have $v(B - B^\bullet) \geqslant vB + n\gamma$, where*

$$B \ := \ (\partial - b_1)\cdots(\partial - b_r) \in K[\partial], \quad B^\bullet \ := \ (\partial - b_1^\bullet)\cdots(\partial - b_r^\bullet) \in K[\partial].$$

Proof. By induction on $r \in \mathbb{N}$. The case $r = 0$ is clear (any $\gamma_0 \in \Gamma^{\geqslant}$ works). Suppose the lemma holds for a certain r. Let $b_1, \dots, b_{r+1} \in K$ and n be given. Set $\beta_i := vb_i$ ($i = 1, \dots, r+1$). Take γ_0 as in the lemma applied to $b_1, \dots, b_r$ and $n+1$ in place of n, and let $\gamma_1 := \gamma_0$ if $b_{r+1} = 0$, $\gamma_1 := \max\{\gamma_0, |\beta_{r+1}|\}$ otherwise. Let $b_1^\bullet, \dots, b_{r+1}^\bullet \in K$ and $\gamma \in \Gamma$ with $\gamma > \gamma_1$ and $v(b_i - b_i^\bullet) \geqslant (n+r+1)\gamma$ for $i = 1, \dots, r+1$. Set

$$B \ := \ (\partial - b_1)\cdots(\partial - b_r), \qquad B^\bullet \ := \ (\partial - b_1^\bullet)\cdots(\partial - b_r^\bullet), \qquad E \ := \ B - B^\bullet.$$

Then

$$B(\partial - b_{r+1}) \ = \ B^\bullet(\partial - b_{r+1}^\bullet) + B^\bullet(b_{r+1}^\bullet - b_{r+1}) + E(\partial - b_{r+1}).$$

Inductively we have $vE \geqslant vB + (n+1)\gamma$. Suppose $E \neq 0$ and $0 \neq b_{r+1} \not\asymp 1$. Then by [ADH, 6.1.5],

$$\begin{aligned} v_E(\beta_{r+1}) - v_B(\beta_{r+1}) &= vE - vB + o(\beta_{r+1}) \\ &\geqslant (n+1)\gamma + o(\beta_{r+1}) \\ &\geqslant n\gamma + |\beta_{r+1}| + o(\beta_{r+1}) > n\gamma. \end{aligned}$$

Hence, using $E(\partial - b_{r+1}) = E\partial - Eb_{r+1}$ and $v(E\partial) = v(E) \neq v_E(\beta_{r+1})$,

$$\begin{aligned} v\big(E(\partial - b_{r+1})\big) &= \min\big\{vE, v_E(\beta_{r+1})\big\} > \min\big\{vB, v_B(\beta_{r+1})\big\} + n\gamma \\ &= v\big(B(\partial - b_{r+1})\big) + n\gamma, \end{aligned}$$

where for the last equality we use $vB \neq v_B(\beta_{r+1})$. Also,

$$v\big(B^\bullet(b^\bullet_{r+1} - b_{r+1})\big) = v_{B^\bullet}\big(v(b^\bullet_{r+1} - b_{r+1})\big) \geqslant v_{B^\bullet}\big((n+r+1)\gamma\big) = v_B\big((n+r+1)\gamma\big)$$

where we use [ADH, 6.1.7] for the last equality. Moreover, by [ADH, 6.1.4],

$$v_B\big((n+r+1)\gamma\big) - n\gamma \geqslant vB + (r+1)\gamma + o(\gamma) > vB \geqslant v\big(B(\partial - b_{r+1})\big).$$

This yields the desired result for $E \neq 0$, $0 \neq b_{r+1} \not\asymp 1$. The cases $E \neq 0$, $b_{r+1} = 0$ and $E = 0$, $0 \neq b_{r+1} \not\asymp 1$ are simpler versions of the above, and so is the case $E \neq 0$, $b_{r+1} \asymp 1$ using [ADH, 5.6.1(i)]. The remaining cases, $E = 0$, $b_{r+1} = 0$ and $E = 0$, $b_{r+1} \asymp 1$, are even simpler to handle. □

Corollary 4.2.2. *Let $a, b_1, \dots, b_r \in K$, $a \neq 0$. Then there exists $\gamma_0 \in \Gamma^{\geqslant}$ such that for all $a^\bullet, b^\bullet_1, \dots, b^\bullet_r \in K$ and $\gamma \in \Gamma$ with $\gamma > \gamma_0$, $v(a - a^\bullet) \geqslant va + \gamma$, and $v(b_i - b^\bullet_i) \geqslant (r+1)\gamma$ for $i = 1, \dots, r$, we have $v(A - A^\bullet) \geqslant vA + \gamma$, where*

$$A := a(\partial - b_1)\cdots(\partial - b_r) \in K[\partial], \quad A^\bullet := a^\bullet(\partial - b^\bullet_1)\cdots(\partial - b^\bullet_r) \in K[\partial].$$

Proof. Take γ_0 as in the previous lemma applied to $b_1, \dots, b_r$ and $n = 1$, and let $B = (\partial - b_1)\cdots(\partial - b_r)$, $A = aB$. Let $a^\bullet, b^\bullet_1, \dots, b^\bullet_r \in K$ and $\gamma \in \Gamma$ be such that $\gamma > \gamma_0$, $v(a - a^\bullet) \geqslant va + \gamma$, and $v(b_i - b^\bullet_i) \geqslant (r+1)\gamma$ for $i = 1, \dots, r$. Set $B^\bullet := (\partial - b^\bullet_1)\cdots(\partial - b^\bullet_r)$, $A^\bullet := a^\bullet B^\bullet$. Then

$$E := A - A^\bullet = a(B - B^\bullet) + (a - a^\bullet)B^\bullet.$$

Lemma 4.2.1 gives $vB^\bullet = vB$, and so

$$v\big(a(B - B^\bullet)\big) \geqslant va + vB + \gamma = vA + \gamma, \quad v\big((a - a^\bullet)B^\bullet\big) = v(a - a^\bullet) + vB \geqslant vA + \gamma,$$

so $vE \geqslant vA + \gamma$. □

In the rest of this subsection we assume $P \in K\{Y\} \setminus K$, set $r := \operatorname{order} P$, and let $\boldsymbol{i}$, $\boldsymbol{j}$ range over $\mathbb{N}^{1+r}$.

Lemma 4.2.3. *For $\delta := v\big(P - P(0)\big)$ and all $h \in \mathcal{O}$ we have $v\big(P_{+h} - P\big) \geqslant \delta + \frac{1}{2}vh$.*

Proof. Note that $\delta \in \Gamma$ and $v(P_{\boldsymbol{j}}) \geqslant \delta$ for all $\boldsymbol{j}$ with $|\boldsymbol{j}| \geqslant 1$. Let $h \in \mathcal{O}^{\neq}$ and $\boldsymbol{i}$ be given; we claim that $v\big((P_{+h})_{\boldsymbol{i}} - P_{\boldsymbol{i}}\big) \geqslant \delta + \frac{1}{2}vh$. By [ADH, (4.3.1)] we have

$$(P_{+h})_{\boldsymbol{i}} = P_{\boldsymbol{i}} + Q(h) \quad \text{where } Q(Y) := \sum_{|\boldsymbol{j}| \geqslant 1} \binom{\boldsymbol{i} + \boldsymbol{j}}{\boldsymbol{i}} P_{\boldsymbol{i}+\boldsymbol{j}}\, Y^{\boldsymbol{j}} \in K\{Y\}.$$

From $Q(0) = 0$ and [ADH, 6.1.4] we obtain

$$v(Q_{\times h}) \geqslant v(Q) + vh + o(vh) \geqslant \delta + \tfrac{1}{2}vh.$$

Together with $v\big(Q(h)\big) \geqslant v(Q_{\times h})$ this yields the lemma. □

Corollary 4.2.4. *Let $f \in K$. Then there exists $\delta \in \Gamma$ such that for all $f^\bullet \in K$ with $f - f^\bullet \prec 1$ we have $v\big(P_{+f^\bullet} - P_{+f}\big) \geqslant \delta + \frac{1}{2}v(f^\bullet - f)$.*

Proof. Take δ as in the preceding lemma with P_{+f} in place of P and $h = f^\bullet - f$. $\square$

Corollary 4.2.5. *Let $a, b_1, \dots, b_r, f \in K$ be such that*

$$A := L_{P_{+f}} = a(\partial - b_1)\cdots(\partial - b_r), \qquad a \neq 0.$$

Then there exists $\gamma_1 \in \Gamma^{\geqslant}$ such that for all $a^\bullet, b_1^\bullet, \dots, b_r^\bullet, f^\bullet \in K$ and $\gamma \in \Gamma$, if

$\gamma > \gamma_1$, $v(a - a^\bullet) \geqslant va + \gamma$, $v(b_i - b_i^\bullet) \geqslant (r+1)\gamma$ $(i = 1, \dots, r)$, *and* $v(f - f^\bullet) \geqslant 4\gamma$,

then

(i) $v\big(P_{+f^\bullet} - P_{+f}\big) \geqslant vA + \gamma$; *and*
(ii) $L_{P_{+f^\bullet}} = a^\bullet(\partial - b_1^\bullet)\cdots(\partial - b_r^\bullet) + E$ *where* $vE \geqslant vA + \gamma$.

Proof. Take γ_0 as in Corollary 4.2.2 applied to $a, b_1, \dots, b_r$, and take δ as in Corollary 4.2.4. Then $\gamma_1 := \max\{\gamma_0, vA - \delta\}$ has the required property. $\square$

In the next result L is a valued differential field extension of K with small derivation such that Γ is cofinal in Γ_L. Then the natural inclusion $K \to L$ extends uniquely to an embedding $K^{\mathrm{c}} \to L^{\mathrm{c}}$ of valued fields by [ADH, 3.2.20]. It is easy to check that this is even an embedding of valued *differential* fields; we identify K^{c} with a valued differential subfield of L^{c} via this embedding.

Corollary 4.2.6. *Let $a, b_1, \dots, b_r \in L^{\mathrm{c}}$ and $f \in K^{\mathrm{c}}$ be such that in $L^{\mathrm{c}}[\partial]$,*

$$A := L_{P_{+f}} = a(\partial - b_1)\cdots(\partial - b_r), \qquad a, f \neq 0, \quad \mathfrak{v} := \mathfrak{v}(A) \prec 1,$$

and let $w \in \mathbb{N}$. Then there are $a^\bullet, b_1^\bullet, \dots, b_r^\bullet \in L$ and $f^\bullet \in K$ such that

$$a^\bullet \sim a, \qquad f^\bullet \sim f, \qquad A^\bullet := L_{P_{+f^\bullet}} \sim A, \qquad \operatorname{order} A^\bullet = r, \qquad \mathfrak{v}(A^\bullet) \sim \mathfrak{v},$$

and such that for $\Delta := \big\{\alpha \in \Gamma_L : \alpha = o\big(v(\mathfrak{v})\big)\big\}$ we have in $L[\partial]$,

$$A^\bullet = a^\bullet(\partial - b_1^\bullet)\cdots(\partial - b_r^\bullet) + E, \qquad E \prec_\Delta \mathfrak{v}^{w+1}A.$$

Proof. Let $\gamma_1 \in \Gamma_L^{\geqslant}$ be as in Corollary 4.2.5 applied to L^{c} in place of K, and take $\gamma_2 \in \Gamma$ such that $\gamma_2 \geqslant \max\{\gamma_1, \frac{1}{4}vf\} + vA$ and $\gamma_2 \geqslant v\big((P_{+f})_{\boldsymbol{i}}\big)$ for all $\boldsymbol{i}$ with $(P_{+f})_{\boldsymbol{i}} \neq 0$. Let $\gamma \in \Gamma$ and $\gamma > \gamma_2$. Then $\gamma - vA > \gamma_1$. By the density of K, L in K^{c}, L^{c}, respectively, we can take $a^\bullet, b_1^\bullet, \dots, b_r^\bullet \in L$ and $f^\bullet \in K$ such that

$$v(a - a^\bullet) \geqslant va + (\gamma - vA), \qquad v(b_i - b_i^\bullet) \geqslant (r+1)(\gamma - vA) \ \text{ for } i = 1, \dots, r,$$

and $v(f - f^\bullet) \geqslant 4(\gamma - vA) > vf$. Then $a^\bullet \sim a$, $f^\bullet \sim f$, and by Corollary 4.2.5,

$$v\big(P_{+f^\bullet} - P_{+f}\big) \geqslant \gamma, \quad A^\bullet := L_{P_{+f^\bullet}} = a^\bullet(\partial - b_1^\bullet)\cdots(\partial - b_r^\bullet) + E, \quad vE \geqslant \gamma.$$

Hence $(P_{+f^\bullet})_{\boldsymbol{i}} \sim (P_{+f})_{\boldsymbol{i}}$ if $(P_{+f})_{\boldsymbol{i}} \neq 0$, and $v\big((P_{+f^\bullet})_{\boldsymbol{i}}\big) > \gamma_2 \geqslant vA$ if $(P_{+f})_{\boldsymbol{i}} = 0$, so $A^\bullet \sim A$, $\operatorname{order} A^\bullet = r$, and $\mathfrak{v}(A^\bullet) \sim \mathfrak{v}$. Choosing γ so that also $\gamma > v(\mathfrak{v}^{w+1}A) + \Delta$ we achieve in addition that $E \prec_\Delta \mathfrak{v}^{w+1}A$. $\square$

Keeping it real. *In this subsection H is a real closed H-asymptotic field with small derivation whose valuation ring is convex, with $\Gamma := v(H^\times) \neq \{0\}$, and K is the asymptotic extension $H[\mathrm{i}]$ of H with $\mathrm{i}^2 = -1$.* Then H^c is real closed and $H^c[\mathrm{i}] = K^c$ as valued field extension of H according to Lemma 4.1.1, and as asymptotic field extension of H by the discussion after Corollary 4.1.6. Using the real splittings from Definition 1.1.5 we show here that we can "preserve the reality of A" in Corollary 4.2.6.

Lemma 4.2.7. *Let $A \in H^c[\partial]$ be of order $r \geqslant 1$ and let $(g_1, \dots, g_r) \in H^c[\mathrm{i}]^r$ be a real splitting of A over $H^c[\mathrm{i}]$. Then for every $\gamma \in \Gamma$ there are $g_1^\bullet, \dots, g_r^\bullet$ in $H[\mathrm{i}]$ such that $v(g_i - g_i^\bullet) > \gamma$ for $i = 1, \dots, r$,*

$$A^\bullet \ := \ (\partial - g_1^\bullet) \cdots (\partial - g_r^\bullet) \in H[\partial],$$

and $(g_1^\bullet, \dots, g_r^\bullet)$ is a real splitting of $A^\bullet$ over $H[\mathrm{i}]$.

Proof. We can reduce to the case where $r = 1$ or $r = 2$. If $r = 1$, then the lemma holds trivially, so suppose $r = 2$. Then again the lemma holds trivially if $g_1, g_2 \in H^c$, so we can assume instead that

$$g_1 \ = \ a - b\mathrm{i} + b^\dagger, \quad g_2 \ = \ a + b\mathrm{i}, \qquad a \in H^c, \ b \in (H^c)^\times.$$

Let $\gamma \in \Gamma$ be given. The density of H in H^c gives $a^\bullet \in H$ with $v(a - a^\bullet) \geqslant \gamma$. Next, choose $\gamma^\bullet \in \Gamma$ such that $\gamma^\bullet \geqslant \max\{\gamma, vb\}$ and $\alpha' > \gamma$ for all nonzero $\alpha > \gamma^\bullet - vb$ in Γ, and take $b^\bullet \in H$ with $v(b - b^\bullet) > \gamma^\bullet$. Then $v(b - b^\bullet) > \gamma$ and $b \sim b^\bullet$. In fact, $b = b^\bullet(1 + \varepsilon)$ where $v\varepsilon + vb = v(b - b^\bullet) > \gamma^\bullet$ and so $v\big((b/b^\bullet)^\dagger\big) = v(\varepsilon') > \gamma$. Set $g_1^\bullet := a^\bullet - b^\bullet\mathrm{i} + b^{\bullet\dagger}$ and $g_2^\bullet := a^\bullet + b^\bullet\mathrm{i}$. Then

$$v(g_1 - g_1^\bullet) \ = \ v\big(a - a^\bullet + (b/b^\bullet)^\dagger + (b^\bullet - b)\mathrm{i}\big) \ > \ \gamma, \quad v(g_2 - g_2^\bullet) \ > \ \gamma,$$
$$(\partial - g_1^\bullet) \cdot (\partial - g_2^\bullet) \ = \ \partial^2 - \big(2a^\bullet + b^{\bullet\dagger}\big)\partial + \big((-a^\bullet)' + a^{\bullet 2} + a^\bullet b^{\bullet\dagger} + b^{\bullet 2}\big) \ \in \ H[\partial].$$

Hence $(g_1^\bullet, g_2^\bullet)$ is a real splitting of $A^\bullet := (\partial - g_1^\bullet)(\partial - g_2^\bullet) \in H[\partial]$. □

In the next two corollaries $a \in (H^c)^\times$ and $b_1, \dots, b_r \in K^c$ are such that

$$A \ := \ a(\partial - b_1) \cdots (\partial - b_r) \in H^c[\partial],$$

$(b_1, \dots, b_r)$ is a real splitting of A over K^c, and $\mathfrak{v} := \mathfrak{v}(A) \prec 1$. We set $\Delta := \Delta(\mathfrak{v})$.

Corollary 4.2.8. *Suppose $A = L_{P_{+f}}$ with $P \in H\{Y\}$ of order $r \geqslant 1$ and f in $(H^c)^\times$. Let $\gamma \in \Gamma$ and $w \in \mathbb{N}$. Then there is $f^\bullet \in H^\times$ such that $v(f^\bullet - f) \geqslant \gamma$,*

$$f^\bullet \ \sim \ f, \quad A^\bullet \ := \ L_{P_{+f^\bullet}} \ \sim \ A, \quad \operatorname{order} A^\bullet \ = \ r, \quad \mathfrak{v}(A^\bullet) \ \sim \ \mathfrak{v}, \tag{4.2.1}$$

and we have $a^\bullet \in H^\times$, $b_1^\bullet, \dots, b_r^\bullet \in K$, and $B^\bullet, E^\bullet \in H[\partial]$ with $A^\bullet = B^\bullet + E^\bullet$, $E^\bullet \prec_\Delta \mathfrak{v}^{w+1} A$, such that

$$B^\bullet \ = \ a^\bullet(\partial - b_1^\bullet) \cdots (\partial - b_r^\bullet), \qquad v(a - a^\bullet), \ v(b_1 - b_1^\bullet), \ \dots \ , v(b_r - b_r^\bullet) \ \geqslant \ \gamma,$$

and $(b_1^\bullet, \dots, b_r^\bullet)$ is a real splitting of $B^\bullet$ over K.

Proof. We apply Corollary 4.2.6 with H, K in the role of K, L, and take γ_1, γ_2 as in the proof of that corollary. We can assume $\gamma > \gamma_2$, so that $\gamma - vA > 0$. The density of H in H^c gives $a^\bullet \in H$ such that $v(a - a^\bullet) \geqslant \max\big\{va + (\gamma - vA), \gamma\big\}$ (so $a^\bullet \sim a$), and Lemma 4.2.7 gives $b_1^\bullet, \dots, b_r^\bullet \in K$ such that $v(b_i - b_i^\bullet) \geqslant \max\big\{(r+1)(\gamma - vA), \gamma\big\}$ for $i = 1, \dots, r$, and $(b_1^\bullet, \dots, b_r^\bullet)$ is a real splitting of

$$B^\bullet \ := \ a^\bullet(\partial - b_1^\bullet) \cdots (\partial - b_r^\bullet) \in H[\partial]$$

over K. Take $f^\bullet \in H$ with $v(f - f^\bullet) \geqslant \max\{4(\gamma - vA), \gamma\}$. Then (4.2.1) follows from the proof of Corollary 4.2.6. We can increase γ so that $\gamma > v(\mathfrak{v}^{w+1}A) + \Delta$, and then we have $A^\bullet - B^\bullet \prec_\Delta \mathfrak{v}^{w+1}A$. $\square$

This result persists after multiplicative conjugation:

Corollary 4.2.9. *Suppose $A = L_{P_{+f,\times\mathfrak{m}}}$ with $P \in H\{Y\}$ of order $r \geqslant 1$, and f in $(H^{\rm c})^\times$, $\mathfrak{m} \in H^\times$. Let $\gamma \in \Gamma$, $w \in \mathbb{N}$. Then there is $f^\bullet \in H^\times$ such that*

$$v(f^\bullet - f) \geqslant \gamma, \quad f^\bullet \sim f, \quad A^\bullet := L_{P_{+f^\bullet,\times\mathfrak{m}}} \sim A, \quad \operatorname{order} A^\bullet = r, \quad \mathfrak{v}(A^\bullet) \sim \mathfrak{v},$$

and we have $a^\bullet \in H^\times$, $b_1^\bullet, \dots, b_r^\bullet \in K$, and $B^\bullet, E^\bullet \in H[\partial]$ with the properties stated in the previous corollary.

Proof. Put $Q := P_{\times\mathfrak{m}} \in H\{Y\}$, $g := f/\mathfrak{m} \in H^{\rm c}$; then $Q_{+g} = P_{+f,\times\mathfrak{m}}$. Applying the previous corollary to Q, g in place of P, f yields $g^\bullet \in H^\times$, $a^\bullet \in H^\times$, and $b_1^\bullet, \dots, b_r^\bullet \in K$ such that $v(g^\bullet - g) \geqslant \gamma - v\mathfrak{m}$,

$$g^\bullet \sim g, \qquad A^\bullet := L_{Q_{+g^\bullet}} \sim A, \qquad \operatorname{order} A^\bullet = r, \qquad \mathfrak{v}(A^\bullet) \sim \mathfrak{v}$$

and $A^\bullet = B^\bullet + E^\bullet$, with $B^\bullet, E^\bullet \in H[\partial]$, $E^\bullet \prec_\Delta \mathfrak{v}^{w+1}A$, and

$$B^\bullet = a^\bullet(\partial - b_1^\bullet)\cdots(\partial - b_r^\bullet), \qquad v(a - a^\bullet),\ v(b_1 - b_1^\bullet),\ \dots\ , v(b_r - b_r^\bullet) \geqslant \gamma,$$

and $(b_1^\bullet, \dots, b_r^\bullet)$ is a real splitting of $B^\bullet$ over K. Therefore $f^\bullet := g^\bullet\mathfrak{m} \in H^\times$ and $a^\bullet, b_1^\bullet, \dots, b_r^\bullet$ have the required properties. $\square$

Strong splitting. *In this subsection H is a real closed H-field with small derivation and asymptotic integration.* Thus $K := H[\mathrm{i}]$ is a d-valued extension of H. Let $A \in K[\partial]^{\neq}$ have order $r \geqslant 1$ and set $\mathfrak{v} := \mathfrak{v}(A)$, and let f, g, h (possibly subscripted) range over K. Recall from Section 1.1 that a splitting of A over K is an r-tuple $(g_1, \dots, g_r)$ such that

$$A = f(\partial - g_1)\cdots(\partial - g_r) \quad \text{where } f \neq 0.$$

We call such a splitting $(g_1, \dots, g_r)$ of A over K **strong** if $\operatorname{Re} g_j \succcurlyeq \mathfrak{v}^\dagger$ for $j = 1, \dots, r$, and we say that A **splits strongly over** K if there is a strong splitting of A over K. This notion is mainly of interest for $\mathfrak{v} \prec 1$, since otherwise $\mathfrak{v} = 1$, and then any splitting of A over K is a strong splitting of A over K.

Lemma 4.2.10. *Let $(g_1, \dots, g_r)$ be a strong splitting of A over K. If $h \neq 0$, then $(g_1, \dots, g_r)$ is a strong splitting of hA over K. If $h \asymp 1$, then $(g_1 - h^\dagger, \dots, g_r - h^\dagger)$ is a strong splitting of Ah over K.*

Proof. The first statement is clear, so suppose $h \asymp 1$. Now use Lemma 1.1.1 and the fact that $\mathfrak{v} \prec 1$ implies $\operatorname{Re} h^\dagger \preccurlyeq h^\dagger \prec \mathfrak{v}^\dagger$. If $\mathfrak{v} = 1$, then use that $\mathfrak{v}(Ah) = 1$ by Corollary 3.1.3. $\square$

Lemma 4.2.11. *Suppose $g \asymp \operatorname{Re} g$. Then $A = \partial - g$ splits strongly over K.*

Proof. Assuming $\mathfrak{v} \prec 1$ gives $\mathfrak{v}' \prec 1$, so $\mathfrak{v}^\dagger \prec 1/\mathfrak{v} \asymp g \asymp \operatorname{Re} g$. $\square$

In particular, every $A \in H[\partial]^{\neq}$ of order 1 splits strongly over K.

Lemma 4.2.12. *Suppose $(g_1, \dots, g_r)$ is a strong splitting of A over K and $\mathfrak{v} \prec^\flat 1$. Let $\phi \preccurlyeq 1$ be active in H and set $h_j := \phi^{-1}\big(g_j - (r-j)\phi^\dagger\big)$ for $j = 1, \dots, r$. Then $(h_1, \dots, h_r)$ is a strong splitting of A^ϕ over $K^\phi = H^\phi[\mathrm{i}]$.*

Proof. By Lemma 1.1.2, $(h_1,\dots,h_r)$ is a splitting of A^ϕ over K^ϕ. We have $\phi^\dagger \prec 1 \preccurlyeq \mathfrak{v}^\dagger$, so $\operatorname{Re} h_j \sim \phi^{-1}\operatorname{Re} g_j \succcurlyeq \phi^{-1}\mathfrak{v}^\dagger$ for $j=1,\dots,r$. Set $\mathfrak{w} := \mathfrak{v}(A^\phi)$ and $\boldsymbol{\delta} := \phi^{-1}\boldsymbol{\partial}$. Lemma 3.1.20 gives $\mathfrak{v}^\dagger \asymp \mathfrak{w}^\dagger$, so $\phi^{-1}\mathfrak{v}^\dagger \asymp \boldsymbol{\delta}(\mathfrak{w})/\mathfrak{w}$. $\square$

In the next two results we assume that for all $q \in \mathbb{Q}^>$ and $\mathfrak{n} \in H^\times$ there is given an element $\mathfrak{n}^q \in H^\times$ such that $(\mathfrak{n}^q)^\dagger = q\mathfrak{n}^\dagger$ (and thus $v(\mathfrak{n}^q) = q\,v(\mathfrak{n})$).

Lemma 4.2.13. *Suppose $(g_1,\dots,g_r)$ is a splitting of A over K, $\mathfrak{v} \prec 1$, $\mathfrak{n} \in H^\times$, and $[\mathfrak{v}] \leqslant [\mathfrak{n}]$. Then for all $q \in \mathbb{Q}^>$ with at most r exceptions, $(g_1 - q\mathfrak{n}^\dagger,\dots,g_r - q\mathfrak{n}^\dagger)$ is a strong splitting of $A\mathfrak{n}^q$ over K.*

Proof. Let $q \in \mathbb{Q}^>$. Then $(g_1 - q\mathfrak{n}^\dagger,\dots,g_r - q\mathfrak{n}^\dagger)$ is a splitting of $A\mathfrak{n}^q$ over K, by Lemma 1.1.1. Moreover, $\big[\mathfrak{v}(A\mathfrak{n}^q)\big] \leqslant [\mathfrak{n}]$, by Lemma 3.1.9, so $\mathfrak{v}(A\mathfrak{n}^q)^\dagger \preccurlyeq \mathfrak{n}^\dagger$. Thus if $\operatorname{Re} g_j \not\sim q\mathfrak{n}^\dagger$ for $j=1,\dots,r$, then $(g_1 - q\mathfrak{n}^\dagger,\dots,g_r - q\mathfrak{n}^\dagger)$ is a strong splitting of $A\mathfrak{n}^q$ over K. $\square$

Corollary 4.2.14. *Let $(P,\mathfrak{m},\widehat{a})$ be a steep slot in K of order $r \geqslant 1$ whose linear part $L := L_{P_{\times\mathfrak{m}}}$ splits over K and such that $\widehat{a} \prec_\Delta \mathfrak{m}$ for $\Delta := \Delta\big(\mathfrak{v}(L)\big)$. Then for all sufficiently small $q \in \mathbb{Q}^>$, any $\mathfrak{n} \asymp |\mathfrak{v}(L)|^q\mathfrak{m}$ in $K^\times$ gives a steep refinement $\big(P,\mathfrak{n},\widehat{a}\big)$ of $(P,\mathfrak{m},\widehat{a})$ whose linear part $L_{P_{\times\mathfrak{n}}}$ splits strongly over K.*

Proof. Note that $|f| \asymp f$ for all f. Lemma 3.3.1 gives $q_0 \in \mathbb{Q}^>$ such that for all $q \in \mathbb{Q}^>$ with $q \leqslant q_0$ and any $\mathfrak{n} \asymp |\mathfrak{v}(L)|^q\mathfrak{m}$, $(P,\mathfrak{n},\widehat{a})$ is a steep refinement of $(P,\mathfrak{m},\widehat{a})$. Now apply Lemma 4.2.13 with L, $\mathfrak{v}(L)$, $|\mathfrak{v}(L)|$ in the respective roles of A, $\mathfrak{v}$, $\mathfrak{n}$, and use Lemma 4.2.10 and the fact that for $\mathfrak{n} \asymp |\mathfrak{v}(L)|^q\mathfrak{m}$ we have $L_{P_{\times\mathfrak{n}}} = L \cdot \mathfrak{n}/\mathfrak{m} = L|\mathfrak{v}(L)|^q h$ with $h \asymp 1$. $\square$

We finish this section with a useful fact on slots in K. Given such a slot $(P,\mathfrak{m},\widehat{a})$, the element $\widehat{a}$ lies in an immediate asymptotic extension of K that might not be of the form $\widehat{H}[i]$ with $\widehat{H}$ an immediate H-field extension of H. By the next lemma we can nevertheless often reduce to this situation, and more:

Lemma 4.2.15. *Suppose H is ω-free. Then every Z-minimal slot in K of positive order is equivalent to a hole $(P,\mathfrak{m},\widehat{b})$ in K with $\widehat{b} \in \widehat{K} = \widehat{H}[i]$ for some immediate ω-free newtonian H-field extension $\widehat{H}$ of H.*

Proof. Let $(P,\mathfrak{m},\widehat{a})$ be a Z-minimal slot in K of order $\geqslant 1$. Take an immediate ω-free newtonian H-field extension $\widehat{H}$ of H; such $\widehat{H}$ exists by (0.7). Then $\widehat{K} = \widehat{H}[i]$ is also newtonian by (0.10). Now apply Corollary 3.2.29 with $L := \widehat{K}$ to obtain $\widehat{b} \in \widehat{K}$ such that $(P,\mathfrak{m},\widehat{b})$ is a hole in K equivalent to $(P,\mathfrak{m},\widehat{a})$. $\square$

4.3. Split-Normal Slots

In this section H is a real closed H-field with small derivation and asymptotic integration. We let $\mathcal{O} := \mathcal{O}_H$ be its valuation ring and $C := C_H$ its constant field. We fix an immediate asymptotic extension $\widehat{H}$ of H with valuation ring $\widehat{\mathcal{O}}$ and an element i of an asymptotic extension of $\widehat{H}$ with $i^2 = -1$. Then $\widehat{H}$ is also an H-field by [ADH, 10.5.8], $i \notin \widehat{H}$ and $K := H[i]$ is an algebraic closure of H. With $\widehat{K} := \widehat{H}[i]$

we have the inclusion diagram

$$\begin{array}{ccc} \widehat{H} & \text{———} & \widehat{K} = \widehat{H}[\mathrm{i}] \\ | & & | \\ H & \text{———} & K = H[\mathrm{i}] \end{array}$$

By [ADH, 3.5.15, 10.5.7], K and $\widehat{K}$ are d-valued with valuation rings $\mathcal{O} + \mathcal{O}\mathrm{i}$ and $\widehat{\mathcal{O}} + \widehat{\mathcal{O}}\mathrm{i}$ and with the same constant field $C[\mathrm{i}]$, and $\widehat{K}$ is an immediate extension of K. Thus H, K, $\widehat{H}$, $\widehat{K}$ have the same H-asymptotic couple (Γ, ψ).

Lemma 4.3.1. *Let $\widehat{a} \in \widehat{H} \setminus H$. Then $Z(H, \widehat{a}) = Z\big(K, \widehat{a}\big) \cap H\{Y\}$.*

Proof. The inclusion "$\supseteq$" is obvious since the Newton degree of a differential polynomial $Q \in H\{Y\}^{\neq}$ does not change when H is replaced by its algebraic closure; see [ADH, 11.1]. Conversely, let $P \in Z(H, \widehat{a})$. Then for all $\mathfrak{v} \in H^{\times}$ and $a \in H$ such that $a - \widehat{a} \prec \mathfrak{v}$ we have $\operatorname{ndeg}_{\prec \mathfrak{v}} H_{+a} \geqslant 1$. Let $\mathfrak{v} \in H^{\times}$ and $z \in K$ be such that $z - \widehat{a} \prec \mathfrak{v}$. Take $a, b \in H$ such that $z = a + b\mathrm{i}$. Then $a - \widehat{a}, b\mathrm{i} \prec \mathfrak{v}$ and hence $\operatorname{ndeg}_{\prec \mathfrak{v}} P_{+z} = \operatorname{ndeg}_{\prec \mathfrak{v}} P_{+a} \geqslant 1$, using [ADH, 11.2.7]. Thus $P \in Z\big(K, \widehat{a}\big)$. □

Corollary 4.3.2. *Let $(P, \mathfrak{m}, \widehat{a})$ be a slot in H with $\widehat{a} \in \widehat{H}$. Then $(P, \mathfrak{m}, \widehat{a})$ is also a slot in K, and if $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal as a slot in K, then $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal as a slot in H. Moreover, $(P, \mathfrak{m}, \widehat{a})$ is a hole in H iff $(P, \mathfrak{m}, \widehat{a})$ is a hole in K, and if $(P, \mathfrak{m}, \widehat{a})$ is a minimal hole in K, then $(P, \mathfrak{m}, \widehat{a})$ is a minimal hole in H.*

Proof. The first three claims are obvious from $\widehat{K}$ being an immediate extension of K and the previous lemma. Suppose $(P, \mathfrak{m}, \widehat{a})$ is minimal as a hole in K. Let $(Q, \mathfrak{n}, \widetilde{b})$ be a hole in H; thus $\widetilde{b} \in \widetilde{H}$ where $\widetilde{H}$ is an immediate asymptotic extension of H. By the first part of the corollary applied to $(Q, \mathfrak{n}, \widetilde{b})$ and $\widetilde{H}$ in place of $(P, \mathfrak{m}, \widehat{a})$ and $\widehat{H}$, respectively, $(Q, \mathfrak{n}, \widetilde{b})$ is also a hole in K. Hence $\mathrm{c}(P) \leqslant \mathrm{c}(Q)$, proving the last claim. □

In the next subsection we define the notion of a *split-normal* slot in H. Later in this section we employ the results of Sections 3.3–4.2 to show, under suitable hypotheses on H, that minimal holes in K of order $\geqslant 1$ give rise to a split-normal Z-minimal slots in H. (Theorem 4.3.9.) We then investigate which kinds of refinements preserve split-normality, and also consider a strengthening of split-normality.

Defining split-normality. *In this subsection b ranges over H and $\mathfrak{m}, \mathfrak{n}$ over $H^{\times}$. Also, $(P, \mathfrak{m}, \widehat{a})$ is a slot in H of order $r \geqslant 1$ with $\widehat{a} \in \widehat{H} \setminus H$ and linear part $L := L_{P_{\times \mathfrak{m}}}$. Set $w := \operatorname{wt}(P)$, so $w \geqslant r$; if* $\operatorname{order} L = r$, *we set $\mathfrak{v} := \mathfrak{v}(L)$.*

Definition 4.3.3. We say that $(P, \mathfrak{m}, \widehat{a})$ is **split-normal** if $\operatorname{order} L = r$, and

- (SN1) $\mathfrak{v} \prec^{\flat} 1$;
- (SN2) $(P_{\times \mathfrak{m}})_{\geqslant 1} = Q + R$ where $Q, R \in H\{Y\}$, Q is homogeneous of degree 1 and order r, L_Q splits over K, and $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} (P_{\times \mathfrak{m}})_1$.

Note that in (SN2) we do not require that $Q = (P_{\times \mathfrak{m}})_1$.

Lemma 4.3.4. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is split-normal. Then $(P, \mathfrak{m}, \widehat{a})$ is normal, and with Q, R as in* (SN2) *we have $(P_{\times \mathfrak{m}})_1 - Q \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} (P_{\times \mathfrak{m}})_1$, so $(P_{\times \mathfrak{m}})_1 \sim Q$.*

Proof. We have $(P_{\times\mathfrak{m}})_1 = Q + R_1$ and $R_1 \preccurlyeq R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$, and thus $(P_{\times\mathfrak{m}})_1 - Q \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$. Now $(P,\mathfrak{m},\widehat{a})$ is normal because $(P_{\times\mathfrak{m}})_{>1} = R_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$. □

If $(P,\mathfrak{m},\widehat{a})$ is normal and $(P_{\times\mathfrak{m}})_1 = Q+R$ where $Q,R \in H\{Y\}$, Q is homogeneous of degree 1 and order r, L_Q splits over K, and $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$, then $(P,\mathfrak{m},\widehat{a})$ is split-normal. Thus if $(P,\mathfrak{m},\widehat{a})$ is normal and L splits over K, then $(P,\mathfrak{m},\widehat{a})$ is split-normal; in particular, if $(P,\mathfrak{m},\widehat{a})$ is normal of order $r=1$, then it is split-normal. If $(P,\mathfrak{m},\widehat{a})$ is split-normal, then so are $(bP,\mathfrak{m},\widehat{a})$ for $b \neq 0$ and $(P_{\times\mathfrak{n}},\mathfrak{m}/\mathfrak{n},\widehat{a}/\mathfrak{n})$. Note also that if $(P,\mathfrak{m},\widehat{a})$ is split-normal, then with Q as in (SN2) we have $\mathfrak{v}(L) \sim \mathfrak{v}(L_Q)$, by Lemma 3.1.1. If $(P,\mathfrak{m},\widehat{a})$ is split-normal and H is λ-free, then $\mathscr{E}^{\mathrm{e}}(L) = \mathscr{E}^{\mathrm{e}}(L_Q)$ with Q as in (SN2), by Lemmas 4.3.4 and 3.1.23.

Lemma 4.3.5. *Suppose $(P,\mathfrak{m},\widehat{a})$ is split-normal and $\phi \preccurlyeq 1$ is active in H and $\phi > 0$ (so H^ϕ is still an H-field). Then the slot $(P^\phi,\mathfrak{m},\widehat{a})$ in H^ϕ is split-normal.*

Proof. We first arrange $\mathfrak{m}=1$. Note that $L_{P^\phi} = L^\phi$ has order r. Put $\mathfrak{w} := \mathfrak{v}(L_{P^\phi})$, and take Q, R as in (SN2). Then $\mathfrak{v} \asymp_{\Delta(\mathfrak{v})} \mathfrak{w} \prec^\flat_\phi 1$ by Lemma 3.1.20. Moreover, $L_{Q^\phi} = L_Q^\phi$ splits over K^ϕ; see [ADH, p. 291] or Lemma 1.1.2. By [ADH, 11.1.4],

$$R^\phi \asymp_{\Delta(\mathfrak{v})} R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}P_1 \asymp_{\Delta(\mathfrak{v})} \mathfrak{w}^{w+1}P_1^\phi,$$

so $(P^\phi,\mathfrak{m},\widehat{a})$ is split-normal. □

Since we need to preserve H being an H-field when compositionally conjugating, we say: $(P^\phi,\mathfrak{m},\widehat{a})$ *is eventually split-normal* if there exists an active ϕ_0 in H such that $(P^\phi,\mathfrak{m},\widehat{a})$ is split-normal for all active $\phi \preccurlyeq \phi_0$ in H with $\phi > 0$. We use this terminology in a similar way with "split-normal" replaced by other properties of slots of order $r \geqslant 1$ in real closed H-fields with small derivation and asymptotic integration, such as "deep" and "deep and split-normal".

Achieving split-normality. *Assume H is ω-free and $(P,\mathfrak{m},\widehat{a})$ is a minimal hole in $K = H[\mathrm{i}]$ of order $r \geqslant 1$, with $\mathfrak{m} \in H^\times$ and $\widehat{a} \in \widehat{K} \setminus K$.* Note that then K is ω-free by [ADH, 11.7.23], K is $(r-1)$-newtonian by Corollary 3.2.3, and K is r-linearly closed by Corollary 3.2.4. In particular, the linear part of $(P,\mathfrak{m},\widehat{a})$ is 0 or splits over K. If $\deg P = 1$, then $r=1$ by Corollary 3.2.8. If $\deg P > 1$, then K and H are r-linearly newtonian by Corollary 3.2.6 and Lemma 1.7.30. In particular, if H is 1-linearly newtonian, then H is r-linearly newtonian. *In this subsection we let a range over K, b, c over H, and $\mathfrak{n}$ over $H^\times$.*

Lemma 4.3.6. *Let $(Q,\mathfrak{n},\widehat{b})$ be a hole in H with $\mathrm{c}(Q) \leqslant \mathrm{c}(P)$ and $\widehat{b} \in \widehat{H}$. Then $\mathrm{c}(Q) = \mathrm{c}(P)$, $(Q,\mathfrak{n},\widehat{b})$ is minimal and remains a minimal hole in K. The linear part of $(Q,\mathfrak{n},\widehat{b})$ is 0 or splits over K, and $(Q,\mathfrak{n},\widehat{b})$ has a refinement $(Q_{+b},\mathfrak{p},\widehat{b}-b)$ (in H) such that $(Q^\phi_{+b},\mathfrak{p},\widehat{b}-b)$ is eventually deep and split-normal.*

Proof. By Corollary 4.3.2, $(Q,\mathfrak{n},\widehat{b})$ is a hole in K, and this hole in K is minimal with $\mathrm{c}(Q) = \mathrm{c}(P)$, since $(P,\mathfrak{m},\widehat{a})$ is minimal. By Corollary 4.3.2 again, $(Q,\mathfrak{n},\widehat{b})$ as a hole in H is also minimal. Since K is r-linearly closed, the linear part of $(Q,\mathfrak{n},\widehat{b})$ is 0 or splits over K. Corollary 3.3.34 gives a refinement $(Q_{+b},\mathfrak{p},\widehat{b}-b)$ of the minimal hole $(Q,\mathfrak{n},\widehat{b})$ in H such that $(Q^\phi_{+b},\mathfrak{p},\widehat{b}-b)$ is deep and normal, eventually. Thus the linear part of $(Q_{+b},\mathfrak{p},\widehat{b}-b)$ is not 0, and as $\mathrm{c}(Q_{+b}) = \mathrm{c}(P)$, this linear part splits over K. Hence for active ϕ in H the linear part of $(Q^{\phi}_{+b}, \mathfrak{p}, \widehat{b} - b)$ splits over $K^{\phi} = H^{\phi}[\mathrm{i}]$. Thus $(Q^{\phi}_{+b}, \mathfrak{p}, \widehat{b} - b)$ is eventually split-normal. $\square$

Now $\widehat{a} = \widehat{b} + \widehat{c}\,\mathrm{i}$ with $\widehat{b}, \widehat{c} \in \widehat{H}$, and $\widehat{b}, \widehat{c} \prec \mathfrak{m}$. Moreover, $\widehat{b} \notin H$ or $\widehat{c} \notin H$. Since $\widehat{a}$ is differentially algebraic over H, so is its conjugate $\widehat{b} - \widehat{c}\,\mathrm{i}$, and therefore its real and imaginary parts $\widehat{b}$ and $\widehat{c}$ are differentially algebraic over H; thus $Z(H, \widehat{b}) \neq \emptyset$ for $\widehat{b} \notin H$, and $Z(H, \widehat{c}) \neq \emptyset$ for $\widehat{c} \notin H$. More precisely:

Lemma 4.3.7. *We have* $\operatorname{trdeg}\big(H\langle\widehat{b}\rangle | H\big) \leqslant 2r$. *If* $\widehat{b} \notin H$, *then* $Z(H, \widehat{b}) \cap H[Y] = \emptyset$, *so* $1 \leqslant \operatorname{order} Q \leqslant 2r$ *for all* $Q \in Z(H, \widehat{b})$ *of minimal complexity. These statements also hold for* $\widehat{c}$ *instead of* $\widehat{b}$.

Proof. The first statement follows from $\widehat{b} \in H\langle \widehat{b} + \widehat{c}\,\mathrm{i}, \widehat{b} - \widehat{c}\,\mathrm{i}\rangle$. Suppose $\widehat{b} \notin H$. If $Q \in Z(H, \widehat{b})$ has minimal complexity, then [ADH, 11.4.8] yields an element f in a proper immediate asymptotic extension of H with $Q(f) = 0$, so $Q \notin H[Y]$. $\square$

Lemma 4.3.8. *Suppose* $\deg P = 1$ *and* $\widehat{b} \notin H$. *Let* $Q \in Z(H, \widehat{b})$ *be of minimal complexity; then either* $\operatorname{order} Q = 1$, *or* $\operatorname{order} Q = 2$, $\deg Q = 1$. *Let* $\widehat{Q} \in H\{Y\}$ *be a minimal annihilator of* $\widehat{b}$ *over* H; *then either* $\operatorname{order} \widehat{Q} = 1$, *or* $\operatorname{order} \widehat{Q} = 2$, $\deg \widehat{Q} = 1$, *and* $L_{\widehat{Q}} \in H[\partial]$ *splits over* K.

Proof. Recall that $r = 1$ by Corollary 3.2.8. Example 1.1.7 and Lemma 1.1.8 give a $\widetilde{Q} \in H\{Y\}$ of degree 1 and order 1 or 2 such that $\widetilde{Q}(\widehat{b}) = 0$ and $L_{\widetilde{Q}}$ splits over K. Then $\mathrm{c}(\widetilde{Q}) = (1,1,1)$ or $\mathrm{c}(\widetilde{Q}) = (2,1,1)$, which proves the claim about Q, using also Lemma 4.3.7. Also, $\widetilde{Q}, \widehat{Q} \in Z(H, \widehat{b})$, hence $\mathrm{c}(Q) \leqslant \mathrm{c}(\widehat{Q}) \leqslant \mathrm{c}(\widetilde{Q})$. If $\mathrm{c}(\widehat{Q}) = \mathrm{c}(\widetilde{Q})$, then $\widehat{Q} = a\widetilde{Q}$ for some $a \in H^{\times}$. The claim about $\widehat{Q}$ now follows easily. $\square$

By Corollary 3.3.34 and Lemma 3.3.23, our minimal hole $(P, \mathfrak{m}, \widehat{a})$ in K has a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ such that eventually $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a} - a)$ is deep and normal. Moreover, as K is r-linearly closed, the linear part of $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a} - a)$, for active ϕ in H, splits over $K^{\phi} = H^{\phi}[\mathrm{i}]$. Our main goal in this subsection is to prove analogues of these facts for suitable Z-minimal slots $(Q, \mathfrak{m}, \widehat{b})$ or $(R, \mathfrak{m}, \widehat{c})$ in H:

Theorem 4.3.9. *If* H *is* 1*-linearly newtonian, then one of the following holds:*

(i) $\widehat{b} \notin H$ *and there exists a* Z*-minimal slot* $(Q, \mathfrak{m}, \widehat{b})$ *in* H *with a refinement* $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ *such that* $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ *is eventually deep and split-normal;*

(ii) $\widehat{c} \notin H$ *and there exists a* Z*-minimal slot* $(R, \mathfrak{m}, \widehat{c})$ *in* H *with a refinement* $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ *such that* $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ *is eventually deep and split-normal.*

Lemmas 4.3.10, 4.3.11 and Corollaries 4.3.13–4.3.16 below are more precise (only Corollary 4.3.15 has H being 1-linearly newtonian as a hypothesis) and together give Theorem 4.3.9. We first deal with the case where $\widehat{b}$ or $\widehat{c}$ is in H:

Lemma 4.3.10. *Suppose* $\widehat{c} \in H$. *Then some hole* $(Q, \mathfrak{m}, \widehat{b})$ *in* H *has the same complexity as* $(P, \mathfrak{m}, \widehat{a})$. *Any such hole* $(Q, \mathfrak{m}, \widehat{b})$ *in* H *is minimal and has a refinement* $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ *such that* $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ *is eventually deep and split-normal.*

Proof. Let $A,B\in H\{Y\}$ be such that $P_{+\widehat{c}i}(Y)=A(Y)+B(Y)\,i$. Then $A(\widehat{b})=B(\widehat{b})=0$. If $A\neq 0$, then $\mathrm{c}(A)\leqslant\mathrm{c}(P)$ gives that $Q:=A$ has the desired property by Lemma 4.3.6. If $B\neq 0$, then likewise $Q:=B$ has the desired property. The rest also follows from that lemma. $\square$

Thus if $\widehat{c}\in H$, we obtain a strong version of (i) in Theorem 4.3.9. Likewise, the next lemma gives a strong version of (ii) in Theorem 4.3.9 if $\widehat{b}\in H$.

Lemma 4.3.11. *Suppose $\widehat{b}\in H$. Then there is a hole $(R,\mathfrak{m},\widehat{c})$ in H with the same complexity as $(P,\mathfrak{m},\widehat{a})$. Every such hole in H is minimal and has a refinement $(R_{+c},\mathfrak{n},\widehat{c}-c)$ such that $(R^{\phi}_{+c},\mathfrak{n},\widehat{c}-c)$ is eventually deep and split-normal.*

This follows by applying Lemma 4.3.10 with $(P,\mathfrak{m},\widehat{a})$ replaced by the minimal hole $\big(P_{\times i},\mathfrak{m},-i\widehat{a}\big)$ in K, which has the same complexity as $(P,\mathfrak{m},\widehat{a})$.

We assume in the rest of this subsection that $\widehat{b},\widehat{c}\notin H$ and that $Q\in Z(H,\widehat{b})$ has minimal complexity. Hence $(Q,\mathfrak{m},\widehat{b})$ is a Z-minimal slot in H, and so is every refinement of $(Q,\mathfrak{m},\widehat{b})$. If $(P_{+a},\mathfrak{n},\widehat{a}-a)$ is a refinement of $(P,\mathfrak{m},\widehat{a})$ and $b=\operatorname{Re}a$, then $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ is a refinement of $(Q,\mathfrak{m},\widehat{b})$. Conversely, if $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ is a refinement of $(Q,\mathfrak{m},\widehat{b})$ and $v\big(\widehat{b}-H\big)\subseteq v(\widehat{c}-H)$, then Lemma 4.1.3 yields a refinement $(P_{+a},\mathfrak{n},\widehat{a}-a)$ of $(P,\mathfrak{m},\widehat{a})$ with $\operatorname{Re}a=b$. Recall from that lemma that $v(\widehat{b}-H)\subseteq v(\widehat{c}-H)$ is equivalent to $v(\widehat{a}-K)=v(\widehat{b}-H)$; in this case, $(P,\mathfrak{m},\widehat{a})$ is special iff $(Q,\mathfrak{m},\widehat{b})$ is special. Recall also that if $(Q,\mathfrak{m},\widehat{b})$ is deep, then so is each of its refinements $(Q_{+b},\mathfrak{m},\widehat{b}-b)$, by Corollary 3.3.8.

Here is a key technical fact underlying Theorem 4.3.9:

Proposition 4.3.12. *Suppose the hole $(P,\mathfrak{m},\widehat{a})$ in K is special, the slot $(Q,\mathfrak{m},\widehat{b})$ in H is normal, and $v\big(\widehat{b}-H\big)\subseteq v(\widehat{c}-H)$. Then some refinement $(Q_{+b},\mathfrak{m},\widehat{b}-b)$ of $(Q,\mathfrak{m},\widehat{b})$ has the property that $(Q^{\phi}_{+b},\mathfrak{m},\widehat{b}-b)$ is eventually split-normal.*

Proof. Replacing $(P,\mathfrak{m},\widehat{a})$, $(Q,\mathfrak{m},\widehat{b})$ by $(P_{\times\mathfrak{m}},1,\widehat{a}/\mathfrak{m})$, $(Q_{\times\mathfrak{m}},1,\widehat{b}/\mathfrak{m})$, respectively, we reduce to the case $\mathfrak{m}=1$; then $\widehat{a},\widehat{b}\prec 1$. Since $\widehat{a}$ is special over $K=H[i]$,

$$\Delta\ :=\ \big\{\delta\in\Gamma:\ |\delta|\in v(\widehat{a}-K)\big\}$$

is a convex subgroup of Γ which is cofinal in $v(\widehat{a}-K)$ and hence in $v(\widehat{b}-H)$, so $\widehat{b}$ is special over H. Compositionally conjugate H, $\widehat{H}$, K, $\widehat{K}$ by a suitable active $\phi\preccurlyeq 1$ in $H^{>}$, and replace P, Q by P^{ϕ}, Q^{ϕ}, to arrange $\Gamma^{\flat}\subseteq\Delta$; in particular, $\Psi\subseteq v(\widehat{b}-H)$ and $\psi(\Delta^{\neq})\subseteq\Delta$. Multiplying P, Q by suitable elements of $H^{\times}$ we also arrange that $P,Q\asymp 1$. By Lemma 4.3.5 it suffices to show that then $(Q,1,\widehat{b})$ has a split-normal refinement $(Q_{+b},1,\widehat{b}-b)$, and this is what we shall do.

Note that H, $\widehat{H}$, K, $\widehat{K}$ have small derivation, so the specializations $\dot{H}$, $\dot{\widehat{H}}$, $\dot{K}$, $\dot{\widehat{K}}$ of H, $\widehat{H}$, K, $\widehat{K}$, respectively, by Δ, are valued differential fields with small derivation. These specializations are asymptotic with asymptotic couple $(\Delta,\psi|\Delta^{\neq})$, and of H-type with asymptotic integration, by [ADH, 9.4.12]; in addition they are d-valued, by [ADH, 10.1.8]. The natural inclusions $\dot{\mathcal{O}}\to\dot{\mathcal{O}}_K$, $\dot{\mathcal{O}}\to\dot{\mathcal{O}}_{\widehat{H}}$, $\dot{\mathcal{O}}_{\widehat{H}}\to\dot{\mathcal{O}}_{\widehat{K}}$, and $\dot{\mathcal{O}}_K\to\dot{\mathcal{O}}_{\widehat{K}}$ induce valued differential field embeddings $\dot{H}\to\dot{K}$, $\dot{H}\to\dot{\widehat{H}}$, $\dot{\widehat{H}}\to\dot{\widehat{K}}$ and $\dot{K}\to\dot{\widehat{K}}$, which we make into inclusions by the usual identifications; see [ADH, pp. 405–406]. By Lemma 4.1.2 and the remarks preceding it, $\dot{H}$ is real closed with convex valuation ring and $\dot{K}$ is an algebraic closure of $\dot{H}$. Moreover, $\dot{\widehat{H}}$ is an immediate extension of $\dot{H}$ and $\dot{\widehat{K}}$ is an immediate extension of $\dot{K}$. Denoting the image of i under the residue morphism $\dot{\mathcal{O}}_{\widehat{K}} \to \dot{\widehat{K}}$ by the same symbol, we then have $\dot{K} = \dot{H}[\mathrm{i}]$, $\dot{\widehat{K}} = \dot{\widehat{H}}[\mathrm{i}]$, and $\mathrm{i} \notin \dot{\widehat{H}}$. This gives the following inclusion diagram:

$$\begin{tikzcd} \dot{\widehat{H}} \arrow[r, dash] \arrow[d, dash] & \dot{\widehat{K}} = \dot{\widehat{H}}[\mathrm{i}] \arrow[d, dash] \\ \dot{H} \arrow[r, dash] & \dot{K} = \dot{H}[\mathrm{i}] \end{tikzcd}$$

Now $\widehat{a} \in \mathcal{O}_{\widehat{K}} \subseteq \dot{\mathcal{O}}_{\widehat{K}}$ and $\widehat{b}, \widehat{c} \in \mathcal{O}_{\widehat{H}} \subseteq \dot{\mathcal{O}}_{\widehat{H}}$, and $\dot{\widehat{a}} = \dot{\widehat{b}} + \dot{\widehat{c}}\,i$, $\operatorname{Re}\dot{\widehat{a}} = \dot{\widehat{b}}$, $\operatorname{Im}\dot{\widehat{a}} = \dot{\widehat{c}}$. For all $a \in \dot{\mathcal{O}}_K$ we have $v(\dot{\widehat{a}} - \dot{a}) = v(\widehat{a} - a) \in \Delta$, hence $\dot{\widehat{a}} \notin \dot{K}$; likewise $v(\widehat{b} - b) \in \Delta$ for all $b \in \dot{\mathcal{O}}$, so $\dot{\widehat{b}} \notin \dot{H}$. Moreover, for all $\delta \in \Delta$ there is an $a \in \dot{\mathcal{O}}_K$ with $v(\dot{\widehat{a}} - \dot{a}) = \delta$; hence $\dot{\widehat{a}}$ is the limit of a c-sequence in $\dot{K}$. This leads us to consider the completions $\dot{H}^{\mathrm{c}}$ and $\dot{K}^{\mathrm{c}}$ of $\dot{H}$ and $\dot{K}$. By [ADH, 4.4.11] and Lemma 4.1.1, these yield an inclusion diagram of valued differential field extensions:

$$\begin{tikzcd} \dot{H}^{\mathrm{c}} \arrow[r, dash] \arrow[d, dash] & \dot{K}^{\mathrm{c}} = \dot{H}^{\mathrm{c}}[\mathrm{i}] \arrow[d, dash] \\ \dot{H} \arrow[r, dash] & \dot{K} = \dot{H}[\mathrm{i}] \end{tikzcd}$$

where $\dot{H}^{\mathrm{c}}$ is real closed with algebraic closure $\dot{K}^{\mathrm{c}} = \dot{H}^{\mathrm{c}}[\mathrm{i}]$. These completions are d-valued by [ADH, 9.1.6]. By Corollary 1.7.5, $\dot{K}$ and $\dot{K}^{\mathrm{c}}$ are ω-free and $(r-1)$-newtonian; thus $\dot{K}^{\mathrm{c}}$ is r-linearly closed by Corollary 1.7.42. We identify the valued differential subfield $\dot{K}\langle \operatorname{Re}\dot{\widehat{a}}, \operatorname{Im}\dot{\widehat{a}}\rangle$ of $\dot{\widehat{K}}$ with its image under the embedding into $\dot{K}^{\mathrm{c}}$ over $\dot{K}$ from Corollary 4.1.7; then $\dot{\widehat{a}} \in \dot{K}^{\mathrm{c}}$ and $\dot{\widehat{b}} = \operatorname{Re}\dot{\widehat{a}} \in \dot{H}^{\mathrm{c}}$. This leads to the next inclusion diagram:

$$\begin{tikzcd} \dot{H}^{\mathrm{c}} \arrow[r, dash] \arrow[d, dash] & \dot{K}^{\mathrm{c}} \arrow[d, dash] \\ \dot{H}\langle\dot{\widehat{b}}\rangle \arrow[d, dash] & \dot{K}\langle\dot{\widehat{a}}\rangle \arrow[d, dash] \\ \dot{H} \arrow[r] & \dot{K} \end{tikzcd}$$

By Corollary 1.5.21, $\dot{P} \in \dot{K}\{Y\}$ is a minimal annihilator of $\dot{\widehat{a}}$ over $\dot{K}$ and has the same complexity as P. Likewise, $\dot{Q} \in \dot{H}\{Y\}$ is a minimal annihilator of $\dot{\widehat{b}}$ over $\dot{H}$ and has the same complexity as Q. Let $s := \operatorname{order} Q = \operatorname{order} \dot{Q}$, so $1 \leqslant s \leqslant 2r$ by Lemma 4.3.7, and the linear part $A \in \dot{H}^{\mathrm{c}}[\partial]$ of $\dot{Q}_{+\dot{\widehat{b}}}$ has order s as well. By [ADH, 5.1.37] applied to $\dot{H}^{\mathrm{c}}$, $\dot{H}$, $\dot{P}$, $\dot{Q}$, $\dot{\widehat{a}}$ in the role of K, F, P, S, f, respectively, A splits over $\dot{K}^{\mathrm{c}} = \dot{H}^{\mathrm{c}}[\mathrm{i}]$, so Lemma 1.1.4 gives a real splitting $(g_1, \dots, g_s)$ of A over $\dot{K}^{\mathrm{c}}$:

$$A \;=\; f(\partial - g_1)\cdots(\partial - g_s), \qquad f, g_1, \dots, g_s \in \dot{K}^{\mathrm{c}},\ f \neq 0.$$

The slot $(Q,1,\widehat{b})$ in H is normal, so $\mathfrak{v}(L_{Q_{+\widehat{b}}}) \sim \mathfrak{v}(L_Q) \prec^\flat 1$ by Lemma 3.1.28, hence $\mathfrak{v}(A) \prec^\flat 1$ in $\dot{K}^c$ by Lemma 3.1.7. Then Corollary 4.2.8 gives $a,b \in \dot{\mathcal{O}}$ and $b_1,\dots,b_s \in \dot{\mathcal{O}}_K$ with $\dot{a},\dot{b} \neq 0$ in $\dot{H}$ such that for the linear part $\widetilde{A} \in \dot{H}[\partial]$ of $\dot{Q}_{+\dot{b}}$,

$$\dot{b} \sim \dot{\widehat{b}}, \qquad \widetilde{A} \sim A, \qquad \operatorname{order}\widetilde{A} = s, \qquad \mathfrak{w} := \mathfrak{v}(\widetilde{A}) \sim \mathfrak{v}(A),$$

and such that for $w := \operatorname{wt}(Q)$ and with $\Delta(\mathfrak{w}) \subseteq \Delta$:

$$\widetilde{A} = \widetilde{B} + \widetilde{E},\ \widetilde{B} = \dot{a}(\partial - \dot{b}_1)\cdots(\partial - \dot{b}_s) \in \dot{H}[\partial], \quad \widetilde{E} \in \dot{H}[\partial], \quad \widetilde{E} \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1}\widetilde{A},$$

and $(\dot{b}_1,\dots,\dot{b}_s)$ is a real splitting of $\widetilde{B}$ over $\dot{K}$. Lemma 1.1.6 shows that we can change $b_1,\dots,b_s$ if necessary, without changing $\dot{b}_1,\dots,\dot{b}_s$, to arrange that $B := a(\partial-b_1)\cdots(\partial-b_s)$ lies in $\dot{\mathcal{O}}[\partial] \subseteq H[\partial]$ and $(b_1,\dots,b_s)$ is a real splitting of B over K. Now $\widehat{b} - b \prec \widehat{b} \prec 1$, so $(Q_{+b},1,\widehat{b}-b)$ is a refinement of the normal slot $(Q,1,\widehat{b})$. Hence $(Q_{+b},1,\widehat{b}-b)$ is normal by Proposition 3.3.25, so $\mathfrak{v} := \mathfrak{v}(L_{Q_{+b}}) \prec^\flat 1$. By Lemma 3.1.7 we have $\dot{\mathfrak{v}} = \mathfrak{w}$, so $\Delta(\mathfrak{v}) = \Delta(\mathfrak{w}) \subseteq \Delta$. Hence in $H[\partial]$:

$$L_{Q_{+b}} = B + E, \quad E \in \dot{\mathcal{O}}[\partial],\ E \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} L_{Q_{+b}}.$$

Thus $(Q_{+b},1,\widehat{b}-b)$ is split-normal. $\square$

Recall from the beginning of this subsection that if $\deg P > 1$, then $K = H[i]$ is r-linearly newtonian; this allows us to remove the assumptions that $(P,\mathfrak{m},\widehat{a})$ is special and $(Q,\mathfrak{m},\widehat{b})$ is normal in Proposition 4.3.12, by reducing to that case:

Corollary 4.3.13. *Suppose* $\deg P > 1$ *and* $v(\widehat{b}-H) \subseteq v(\widehat{c}-H)$. *Then* $(Q,\mathfrak{m},\widehat{b})$ *has a special refinement* $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ *such that* $(Q^{\phi}_{+b},\mathfrak{n},\widehat{b}-b)$ *is eventually deep and split-normal.*

Proof. By Lemmas 3.2.26 and 3.3.23, the hole $(P,\mathfrak{m},\widehat{a})$ in K has a quasilinear refinement $(P_{+a},\mathfrak{n},\widehat{a}-a)$. (The use of Lemma 3.3.23 is because we require $\mathfrak{n} \in H^\times$.) Let $b = \operatorname{Re} a$. Then, using Lemma 4.1.3 for the second equality,

$$v\big((\widehat{a}-a)-K\big) = v(\widehat{a}-K) = v(\widehat{b}-H) = v\big((\widehat{b}-b)-H\big),$$

and $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ is a Z-minimal refinement of $(Q,\mathfrak{m},\widehat{b})$. We replace $(P,\mathfrak{m},\widehat{a})$ and $(Q,\mathfrak{m},\widehat{b})$ by $(P_{+a},\mathfrak{n},\widehat{a}-a)$ and $(Q_{+b},\mathfrak{n},\widehat{b}-b)$, respectively, to arrange that the hole $(P,\mathfrak{m},\widehat{a})$ in K is quasilinear. Then by Proposition 1.5.12 and K being r-linearly newtonian, $(P,\mathfrak{m},\widehat{a})$ is special. Hence $(Q,\mathfrak{m},\widehat{b})$ is also special, so Proposition 3.3.36 gives a refinement $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ of $(Q,\mathfrak{m},\widehat{b})$ and an active $\phi_0 \in H^>$ such that $(Q^{\phi_0}_{+b},\mathfrak{n},\widehat{b}-b)$ is deep and normal. Refinements of $(P,\mathfrak{m},\widehat{a})$ remain quasilinear, by Corollary 3.2.23. Since $v(\widehat{b}-H) \subseteq v(\widehat{c}-H)$ we have a refinement $(P_{+a},\mathfrak{n},\widehat{a}-a)$ of $(P,\mathfrak{m},\widehat{a})$ with $\operatorname{Re} a = b$. Then by Lemma 3.2.35 the minimal hole $(P^{\phi_0}_{+a},\mathfrak{n},\widehat{a}-a)$ in $H^{\phi_0}[i]$ is special. Now apply Proposition 4.3.12 with H^{ϕ_0}, $(P^{\phi_0}_{+a},\mathfrak{n},\widehat{a}-a)$, $(Q^{\phi_0}_{+b},\mathfrak{n},\widehat{b}-b)$ in place of H, $(P,\mathfrak{m},\widehat{a})$, $(Q,\mathfrak{m},\widehat{b})$, respectively: it gives $b_0 \in H$ and a refinement

$$\big((Q^{\phi_0}_{+b})_{+b_0},\mathfrak{n},(\widehat{b}-b)-b_0\big) = \big(Q^{\phi_0}_{+(b+b_0)},\mathfrak{n},\widehat{b}-(b+b_0)\big)$$

of $(Q^{\phi_0}_{+b},\mathfrak{n},\widehat{b}-b)$, and thus a refinement $\big(Q_{+(b+b_0)},\mathfrak{n},\widehat{b}-(b+b_0)\big)$ of $(Q_{+b},\mathfrak{n},\widehat{b}-b)$, such that $\big(Q^{\phi}_{+(b+b_0)},\mathfrak{n},\widehat{b}-(b+b_0)\big)$ is eventually split-normal. By the remark before Proposition 4.3.12, $\big(Q^{\phi}_{+(b+b_0)},\mathfrak{n},\widehat{b}-(b+b_0)\big)$ is also eventually deep. $\square$

Recall that $v(\widehat{b}-H)\subseteq v(\widehat{c}-H)$ or $v(\widehat{c}-H)\subseteq v(\widehat{b}-H)$. The following corollary concerns the second case:

Corollary 4.3.14. *If* $\deg P>1$, $v(\widehat{c}-H)\subseteq v(\widehat{b}-H)$, *and* $R\in Z(H,\widehat{c})$ *has minimal complexity, then the* Z*-minimal slot* $(R,\mathfrak{m},\widehat{c})$ *in* H *has a special refinement* $(R_{+c},\mathfrak{n},\widehat{c}-c)$ *such that* $(R^{\phi}_{+c},\mathfrak{n},\widehat{c}-c)$ *is eventually deep and split-normal.*

Proof. Apply Corollary 4.3.13 to the minimal hole $(P_{\times\mathrm{i}},\mathfrak{m},-\mathrm{i}\widehat{a})$ in $H[\mathrm{i}]$. $\square$

In the next two corollaries we handle the case $\deg P=1$. Recall from Lemma 4.3.8 that then $\operatorname{order}Q=1$ or $\operatorname{order}Q=2$, $\deg Q=1$. Theorem 3.3.33 gives:

Corollary 4.3.15. *Suppose* H *is* 1*-linearly newtonian and* $\operatorname{order}Q=1$. *Then the slot* $(Q,\mathfrak{m},\widehat{b})$ *in* H *has a refinement* $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ *such that* $(Q^{\phi}_{+b},\mathfrak{n},\widehat{b}-b)$ *is eventually deep and split-normal.*

Corollary 4.3.16. *Suppose* $\deg P=1$ *and* $\operatorname{order}Q=2$, $\deg Q=1$. *Let* $\widehat{Q}\in H\{Y\}$ *be a minimal annihilator of* $\widehat{b}$ *over* H. *Then* $\big(\widehat{Q},\mathfrak{m},\widehat{b}\big)$ *is a* Z*-minimal hole in* H *and has a refinement* $\big(\widehat{Q}_{+b},\mathfrak{n},\widehat{b}-b\big)$ *such that* $\big(\widehat{Q}^{\phi}_{+b},\mathfrak{n},\widehat{b}-b\big)$ *is eventually deep and split-normal.*

Proof. By the proof of Lemma 4.3.8 we have $\mathrm{c}(Q)=\mathrm{c}(\widehat{Q})$ (hence $\big(\widehat{Q},\mathfrak{m},\widehat{b}\big)$ is a Z-minimal hole in H) and $L_{\widehat{Q}}$ splits over $H[\mathrm{i}]$. Corollary 3.3.12 gives a refinement $\big(\widehat{Q}_{+b},\mathfrak{n},\widehat{b}-b\big)$ of $\big(\widehat{Q},\mathfrak{m},\widehat{b}\big)$ whose linear part has Newton weight 0 and such that the slot $\big(\widehat{Q}^{\phi}_{+b},\mathfrak{n},\widehat{b}-b\big)$ in H^{ϕ} is deep, eventually. Moreover, by Lemmas 3.3.17 and 3.2.31, $\big(\widehat{Q}^{\phi}_{+b},\mathfrak{n},\widehat{b}-b\big)$ is normal and its linear part splits over $H^{\phi}[\mathrm{i}]$, eventually. Thus $\big(\widehat{Q}^{\phi}_{+b},\mathfrak{n},\widehat{b}-b\big)$ is eventually deep and split-normal. $\square$

This concludes the proof of Theorem 4.3.9.

Split-normality and refinements. We now study the behavior of split-normality under refinements. In this subsection a ranges over H and $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{v}$ range over $H^{\times}$. Let $(P,\mathfrak{m},\widehat{a})$ be a slot in H of order $r\geqslant 1$ with $\widehat{a}\in\widehat{H}\setminus H$, and $L:=L_{P_{\times\mathfrak{m}}}$, $w:=\operatorname{wt}(P)$. Here is the split-normal analogue of Lemma 3.3.19:

Lemma 4.3.17. *Suppose* $\operatorname{order}(L)=r$ *and* $\mathfrak{v}$ *is such that* (SN1) *and* (SN2) *hold, and* $\mathfrak{v}(L)\asymp_{\Delta(\mathfrak{v})}\mathfrak{v}$. *Then* $(P,\mathfrak{m},\widehat{a})$ *is split-normal.*

Proof. Same as that of 3.3.19, but with R as in (SN2) instead of $(P_{\times\mathfrak{m}})_{>1}$. $\square$

Now split-normal analogues of Propositions 3.3.25 and 3.3.26:

Lemma 4.3.18. *Suppose* $(P,\mathfrak{m},\widehat{a})$ *is split-normal. Let a refinement* $(P_{+a},\mathfrak{m},\widehat{a}-a)$ *of* $(P,\mathfrak{m},\widehat{a})$ *be given. Then* $(P_{+a},\mathfrak{m},\widehat{a}-a)$ *is also split-normal.*

Proof. As in the proof of Proposition 3.3.25 we arrange $\mathfrak{m}=1$ and show for $\mathfrak{v}:=\mathfrak{v}(L_P)$, using Lemmas 3.1.28 and 4.3.4, that $\operatorname{order}(L_{P_{+a}})=r$ and

$$(P_{+a})_1\sim_{\Delta(\mathfrak{v})}P_1,\quad \mathfrak{v}(L_{P_{+a}})\sim_{\Delta(\mathfrak{v})}\mathfrak{v},\quad (P_{+a})_{>1}\prec_{\Delta(\mathfrak{v})}\mathfrak{v}^{w+1}(P_{+a})_1.$$

Now take Q, R as in (SN2) for $\mathfrak{m}=1$. Then $P_1 = Q + R_1$, and so by Lemma 3.1.29 for $A = L_Q$ we obtain $(P_{+a})_1 - Q \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{+a})_1$, and thus $(P_{+a})_{\geqslant 1} - Q \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{+a})_1$. Hence (SN2) holds with $\mathfrak{m}=1$ and P_{+a} instead of P. Thus the slot $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ in H is split-normal by Lemma 4.3.17. $\square$

Lemma 4.3.19. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is split-normal,* $\widehat{a} \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$*, and* $[\mathfrak{n}/\mathfrak{m}] \leqslant [\mathfrak{v}]$, $\mathfrak{v} := \mathfrak{v}(L)$*. Then the refinement* $(P, \mathfrak{n}, \widehat{a})$ *of* $(P, \mathfrak{m}, \widehat{a})$ *is split-normal: if* $\mathfrak{m}$*,* P*,* Q*,* $\mathfrak{v}$ *are as in* (SN2)*, then* (SN2) *holds with* $\mathfrak{n}$*,* $Q_{\times \mathfrak{n}/\mathfrak{m}}$*,* $R_{\times \mathfrak{n}/\mathfrak{m}}$*,* $\mathfrak{v}(L_{P_{\times \mathfrak{n}}})$ *in place of* $\mathfrak{m}$*,* Q*,* R*,* $\mathfrak{v}$*.*

Proof. Set $\widetilde{L} := L_{P_{\times \mathfrak{n}}}$. Lemma 3.3.1 gives $\operatorname{order}(\widetilde{L}) = r$ and $\mathfrak{v}(\widetilde{L}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. Thus $(P_{\times \mathfrak{n}})_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times \mathfrak{n}})_1$ by Proposition 3.3.26. Now arrange $\mathfrak{m}=1$ in the usual way, and take Q, R as in (SN2) for $\mathfrak{m}=1$. Then

$$(P_{\times \mathfrak{n}})_1 \;=\; (P_1)_{\times \mathfrak{n}} \;=\; Q_{\times \mathfrak{n}} + (R_1)_{\times \mathfrak{n}}, \qquad (P_{\times \mathfrak{n}})_{>1} \;=\; (R_{\times \mathfrak{n}})_{>1} \;=\; (R_{>1})_{\times \mathfrak{n}}$$

by [ADH, 4.3], where $Q_{\times \mathfrak{n}}$ is homogeneous of degree 1 and order r, and $L_{Q_{\times \mathfrak{n}}} = L_Q \mathfrak{n}$ splits over K. Using [ADH, 4.3, 6.1.3] and $[\mathfrak{n}] \leqslant [\mathfrak{v}]$ we obtain

$$(R_1)_{\times \mathfrak{n}} \;\asymp_{\Delta(\mathfrak{v})}\; \mathfrak{n} R_1 \;\preccurlyeq\; \mathfrak{n} R \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{n}\mathfrak{v}^{w+1} P_1 \;\asymp_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{w+1}(P_1)_{\times \mathfrak{n}} \;=\; \mathfrak{v}^{w+1}(P_{\times \mathfrak{n}})_1.$$

Hence (SN2) holds for $\mathfrak{n}, Q_{\times \mathfrak{n}}, R_{\times \mathfrak{n}}, \mathfrak{v}(\widetilde{L})$ in place of $\mathfrak{m}, Q, R, \mathfrak{v}$. $\square$

Recall our standing assumption in this section that H is a real closed H-field. Thus H is d-valued, and for all $\mathfrak{n}$ and $q \in \mathbb{Q}^{>}$ we have $\mathfrak{n}^q \in H^{\times}$ such that $(\mathfrak{n}^q)^{\dagger} = q\mathfrak{n}^{\dagger}$. *In the rest of this section we fix such an* $\mathfrak{n}^q$ *for all* $\mathfrak{n}$ *and* $q \in \mathbb{Q}^{>}$. Now we upgrade Corollary 3.3.31 with "split-normal" instead of "normal":

Lemma 4.3.20. *Suppose* $\mathfrak{m} = 1$*,* $(P, 1, \widehat{a})$ *is split-normal,* $\widehat{a} \prec \mathfrak{n} \prec 1$*, and for* $\mathfrak{v} := \mathfrak{v}(L_P)$ *we have* $[\mathfrak{n}^{\dagger}] < [\mathfrak{v}] < [\mathfrak{n}]$*. Then* $(P, \mathfrak{n}^q, \widehat{a})$ *is a split-normal refinement of* $(P, 1, \widehat{a})$ *for all but finitely many* $q \in \mathbb{Q}$ *with* $0 < q < 1$*.*

Proof. Corollary 3.3.31 gives that $(P, \mathfrak{n}^q, \widehat{a})$ is a normal refinement of $(P, 1, \widehat{a})$ for all but finitely many $q \in \mathbb{Q}$ with $0 < q < 1$. Take Q, R as in (SN2) for $\mathfrak{m} = 1$. Then $L = L_Q + L_R$ where L_Q splits over $H[i]$ and $L_R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} L$, for $\mathfrak{v} := \mathfrak{v}(L)$. Applying Corollary 3.1.19 to $A := L$, $B := L_R$ we obtain: $L_R \mathfrak{n}^q \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1} L\mathfrak{n}^q$, $\mathfrak{w} := \mathfrak{v}(L\mathfrak{n}^q)$, for all but finitely many $q \in \mathbb{Q}^{>}$.

Let $q \in \mathbb{Q}$ be such that $0 < q < 1$, $(P, \mathfrak{n}^q, \widehat{a})$ is a normal refinement of $(P, 1, \widehat{a})$, and $L_R \mathfrak{n}^q \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1} L\mathfrak{n}^q$, with $\mathfrak{w}$ as above. Then $(P_{\times \mathfrak{n}^q})_1 = Q_{\times \mathfrak{n}^q} + (R_1)_{\times \mathfrak{n}^q}$ where $Q_{\times \mathfrak{n}^q}$ is homogeneous of degree 1 and order r, $L_{Q_{\times \mathfrak{n}^q}} = L_Q \mathfrak{n}^q$ splits over $H[i]$, and $(R_1)_{\times \mathfrak{n}^q} \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1}(P_{\times \mathfrak{n}^q})_1$ for $\mathfrak{w} := \mathfrak{v}(L_{P_{\times \mathfrak{n}^q}})$. Since $(P, \mathfrak{n}^q, \widehat{a})$ is normal, we also have $(P_{\times \mathfrak{n}^q})_{>1} \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1}(P_{\times \mathfrak{n}^q})_1$. Thus $(P, \mathfrak{n}^q, \widehat{a})$ is split-normal. $\square$

Remark. We do not know if in this last lemma we can drop the assumption $[\mathfrak{n}^{\dagger}] < [\mathfrak{v}]$.

Strengthening split-normality. *In this subsection* a, b *range over* H *and* $\mathfrak{m}, \mathfrak{n}$ *over* $H^{\times}$*, and* $(P, \mathfrak{m}, \widehat{a})$ *is a slot in* H *of order* $r \geqslant 1$ *and weight* $w := \operatorname{wt}(P)$*, so* $w \geqslant 1$*, and* $L := L_{P_{\times \mathfrak{m}}}$*. If* $\operatorname{order} L = r$*, we set* $\mathfrak{v} := \mathfrak{v}(L)$*.*

With an eye towards later use in connection with fixed point theorems over Hardy fields we strengthen here the concept of split-normality; in the next subsection we show how to improve Theorem 4.3.9 accordingly. See the last subsection of Section 4.2 for the notion of strong splitting.

Definition 4.3.21. Call $(P,\mathfrak{m},\widehat{a})$ **almost strongly split-normal** if $\operatorname{order} L = r$, $\mathfrak{v} \prec^\flat 1$, and the following strengthening of (SN2) holds:

(SN2as) $(P_{\times\mathfrak{m}})_{\geqslant 1} = Q+R$ where $Q,R \in H\{Y\}$, Q is homogeneous of degree 1 and order r, L_Q splits strongly over K, and $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$.

We say that $(P,\mathfrak{m},\widehat{a})$ is **strongly split-normal** if $\operatorname{order} L = r$, $\mathfrak{v} \prec^\flat 1$, and the following condition is satisfied:

(SN2s) $P_{\times\mathfrak{m}} = Q+R$ where $Q,R \in H\{Y\}$, Q is homogeneous of degree 1 and order r, L_Q splits strongly over K, and $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$.

To facilitate use of (SN2s) we observe:

Lemma 4.3.22. *Suppose $(P,\mathfrak{m},\widehat{a})$ is strongly split-normal and $P_{\times\mathfrak{m}} = Q+R$ as in* (SN2s). *Then $Q \sim (P_{\times\mathfrak{m}})_1$, $\mathfrak{v}_Q := \mathfrak{v}(L_Q) \sim \mathfrak{v}$, so $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}_Q^{w+1}Q$.*

Proof. We have $(P_{\times\mathfrak{m}})_1 = Q+R_1$, so $Q = (P_{\times\mathfrak{m}})_1 - R_1$ with $R_1 \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$. Now apply Lemma 3.1.1 to $A := L$ and $B := -L_{R_1}$. ☐

If $(P,\mathfrak{m},\widehat{a})$ is almost strongly split-normal, then $(P,\mathfrak{m},\widehat{a})$ is split-normal and hence normal by Lemma 4.3.4. If $(P,\mathfrak{m},\widehat{a})$ is normal and L splits strongly over K, then $(P,\mathfrak{m},\widehat{a})$ is almost strongly split-normal; in particular, if $(P,\mathfrak{m},\widehat{a})$ is normal of order $r=1$, then $(P,\mathfrak{m},\widehat{a})$ is almost strongly split-normal, by Lemma 4.2.11. Moreover:

Lemma 4.3.23. *The following are equivalent:*

(i) $(P,\mathfrak{m},\widehat{a})$ *is strongly split-normal;*
(ii) $(P,\mathfrak{m},\widehat{a})$ *is almost strongly split-normal and strictly normal;*
(iii) $(P,\mathfrak{m},\widehat{a})$ *is almost strongly split-normal and $P(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_1)_{\times\mathfrak{m}}$.*

Proof. Suppose $(P,\mathfrak{m},\widehat{a})$ is strongly split-normal, and let Q, R be as in (SN2s). Then $(P_{\times\mathfrak{m}})_{\geqslant 1} = Q + R_{\geqslant 1}$, L_Q splits strongly over K, and $R_{\geqslant 1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$. Hence $(P,\mathfrak{m},\widehat{a})$ is almost strongly split-normal, and thus normal. Also $P(0) = R(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$, so $(P,\mathfrak{m},\widehat{a})$ is strictly normal. This shows (i) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (iii) is clear. For (iii) $\Rightarrow$ (i) suppose $(P,\mathfrak{m},\widehat{a})$ is almost strongly split-normal and $P(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_1)_{\times\mathfrak{m}}$. Take Q, R as in (SN2as). Then $P_{\times\mathfrak{m}} = Q + \widetilde{R}$ where $\widetilde{R} := P(0) + R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_1)_{\times\mathfrak{m}}$. Thus $(P,\mathfrak{m},\widehat{a})$ is strongly split-normal. ☐

Corollary 4.3.24. *If L splits strongly over K, then*

$$(P,\mathfrak{m},\widehat{a}) \text{ is strongly split-normal} \iff (P,\mathfrak{m},\widehat{a}) \text{ is strictly normal.}$$

The following diagram summarizes some implications between these variants of normality, for slots $(P,\mathfrak{m},\widehat{a})$ in H of order $r \geqslant 1$:

$$\begin{tikzcd}
\text{strongly split-normal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{almost strongly split-normal} \arrow[r, Rightarrow] & \text{split-normal} \arrow[d, Rightarrow] \\
\text{strictly normal} \arrow[rr, Rightarrow] & & \text{normal}
\end{tikzcd}$$

If $(P,\mathfrak{m},\widehat{a})$ is almost strongly split-normal, then so are $(bP,\mathfrak{m},\widehat{a})$ for $b \neq 0$ and $(P_{\times\mathfrak{n}},\mathfrak{m}/\mathfrak{n},\widehat{a}/\mathfrak{n})$, and likewise with "strongly" in place of "almost strongly".

Here is a version of Lemma 4.3.18 for (almost) strong split-normality:

Lemma 4.3.25. *Suppose $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ refines $(P, \mathfrak{m}, \widehat{a})$. If $(P, \mathfrak{m}, \widehat{a})$ is almost strongly split-normal, then so is $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$. If $(P, \mathfrak{m}, \widehat{a})$ is strongly split-normal, Z-minimal, and $\widehat{a}-a \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+w+1}\mathfrak{m}$, then $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ is strongly split-normal.*

Proof. The first part follows from Lemma 4.3.18 and its proof. In combination with Lemmas 3.3.42 and 4.3.23, this also yields the second part. $\square$

Lemma 4.3.26. *Suppose that $(P, \mathfrak{m}, \widehat{a})$ is split-normal and $\widehat{a} \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$. Then for all sufficiently small $q \in \mathbb{Q}^{>}$, any $\mathfrak{n} \asymp \mathfrak{v}^q\mathfrak{m}$ yields an almost strongly split-normal refinement $(P, \mathfrak{n}, \widehat{a})$ of $(P, \mathfrak{m}, \widehat{a})$.*

Proof. We arrange $\mathfrak{m} = 1$, so $\widehat{a} \prec_{\Delta(\mathfrak{v})} 1$. Take Q, R as in (SN2) with $\mathfrak{m} = 1$, and take $q_0 \in \mathbb{Q}^{>}$ such that $\widehat{a} \prec \mathfrak{v}^{q_0} \prec 1$. By Lemma 4.2.13 we can decrease q_0 so that for all $q \in \mathbb{Q}$ with $0 < q \leqslant q_0$ and any $\mathfrak{n} \asymp \mathfrak{v}^q$, $L_{Q_{\times\mathfrak{n}}} = L_Q\mathfrak{n}$ splits strongly over K. Suppose $q \in \mathbb{Q}$, $0 < q \leqslant q_0$, and $\mathfrak{n} \asymp \mathfrak{v}^q$. Then $(P, \mathfrak{n}, \widehat{a})$ is an almost strongly split-normal refinement of $(P, 1, \widehat{a})$, by Lemma 4.3.19. $\square$

Corollary 4.3.27. *Suppose that $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, deep, and split-normal. Then $(P, \mathfrak{m}, \widehat{a})$ has a refinement which is deep and almost strongly split-normal.*

Proof. Lemma 3.3.13 gives a such that $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$. By Corollary 3.3.8, the refinement $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ of $(P, \mathfrak{m}, \widehat{a})$ is deep with $\mathfrak{v}(L_{P_{+a,\times\mathfrak{m}}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$, and by Lemma 4.3.18 it is also split-normal. Now apply Lemma 4.3.26 to $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ in place of $(P, \mathfrak{m}, \widehat{a})$ and again use Corollary 3.3.8. $\square$

We now turn to the behavior of these properties under compositional conjugation.

Lemma 4.3.28. *Let ϕ be active in H with $0 < \phi \preccurlyeq 1$. If $(P, \mathfrak{m}, \widehat{a})$ is almost strongly split-normal, then so is the slot $(P^\phi, \mathfrak{m}, \widehat{a})$ in H^ϕ. Likewise with "strongly" in place of "almost strongly".*

Proof. We arrange $\mathfrak{m} = 1$, assume $(P, \mathfrak{m}, \widehat{a})$ is almost strongly split-normal, and take Q, R as in (SN2as). The proof of Lemma 4.3.5 shows that with $\mathfrak{w} := \mathfrak{v}(L_{P^\phi})$ we have $\mathfrak{w} \prec^\flat_\phi 1$ and $(P^\phi)_{\geqslant 1} = Q^\phi + R^\phi$ where $Q^\phi \in H^\phi\{Y\}$ is homogeneous of degree 1 and order r, L_{Q^ϕ} splits over $H^\phi[\mathrm{i}]$, and $R^\phi \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1}(P^\phi)_1$. By Lemma 4.2.12, $L_{Q^\phi} = L_Q^\phi$ even splits strongly over $H[\mathrm{i}]$. Hence $(P^\phi, \mathfrak{m}, \widehat{a})$ is almost strongly split-normal. The rest follows from Lemma 4.3.23 and the fact that if $(P, \mathfrak{m}, \widehat{a})$ is strictly normal, then so is $(P^\phi, \mathfrak{m}, \widehat{a})$. $\square$

If H is $\upomega$-free and r-linearly newtonian, then by Corollary 3.3.48, every Z-minimal slot in H of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that the slot $(P^\phi, \mathfrak{m}, \widehat{a})$ in H^ϕ is eventually deep and strictly normal. Corollary 4.3.30 of the next lemma is a variant of this fact for strong split-normality.

Lemma 4.3.29. *Assume H is $\upomega$-free and r-linearly newtonian, and every $A \in H[\partial]$ of order r splits over K. Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal. Then there is a refinement $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ of $(P, \mathfrak{m}, \widehat{a})$ and an active ϕ in H with $0 < \phi \preccurlyeq 1$ such that $(P^\phi_{+a}, \mathfrak{n}, \widehat{a}-a)$ is deep and strictly normal, and its linear part splits strongly over K^ϕ (so $(P^\phi_{+a}, \mathfrak{n}, \widehat{a}-a)$ is strongly split-normal by Corollary 4.3.24).*

Proof. For any active ϕ in H with $0 < \phi \preccurlyeq 1$ we may replace H, $(P, \mathfrak{m}, \widehat{a})$ by H^ϕ, $(P^\phi, \mathfrak{m}, \widehat{a})$, respectively. We may also replace $(P, \mathfrak{m}, \widehat{a})$ by any of its refinements. Now Theorem 3.3.33 gives a refinement $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ of $(P, \mathfrak{m}, \widehat{a})$ and an active ϕ in H such that $0 < \phi \preccurlyeq 1$ and $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a}-a)$ is deep and normal. Replacing H, $(P, \mathfrak{m}, \widehat{a})$ by H^{ϕ}, $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a}-a)$, respectively, we thus arrange that $(P, \mathfrak{m}, \widehat{a})$ itself is deep and normal. We show that then the lemma holds with $\phi = 1$. For this we first replace $(P, \mathfrak{m}, \widehat{a})$ by a suitable refinement $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ to arrange by Corollary 3.3.47 that $(P, \mathfrak{m}, \widehat{a})$ is strictly normal and $\widehat{a} \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$. Now L splits over K, so by Corollary 4.2.14, for sufficiently small $q \in \mathbb{Q}^{>}$, any $\mathfrak{n} \asymp |\mathfrak{v}|^q \mathfrak{m}$ gives a refinement $(P, \mathfrak{n}, \widehat{a})$ of $(P, \mathfrak{m}, \widehat{a})$ whose linear part $L_{P_{\times\mathfrak{n}}}$ has order r and splits strongly over K. For each such $\mathfrak{n}$, $(P, \mathfrak{n}, \widehat{a})$ is deep by Corollary 3.3.8, and for some such $\mathfrak{n}$, $(P, \mathfrak{n}, \widehat{a})$ is also strictly normal, by Remark 3.3.45. ☐

The previous lemma in combination with Lemma 4.3.28 yields:

Corollary 4.3.30. *With the same assumptions on H, K as in Lemma 4.3.29, every Z-minimal slot in H of order r has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is eventually deep and strongly split-normal.*

For $r = 1$ the splitting assumption is automatically satisfied (and this is the case most relevant later). We do not know whether "every $A \in H[\partial]^{\neq}$ of order $\leqslant r$ splits over K" is strictly weaker than "K is r-linearly closed".

Achieving strong split-normality. We make the same assumptions as in the subsection *Achieving split-normality*: H *is* $\boldsymbol{\omega}$*-free and* $(P, \mathfrak{m}, \widehat{a})$ *is a minimal hole in* $K = H[\mathrm{i}]$ *of order* $r \geqslant 1$, *with* $\mathfrak{m} \in H^{\times}$ *and* $\widehat{a} \in \widehat{K} \setminus K$. Recall that K is also $\boldsymbol{\omega}$-free [ADH, 11.7.23]. We have

$$\widehat{a} \ = \ \widehat{b} + \widehat{c}\,i, \qquad \widehat{b}, \widehat{c} \in \widehat{H}.$$

We let a range over K, b, c over H, and $\mathfrak{n}$ over $H^{\times}$. In connection with the next two lemmas we note that given an active ϕ in H with $0 < \phi \preccurlyeq 1$, if $(P, \mathfrak{m}, \widehat{a})$ is normal (strictly normal, respectively), then so is $(P^{\phi}, \mathfrak{m}, \widehat{a})$, by Lemma 3.3.20 (Lemma 3.3.40, respectively); moreover, if the linear part of $(P, \mathfrak{m}, \widehat{a})$ splits strongly over K, then the linear part of $(P^{\phi}, \mathfrak{m}, \widehat{a})$ splits strongly over $K^{\phi} = H^{\phi}[\mathrm{i}]$, by Lemma 4.2.12. Here is a "complex" version of Lemma 4.3.29, with a similar proof:

Lemma 4.3.31. *For some refinement $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ of $(P, \mathfrak{m}, \widehat{a})$ and active ϕ in H with $0 < \phi \preccurlyeq 1$, the hole $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a}-a)$ in K^{ϕ} is deep and normal, its linear part splits strongly over K^{ϕ}, and it is moreover strictly normal if* $\deg P > 1$.

Proof. For any active ϕ in H with $0 < \phi \preccurlyeq 1$ we may replace H and $(P, \mathfrak{m}, \widehat{a})$ by H^{ϕ} and the minimal hole $(P^{\phi}, \mathfrak{m}, \widehat{a})$ in K^{ϕ}. We may also replace $(P, \mathfrak{m}, \widehat{a})$ by any of its refinements $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$. As noted before Theorem 4.3.9, Corollary 3.3.34 and Lemma 3.3.23 give a refinement $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ of $(P, \mathfrak{m}, \widehat{a})$ and an active ϕ in H with $0 < \phi \preccurlyeq 1$ such that $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a}-a)$ is deep and normal. Replacing H, $(P, \mathfrak{m}, \widehat{a})$ by H^{ϕ}, $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a}-a)$, respectively, we thus arrange that $(P, \mathfrak{m}, \widehat{a})$ itself is deep and normal. We show that then the lemma holds with $\phi = 1$.

Set $L := L_{P_{\times\mathfrak{m}}}$ and $\mathfrak{v} := \mathfrak{v}(L)$. Lemma 3.3.13 gives a with $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$. If $\deg P > 1$, then K is r-linearly newtonian and we use Corollary 3.3.16 to take a such that even $\widehat{a} - a \preccurlyeq \mathfrak{v}^{w+2}\mathfrak{m}$. Replacing $(P, \mathfrak{m}, \widehat{a})$ by $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$, we thus arrange by Lemma 3.3.7 and Proposition 3.3.25 that $\widehat{a} \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$, and also by Lemma 3.3.46 that $(P, \mathfrak{m}, \widehat{a})$ is strictly normal if $\deg P > 1$. Now L splits over K, since K is r-linearly closed by Corollary 3.2.4. Then by Corollary 4.2.14, for sufficiently small $q \in \mathbb{Q}^{>}$, any $\mathfrak{n} \asymp |\mathfrak{v}|^q \mathfrak{m}$ gives a refinement $(P, \mathfrak{n}, \widehat{a})$ of $(P, \mathfrak{m}, \widehat{a})$ whose linear part $L_{P_{\times\mathfrak{n}}}$ splits strongly over K. For such $\mathfrak{n}$, $(P,\mathfrak{n},\widehat{a})$ is deep by Lemma 3.3.7 and normal by Proposition 3.3.26. If $(P,\mathfrak{m},\widehat{a})$ is strictly normal, then for some such $\mathfrak{n}$, $(P,\mathfrak{n},\widehat{a})$ is also strictly normal, thanks to Lemma 3.3.44. $\square$

The following version of Lemma 4.3.31 also encompasses linear $(P,\mathfrak{m},\widehat{a})$:

Lemma 4.3.32. *Suppose $\partial K = K$ and $\mathrm{I}(K) \subseteq K^\dagger$. Then there is a refinement $(P_{+a},\mathfrak{n},\widehat{a}-a)$ of $(P,\mathfrak{m},\widehat{a})$ and an active ϕ in H with $0<\phi\preccurlyeq 1$ such that the hole $(P^\phi_{+a},\mathfrak{n},\widehat{a}-a)$ in K^ϕ is deep and strictly normal, and its linear part splits strongly over K^ϕ.*

Proof. Thanks to Lemma 4.3.31 we need only consider the case $\deg P = 1$. Then we have $r=1$ by Corollary 3.2.8. (See now the remark following this proof.) As in the proof of Lemma 4.3.31 we may replace H and $(P,\mathfrak{m},\widehat{a})$ for any active $\phi\preccurlyeq 1$ in $H^>$ by H^ϕ and $(P^\phi,\mathfrak{m},\widehat{a})$, and also $(P,\mathfrak{m},\widehat{a})$ by any of its refinements $(P_{+a},\mathfrak{n},\widehat{a}-a)$. Recall here that $\mathfrak{n}\in H^\times$. Hence using a remark preceding Lemma 3.3.39 and Corollary 3.5.17 we arrange that $(P,\mathfrak{m},\widehat{a})$ is strictly normal, and thus balanced and deep. We show that then the lemma holds with $\phi=1$.

Set $L := L_{P_{\times\mathfrak{m}}}$, $\mathfrak{v}:=\mathfrak{v}(L)$. Lemmas 3.5.9 and 3.5.10 yield an a with $\widehat{a}-a\preccurlyeq\mathfrak{v}^4\mathfrak{m}$. Replacing $(P,\mathfrak{m},\widehat{a})$ by $(P_{+a},\mathfrak{m},\widehat{a}-a)$ arranges that $\widehat{a}\prec_{\Delta(\mathfrak{v})}\mathfrak{m}$, by Lemmas 3.3.7 and 3.3.41. As in the proof of Lemma 4.3.31, for sufficiently small $q\in\mathbb{Q}^>$, any $\mathfrak{n}\asymp|\mathfrak{v}|^q\mathfrak{m}$ now gives a strictly normal and deep refinement $(P,\mathfrak{n},\widehat{a})$ of $(P,\mathfrak{m},\widehat{a})$ whose linear part splits strongly over K. $\square$

Remark. Suppose we replace our standing assumption that H is $\boldsymbol{\omega}$-free and $(P,\mathfrak{m},\widehat{a})$ is a minimal hole in K by the assumption that H is $\boldsymbol{\lambda}$-free and $(P,\mathfrak{m},\widehat{a})$ is a slot in K of order and degree 1 (so K is $\boldsymbol{\lambda}$-free by [ADH, 11.6.8] and $(P,\mathfrak{m},\widehat{a})$ is Z-minimal). Then Lemma 4.3.32 goes through with "hole" replaced by "slot". Its proof also goes through with the references to Lemmas 3.3.7 and 3.3.41 replaced by references to Corollary 3.3.8 and Lemma 3.3.42. The end of that proof refers to the end of the proof of Lemma 4.3.31, and there one should replace Proposition 3.3.26 by Corollary 3.3.27, and Lemma 3.3.44 by Remark 3.3.45.

In the remainder of this subsection we prove the following variant of Theorem 4.3.9:

Theorem 4.3.33. *If H is 1-linearly newtonian, then one of the following holds:*

(i) *$\widehat{b}\notin H$ and there exists a Z-minimal slot $(Q,\mathfrak{m},\widehat{b})$ in H with a refinement $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ such that $(Q^\phi_{+b},\mathfrak{n},\widehat{b}-b)$ is eventually deep and almost strongly split-normal;*

(ii) *$\widehat{c}\notin H$ and there exists a Z-minimal slot $(R,\mathfrak{m},\widehat{c})$ in H with a refinement $(R_{+c},\mathfrak{n},\widehat{c}-c)$ such that $(R^\phi_{+c},\mathfrak{n},\widehat{c}-c)$ is eventually deep and almost strongly split-normal.*

Moreover, if H is 1-linearly newtonian and either $\deg P>1$*, or $\widehat{b}\notin H$ and $Z(H,\widehat{b})$ contains an element of order 1, or $\widehat{c}\notin H$ and $Z(H,\widehat{c})$ contains an element of order 1, then* (i) *holds with "almost" omitted, or* (ii) *holds with "almost" omitted.*

Towards the proof of this theorem we first show:

Lemma 4.3.34. *Suppose $\widehat{b}\notin H$ and $(Q,\mathfrak{m},\widehat{b})$ is a Z-minimal slot in H with a refinement $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ such that $(Q^\phi_{+b},\mathfrak{n},\widehat{b}-b)$ is eventually deep and split-normal. Then $(Q,\mathfrak{m},\widehat{b})$ has a refinement $(Q_{+b},\mathfrak{n},\widehat{b}-b)$ such that $(Q^\phi_{+b},\mathfrak{n},\widehat{b}-b)$ is eventually deep and almost strongly split-normal.*

Proof. Let $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ be a refinement of $(Q, \mathfrak{m}, \widehat{b})$ and let ϕ_0 be active in H such that $0 < \phi_0 \preccurlyeq 1$ and $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$ is deep and split-normal. Then Corollary 4.3.27 yields a refinement $\big((Q^{\phi_0}_{+b})_{+b_0}, \mathfrak{n}_0, (\widehat{b} - b) - b_0\big)$ of $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$ which is deep and almost strongly split-normal. Hence

$$\big((Q_{+b})_{+b_0}, \mathfrak{n}_0, (\widehat{b} - b) - b_0\big) \ = \ \big(Q_{+(b+b_0)}, \mathfrak{n}_0, \widehat{b} - (b + b_0)\big)$$

is a refinement of $(Q, \mathfrak{m}, \widehat{b})$, and $\big(Q^{\phi}_{+(b+b_0)}, \mathfrak{n}_0, \widehat{b} - (b + b_0)\big)$ is eventually deep and almost strongly split-normal by Lemma 4.3.28. □

Likewise:

Lemma 4.3.35. *Suppose $\widehat{c} \notin H$, and $(R, \mathfrak{m}, \widehat{c})$ is a Z-minimal slot in H with a refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ is eventually deep and split-normal. Then $(R, \mathfrak{m}, \widehat{c})$ has a refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ is eventually deep and almost strongly split-normal.*

Theorem 4.3.9 and the two lemmas above give the first part of Theorem 4.3.33. We break up the proof of the "moreover" part into several cases, along the lines of the proof of Theorem 4.3.9. We begin with the case where $\widehat{b} \in H$ or $\widehat{c} \in H$.

Lemma 4.3.36. *Suppose H is 1-linearly newtonian, $\widehat{b} \notin H$, $(Q, \mathfrak{m}, \widehat{b})$ is a Z-minimal slot in H of order r, and some refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ of $(Q, \mathfrak{m}, \widehat{b})$ is such that $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ is eventually deep and split-normal. Then $(Q, \mathfrak{m}, \widehat{b})$ has a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ with $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ eventually deep and strongly split-normal.*

Proof. Lemma 4.3.34 gives a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ of $(Q, \mathfrak{m}, \widehat{b})$ with $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ eventually deep and almost strongly split-normal. We upgrade this to "strongly split-normal" as follows: Take active ϕ_0 in H with $0 < \phi_0 \preccurlyeq 1$ such that the slot $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$ in H^{ϕ_0} is deep and almost strongly split-normal. Now H is 1-linearly newtonian, hence r-linearly newtonian. Therefore Corollary 3.3.47 yields a deep and strictly normal refinement $\big((Q^{\phi_0}_{+b})_{+b_0}, \mathfrak{n}, (\widehat{b} - b) - b_0\big)$ of $\big(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b\big)$. By Lemma 4.3.25, this refinement is still almost strongly split-normal, thus strongly split-normal by Lemma 4.3.23. Then by Lemma 4.3.28, $\big(Q_{+(b+b_0)}, \mathfrak{n}, \widehat{b} - (b + b_0)\big)$ is a refinement of $(Q, \mathfrak{m}, \widehat{b})$ such that $\big(Q^{\phi}_{+(b+b_0)}, \mathfrak{n}, \widehat{b} - (b + b_0)\big)$ is eventually deep and strongly split-normal. □

Lemmas 4.3.10 and 4.3.36 give the following:

Corollary 4.3.37. *Suppose H is 1-linearly newtonian and $\widehat{c} \in H$. Then there is a hole $(Q, \mathfrak{m}, \widehat{b})$ in H of the same complexity as $(P, \mathfrak{m}, \widehat{a})$. Every such hole $(Q, \mathfrak{m}, \widehat{b})$ in H is minimal and has a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ such that $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ is eventually deep and strongly split-normal.*

Just as Lemma 4.3.10 gave rise to Lemma 4.3.11, Corollary 4.3.37 leads to:

Corollary 4.3.38. *Suppose H is 1-linearly newtonian and $\widehat{b} \in H$. Then there is a hole $(R, \mathfrak{m}, \widehat{c})$ in H of the same complexity as $(P, \mathfrak{m}, \widehat{a})$. Every such hole in H is minimal and has a refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ is eventually deep and strongly split-normal.*

In the following two lemmas we assume that $\widehat{b}, \widehat{c} \notin H$. Let $Q \in Z(H, \widehat{b})$ be of minimal complexity, so $(Q, \mathfrak{m}, \widehat{b})$ is a Z-minimal slot in H, as is each of its refinements. The next lemma strengthens Corollary 4.3.13:

Lemma 4.3.39. *Suppose* $\deg P > 1$ *and* $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$. *Then* $(Q, \mathfrak{m}, \widehat{b})$ *has a refinement* $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ *such that* $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ *is eventually deep and strongly split-normal.*

Proof. Corollary 4.3.13 and Lemma 4.3.34 give a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ of $(Q, \mathfrak{m}, \widehat{b})$ and an active ϕ_0 in H with $0 < \phi_0 \preccurlyeq 1$ such that the slot $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$ in H^{ϕ_0} is deep and almost strongly split-normal. From $\deg P > 1$ we obtain that H is r-linearly newtonian. Now argue as in the proof of Lemma 4.3.36. ☐

Similarly we obtain a strengthening of Corollary 4.3.14, using that corollary and Lemma 4.3.35 in place of Corollary 4.3.13 and Lemma 4.3.34 in the proof:

Lemma 4.3.40. *If* $\deg P > 1$, $v(\widehat{c} - H) \subseteq v(\widehat{b} - H)$, *and* $R \in Z(H, \widehat{c})$ *has minimal complexity, then the Z-minimal slot* $(R, \mathfrak{m}, \widehat{c})$ *in* H *has a refinement* $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ *such that* $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ *is eventually deep and strongly split-normal.*

We now prove the "moreover" part of Theorem 4.3.33. Thus, suppose H is 1-linearly newtonian. If $\widehat{b} \in H$, then $\widehat{c} \notin H$ and Corollary 4.3.38 yields a strong version of (ii) with "almost" omitted. Likewise, if $\widehat{c} \in H$, then $\widehat{b} \notin H$ and Corollary 4.3.37 yields a strong version of (i), with "almost" omitted. In the rest of the proof we assume $\widehat{b}, \widehat{c} \notin H$. By Lemma 4.1.3 we have $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$ or $v(\widehat{c} - H) \subseteq v(\widehat{b} - H)$, and thus Lemmas 4.3.39 and 4.3.40 take care of the case $\deg P > 1$. If $Z(H, \widehat{b})$ contains an element of order 1, and $Q \in Z(H, \widehat{b})$ has minimal complexity, then $\operatorname{order} Q = 1$ by Lemma 4.3.7, so Corollary 4.3.30 and the remark following it yield (i) with "almost" omitted. Likewise, if $Z(H, \widehat{c})$ contains an element of order 1, then (ii) holds with "almost" omitted. ☐

Revisiting newtonianity. We now use our results about isolated holes and split-normality to obtain with Corollary 4.3.41 a sharper first-order characterization of newtonianity than provided by our definition of this notion in [ADH].

Let H be a real closed H-field with small derivation and asymptotic integration. Let $P \in H\{Y\}^{\neq}$ have order $r \geqslant 1$ and weight w. Just for the next corollary, call P **strongly split-normal** if the following conditions are satisfied:

(1) L_P has order r and $\mathfrak{v} := \mathfrak{v}(L_P) \prec^{\flat} 1$; and
(2) $P = Q + R$ where $Q \in H\{Y\}$ is homogeneous of degree 1, $\operatorname{order} Q = r$, L_Q splits strongly over K, and $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} P_1$.

Call P **eventually deep and strongly split-normal** if $\operatorname{ndeg} S_P = 0$ and for all small enough active $\phi > 0$ in H, the differential polynomial $P^{\phi} \in H^{\phi}\{Y\}$ is strongly split-normal with respect to H^{ϕ}. Note: $\operatorname{ndeg} P = \operatorname{nmul} P = 1$ for such P.

Corollary 4.3.41. *Assume H is ω-free. Then the following are equivalent:*

(i) *H is newtonian;*
(ii) *$H[i]$ is 1-linearly newtonian and every eventually deep and strongly split-normal P in $H\{Y\}^{\neq}$ of order $\geqslant 1$ has a zero $y \prec 1$ in H.*

Proof. The direction (i) ⇒ (ii) is clear from (0.10) and [ADH, 14.2.11]. For (ii) ⇒ (i), suppose H is not newtonian and $H[i]$ is 1-linearly newtonian. By Proposition 3.4.9

it is enough to show that then H has an isolated hole $(Q, \mathfrak{m}, \widehat{b})$ such that for all small enough active $\phi > 0$ in H the hole $(Q^\phi, \mathfrak{m}, \widehat{b})$ in H^ϕ is deep and strongly split-normal. We set $K := H[\mathrm{i}]$ and let b, c range over H.

Lemma 3.2.1 and subsequent remarks give a minimal hole $(P, \mathfrak{m}, \widehat{a})$ in K of order $r \geqslant 1$, where $\mathfrak{m} \in H^\times$. Then $\deg P > 1$ by Corollary 3.2.8. By Lemma 4.2.15 we arrange that $\widehat{a} \in \widehat{K} := \widehat{H}[\mathrm{i}]$ where $\widehat{H}$ is an immediate $\boldsymbol{\omega}$-free newtonian H-field extension of H, so $\widehat{a} = \widehat{b} + \widehat{c}\mathrm{i}$ with $\widehat{b}, \widehat{c} \in \widehat{H}$. Then $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$ or $v(\widehat{c} - H) \subseteq v(\widehat{b} - H)$ by Lemma 4.1.3; we assume $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$. (The other case is similar.) The equivalence (i)$\Leftrightarrow$(iii) of that lemma then gives $\widehat{b} \notin H$.

Take $Q \in Z(H, \widehat{b})$ of minimal complexity. Then $(Q, \mathfrak{m}, \widehat{b})$ is a Z-minimal slot in H, of positive order by Lemma 4.3.7. Given any refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ of $(Q, \mathfrak{m}, \widehat{b})$, Lemma 4.1.3 gives c with $v(\widehat{a} - a) = v(\widehat{b} - b)$ for $a := b + c\mathrm{i}$, and we may then replace $(P, \mathfrak{m}, \widehat{a})$ and $(Q, \mathfrak{m}, \widehat{b})$ by $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ and $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$, respectively, whenever convenient. Likewise, for any active ϕ in H with $0 < \phi \preccurlyeq 1$, we can also replace H, K, $(P, \mathfrak{m}, \widehat{a})$, $(Q, \mathfrak{m}, \widehat{b})$ by H^ϕ, K^ϕ, $(P^\phi, \mathfrak{m}, \widehat{a})$, $(Q^\phi, \mathfrak{m}, \widehat{b})$.

Suppose now that $\widehat{c} \notin H$. Use Corollary 4.3.13 to arrange that $(Q, \mathfrak{m}, \widehat{b})$ is normal. Next, use Proposition 3.4.6 to arrange that $(Q, \mathfrak{m}, \widehat{b})$ is isolated, but possibly no longer normal. Being isolated persists under refinement, so we can use Lemma 4.3.39 to arrange that $(Q^\phi, \mathfrak{m}, \widehat{b})$ is eventually deep and strongly split-normal. With Lemma 3.2.14, changing $\widehat{b}$ if necessary, we arrange that $(Q, \mathfrak{m}, \widehat{b})$ is an isolated hole in H, not just an isolated slot in H, thus achieving our goal.

Finally, suppose that $\widehat{c} \in H$. Then use Corollary 4.3.37 and Proposition 3.4.6 to choose Q such that $(Q, \mathfrak{m}, \widehat{b})$ is a minimal and isolated hole in H with the property that $(Q^\phi, \mathfrak{m}, \widehat{b})$ is eventually deep and strongly split-normal. $\square$

4.4. Ultimate Slots

In this section H is a Liouville closed H-field with small derivation, $\widehat{H}$ is an immediate asymptotic extension of H, and i is an element of an asymptotic extension of $\widehat{H}$ with $\mathrm{i}^2 = -1$. Then $\widehat{H}$ is an H-field, $\mathrm{i} \notin \widehat{H}$, $K := H[\mathrm{i}]$ is an algebraic closure of H, and $\widehat{K} := \widehat{H}[\mathrm{i}]$ is an immediate d-valued extension of K. (See the beginning of Section 4.3.) Let C be the constant field of H, let $\mathcal{O}$ denote the valuation ring of H and Γ its value group. Accordingly, the constant field of K is $C_K = C[\mathrm{i}]$ and the valuation ring of K is $\mathcal{O}_K = \mathcal{O} + \mathcal{O}i$. Let $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{w}$ range over $H^\times$ and ϕ over the elements of $H^>$ which are active in H (and hence in K).

In Section 1.2 we introduced

$$W := \{\mathrm{wr}(a, b) : a, b \in H,\ a^2 + b^2 = 1\}.$$

Note that W is a subspace of the $\mathbb{Q}$-linear space H, because $Wi = S^\dagger$ where

$$S := \{a + bi : a, b \in H,\ a^2 + b^2 = 1\}$$

is a divisible subgroup of $K^\times$. We have $K^\dagger = H + Wi$ by Lemma 1.2.4. Thus there exists a complement Λ of the subspace $K^\dagger$ of K such that $\Lambda \subseteq Hi$, and in this section we fix such Λ and let λ range over Λ. Let $\mathrm{U} = K\big[\mathrm{e}(\Lambda)\big]$ be the universal exponential extension of K defined in Section 2.2.

For $A \in K[\partial]^{\neq}$ we have its set $\mathscr{E}^{\mathrm{u}}(A) \subseteq \Gamma$ of ultimate exceptional values, which a-priori might depend on our choice of Λ. We now make good on a promise from

Section 2.5 by showing under the mild assumption $\mathrm{I}(K) \subseteq K^\dagger$ and with our restriction $\Lambda \subseteq Hi$ there is no such dependence:

Corollary 4.4.1. *Suppose $\mathrm{I}(K) \subseteq K^\dagger$. Then for $A \in K[\partial]^{\neq}$, the status of A being terminal does not depend on the choice of Λ, and the set $\mathscr{E}^{\mathrm{u}}(A)$ of ultimate exceptional values of A also does not depend on this choice.*

Proof. Let $\Lambda^* \subseteq Hi$ also be a complement of $K^\dagger$. Let $\lambda \mapsto \lambda^*$ be the $\mathbb{Q}$-linear bijection $\Lambda \to \Lambda^*$ with $\lambda - \lambda^* \in Wi$ for all λ. Then by Lemmas 1.2.8 and 1.2.13,

$$\lambda - \lambda^* \in \mathrm{I}(H)\mathrm{i} \ \subseteq\ \mathrm{I}(K) \ \subseteq\ (\mathcal{O}_K^\times)^\dagger$$

for all λ. Now use Lemma 2.5.6 and Corollary 2.5.7. □

Corollary 4.4.2. *Suppose $\mathrm{I}(K) \subseteq K^\dagger$. Let $A = \partial - g \in K[\partial]$ where $g \in K$ and let $\mathfrak{g} \in H^\times$ be such that $\mathfrak{g}^\dagger = \operatorname{Re} g$. Then*

$$\mathscr{E}^{\mathrm{u}}(A) \ =\ v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) \ =\ \{v\mathfrak{g}\}.$$

In particular, if $\operatorname{Re} g \in \mathrm{I}(H)$, then $\mathscr{E}^{\mathrm{u}}(A) = \{0\}$.

Proof. Let $f \in K^\times$ and λ be such that $g = f^\dagger + \lambda$. Then

$$\mathscr{E}^{\mathrm{u}}(A) \ =\ v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) \ =\ \{vf\}$$

by Lemma 2.5.12 and its proof. Recall that $K^\dagger = H + \mathrm{I}(H)\mathrm{i}$ by Lemma 1.2.13 and remarks preceding it, so $g \in K^\dagger$ iff $\operatorname{Im} g \in \mathrm{I}(H)$. Consider first the case $g \notin K^\dagger$. Then by Corollary 4.4.1 we can change Λ if necessary to arrange $\lambda := (\operatorname{Im} g)\mathrm{i} \in \Lambda$ so that we can take $f := \mathfrak{g}$ in the above. Now suppose $g \in K^\dagger$. Then $g = (\mathfrak{g}h)^\dagger$ where $h \in K^\times$, $h^\dagger = (\operatorname{Im} g)\mathrm{i}$. Then we can take $f := \mathfrak{g}h$, $\lambda := 0$, and we have $h \asymp 1$ since $h^\dagger \in \mathrm{I}(H)\mathrm{i} \subseteq \mathrm{I}(K)$. □

Corollary 4.4.3. *Suppose $\mathrm{I}(K) \subseteq K^\dagger$, and let F be a Liouville closed H-field extension of H, and $L := F[\mathrm{i}]$. Then the subspace $L^\dagger$ of the $\mathbb{Q}$-linear space L has a complement Λ_L with $\Lambda \subseteq \Lambda_L \subseteq F\mathrm{i}$. For any such Λ_L and $A \in K[\partial]^{\neq}$ we have $\mathscr{E}^{\mathrm{e}}(A_\lambda) = \mathscr{E}^{\mathrm{e}}_L(A_\lambda) \cap \Gamma$ for all λ, and thus $\mathscr{E}^{\mathrm{u}}(A) \subseteq \mathscr{E}^{\mathrm{u}}_L(A)$, where $\mathscr{E}^{\mathrm{u}}_L(A)$ is the set of ultimate exceptional values of $A \in L[\partial]^{\neq}$ with respect to Λ_L.*

Proof. By the remarks at the beginning of this subsection applied to F, L in place of H, K we have $L^\dagger = F + W_F\mathrm{i}$ where W_F is a subspace of the $\mathbb{Q}$-linear space F. Also $K^\dagger = H + \mathrm{I}(H)\mathrm{i}$ by Lemma 1.2.13, and $L^\dagger \cap K = K^\dagger$ by Lemma 2.5.22. This yields a complement Λ_L of $L^\dagger$ in L with $\Lambda \subseteq \Lambda_L \subseteq F\mathrm{i}$. Since H is Liouville closed and hence λ-free by [ADH, 11.6.2], its algebraic closure K is λ-free by [ADH, 11.6.8]. Now the rest follows from remarks preceding Lemma 2.5.10. □

Given $A \in K[\partial]^{\neq}$, let $\mathscr{E}^{\mathrm{u}}(A^\phi)$ be the set of ultimate exceptional values of the linear differential operator $A^\phi \in K^\phi[\delta]$, $\delta = \phi^{-1}\partial$, with respect to $\Lambda^\phi = \phi^{-1}\Lambda$. We summarize some properties of ultimate exceptional values used later in this section:

Lemma 4.4.4. *Let $A \in K[\partial]^{\neq}$ have order r. Then for all $b \in K^\times$ and all ϕ,*

$$\mathscr{E}^{\mathrm{u}}(bA) \ =\ \mathscr{E}^{\mathrm{u}}(A), \quad \mathscr{E}^{\mathrm{u}}(Ab) \ =\ \mathscr{E}^{\mathrm{u}}(A) - vb, \quad \mathscr{E}^{\mathrm{u}}(A^\phi) \ =\ \mathscr{E}^{\mathrm{u}}(A).$$

Moreover, if $\mathrm{I}(K) \subseteq K^\dagger$, then:

(i) $|\mathscr{E}^{\mathrm{u}}(A)| \leqslant r$;

(ii) $\dim_{C[\mathrm{i}]} \ker_{\mathrm{U}} A = r \implies \mathscr{E}^{\mathrm{u}}(A) = v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A)$;

(iii) *under the assumption that* $\mathfrak{v} := \mathfrak{v}(A) \prec^{\flat} 1$ *and* $B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+1} A$ *where* $B \in K[\partial]$ *has order* $\leqslant r$, *we have* $\mathscr{E}^{\mathrm{u}}(A+B) = \mathscr{E}^{\mathrm{u}}(A)$;

(iv) *for* $r=1$ *we have* $|\mathscr{E}^{\mathrm{u}}(A)| = 1$ *and* $\mathscr{E}^{\mathrm{u}}(A) = v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A)$.

Proof. For the displayed equalities, see Remark 2.5.8. Now assume $\mathrm{I}(K) \subseteq K^{\dagger}$. Then $K^{\dagger} = H + \mathrm{I}(H)i$, so (i) and (ii) follow from Proposition 2.5.24 and (iii) from Proposition 3.1.27. Corollary 4.4.2 yields (iv). $\square$

Recall from Lemma 1.2.9 that if K is 1-linearly newtonian, then $\mathrm{I}(K) \subseteq K^{\dagger}$.

Suppose $\mathrm{I}(K) \subseteq K^{\dagger}$. Then $K^{\dagger} = H + \mathrm{I}(H)i$, so our Λ has the form $\Lambda_H i$ with Λ_H a complement of $\mathrm{I}(H)$ in H. Conversely, any complement Λ_H of $\mathrm{I}(H)$ in H yields a complement $\Lambda = \Lambda_H i$ of $K^{\dagger}$ in K with $\Lambda \subseteq Hi$. Now $\mathrm{I}(H)$ is a C-linear subspace of H, so $\mathrm{I}(H)$ has a complement Λ_H in H that is a C-linear subspace of H, and then $\Lambda := \Lambda_H i$ is also a C-linear subspace of K.

Lemma 4.4.5. *Suppose* $\mathrm{I}(K) \subseteq K^{\dagger}$ *and* $g \in K$, $g - \lambda \in K^{\dagger}$. *Then*

$$\operatorname{Im} g \in \mathrm{I}(H) \iff \lambda = 0, \qquad \operatorname{Im} g \notin \mathrm{I}(H) \implies \lambda \sim (\operatorname{Im} g)i.$$

Proof. Recall that $\Lambda = \Lambda_H i$ where Λ_H is a complement of $\mathrm{I}(H)$ in H, so $\lambda = \lambda_H i$ where $\lambda_H \in \Lambda_H$. Also, $K^{\dagger} = H \oplus \mathrm{I}(H)i$, hence $\operatorname{Im}(g) - \lambda_H \in \mathrm{I}(H)$; this proves the displayed equivalence. Suppose $\operatorname{Im} g \notin \mathrm{I}(H)$; since $\mathrm{I}(H)$ is an $\mathcal{O}_H$-submodule of H and $\lambda_H \notin \mathrm{I}(H)$, we then have $\operatorname{Im}(g) - \lambda_H \prec \lambda_H$, so $\lambda = \lambda_H i \sim \operatorname{Im}(g)i$. $\square$

Corollary 4.4.6. *Suppose* $\mathrm{I}(K) \subseteq K^{\dagger}$, $A \in K[\partial]^{\neq}$ *has order* r, $\dim_{C[i]} \ker_{\mathrm{U}} A = r$, *and* λ *is an eigenvalue of* A *with respect to* Λ. *Then* $\lambda \preccurlyeq \mathfrak{v}(A)^{-1}$.

Proof. Take $f \neq 0$ and $g_1, \dots, g_r$ in K with $A = f(\partial - g_1)\cdots(\partial - g_r)$. By Corollary 3.1.6 we have $g_1, \dots, g_r \preccurlyeq \mathfrak{v}(A)^{-1}$, and so Corollary 2.4.6 gives $j \in \{1, \dots, r\}$ with $g_j - \lambda \in K^{\dagger}$. Now use Lemma 4.4.5. $\square$

Ultimate slots in H. *In this subsection* a, b *range over* H. Also, $(P, \mathfrak{m}, \widehat{a})$ is a slot in H of order $r \geqslant 1$, where $\widehat{a} \in \widehat{H} \setminus H$. Recall that $L_{P_{\times\mathfrak{m}}} = L_P \mathfrak{m}$, so if $(P, \mathfrak{m}, \widehat{a})$ is normal, then L_P has order r.

Corollary 4.4.7. *Suppose* $\mathrm{I}(K) \subseteq K^{\dagger}$ *and the slot* $(P, \mathfrak{m}, \widehat{a})$ *is split-normal with linear part* $L := L_{P_{\times\mathfrak{m}}}$. *Then with* Q *and* R *as in* (SN2) *we have* $\mathscr{E}^{\mathrm{u}}(L) = \mathscr{E}^{\mathrm{u}}(L_Q)$.

This follows from Lemmas 4.3.4 and 4.4.4(iii). In a similar vein we have an analogue of Lemma 3.3.24:

Lemma 4.4.8. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is normal and* $a \prec \mathfrak{m}$. *Then* L_P *and* $L_{P_{+a}}$ *have order* r, *and if* $\mathrm{I}(K) \subseteq K^{\dagger}$, *then* $\mathscr{E}^{\mathrm{u}}(L_P) = \mathscr{E}^{\mathrm{u}}(L_{P_{+a}})$.

Proof. We have $L_{P_{\times\mathfrak{m}}} = L_P \mathfrak{m}$ and $L_{P_{+a,\times\mathfrak{m}}} = L_{P_{\times\mathfrak{m},+a/\mathfrak{m}}} = L_{P_{+a}}\mathfrak{m}$. The slot $(P_{\times\mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$ in H is normal and $a/\mathfrak{m} \prec 1$. Lemma 3.1.29 applied to $\widehat{H}$, $P_{\times\mathfrak{m}}$, $\widehat{a}/\mathfrak{m}$ in place of K, P, a, respectively, gives: L_P and $L_{P_{+a}}$ have order r, and

$$L_P\mathfrak{m} - L_{P_{+a}}\mathfrak{m} \;=\; L_{P_{\times\mathfrak{m}}} - L_{P_{\times\mathfrak{m},+a/\mathfrak{m}}} \;\prec_{\Delta(\mathfrak{v})}\; \mathfrak{v}^{r+1} L_P\mathfrak{m}$$

where $\mathfrak{v} := \mathfrak{v}(L_P\mathfrak{m}) \prec^{\flat} 1$ by (N1). Suppose now that $\mathrm{I}(K) \subseteq K^{\dagger}$. Then

$$\mathscr{E}^{\mathrm{u}}(L_P) \;=\; \mathscr{E}^{\mathrm{u}}(L_P\mathfrak{m}) + v(\mathfrak{m}) \;=\; \mathscr{E}^{\mathrm{u}}(L_{P_{+a}}\mathfrak{m}) + v(\mathfrak{m}) \;=\; \mathscr{E}^{\mathrm{u}}(L_{P_{+a}})$$

by Lemma 4.4.4(iii). $\square$

The notion introduced below is modeled on that of “isolated slot” (Definition 3.4.1):

Definition 4.4.9. Call $(P, \mathfrak{m}, \widehat{a})$ **ultimate** if for all $a \prec \mathfrak{m}$,

$$\operatorname{order}(L_{P_{+a}}) = r \quad \text{and} \quad \mathscr{E}^{\mathrm{u}}(L_{P_{+a}}) \cap v(\widehat{a} - H) \ < \ v(\widehat{a} - a);$$

equivalently, for all $a \prec \mathfrak{m}$: $\operatorname{order}(L_{P_{+a}}) = r$ and whenever $\mathfrak{w} \preccurlyeq \widehat{a} - a$ is such that $v(\mathfrak{w}) \in \mathscr{E}^{\mathrm{u}}(L_{P_{+a}})$, then $\mathfrak{w} \prec \widehat{a} - b$ for all b. (Thus if $(P, \mathfrak{m}, \widehat{a})$ is ultimate, then it is isolated.)

If $(P, \mathfrak{m}, \widehat{a})$ is ultimate, then so is every equivalent slot in H and $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$, as well as the slot $(P^{\phi}, \mathfrak{m}, \widehat{a})$ in H^{ϕ} (by Lemma 4.4.4). The proofs of the next two lemma are like those of their "isolated" versions, Lemmas 3.4.2 and 3.4.3:

Lemma 4.4.10. *If $(P, \mathfrak{m}, \widehat{a})$ is ultimate, then so is any of its refinements.*

Lemma 4.4.11. *If $(P, \mathfrak{m}, \widehat{a})$ is ultimate, then so is any of its multiplicative conjugates.*

The ultimate condition is most useful in combination with other properties:

Lemma 4.4.12. *If $\mathrm{I}(K) \subseteq K^{\dagger}$ and $(P, \mathfrak{m}, \widehat{a})$ is normal, then*

$$(P, \mathfrak{m}, \widehat{a}) \text{ is ultimate} \quad \Longleftrightarrow \quad \mathscr{E}^{\mathrm{u}}(L_P) \cap v(\widehat{a} - H) \leqslant v\mathfrak{m}.$$

Proof. Use Lemma 4.4.8 and the equivalence $\widehat{a} - a \prec \mathfrak{m} \Leftrightarrow a \prec \mathfrak{m}$. □

The "ultimate" version of Lemma 3.4.5 has the same proof:

Lemma 4.4.13. *If $\deg P = 1$, then*

$$(P, \mathfrak{m}, \widehat{a}) \text{ is ultimate} \quad \Longleftrightarrow \quad \mathscr{E}^{\mathrm{u}}(L_P) \cap v(\widehat{a} - H) \leqslant v\mathfrak{m}.$$

The next proposition is the "ultimate" version of Proposition 3.4.6:

Proposition 4.4.14. *Suppose $\mathrm{I}(K) \subseteq K^{\dagger}$, and $(P, \mathfrak{m}, \widehat{a})$ is normal. Then $(P, \mathfrak{m}, \widehat{a})$ has an ultimate refinement.*

Proof. Suppose $(P, \mathfrak{m}, \widehat{a})$ is not already ultimate. Then Lemma 4.4.12 gives γ with

$$\gamma \in \mathscr{E}^{\mathrm{u}}(L_P) \cap v(\widehat{a} - H), \quad \gamma > v\mathfrak{m}.$$

Lemma 4.4.4(i) gives $|\mathscr{E}^{\mathrm{u}}(L_P)| \leqslant r$, so we can take

$$\gamma \ := \ \max \mathscr{E}^{\mathrm{u}}(L_P) \cap v(\widehat{a} - H),$$

and then $\gamma > v\mathfrak{m}$. Take a and $\mathfrak{n}$ with $v(\widehat{a} - a) > \gamma = v(\mathfrak{n})$; then $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is a refinement of $(P, \mathfrak{m}, \widehat{a})$ and $a \prec \mathfrak{m}$. Let $b \prec \mathfrak{n}$; then $a + b \prec \mathfrak{m}$, so by Lemma 4.4.8,

$$\operatorname{order}(L_{(P_{+a})_{+b}}) \ = \ r, \qquad \mathscr{E}^{\mathrm{u}}(L_{(P_{+a})_{+b}}) \ = \ \mathscr{E}^{\mathrm{u}}(L_P).$$

Also $v\big((\widehat{a} - a) - b\big) > \gamma$, hence

$$\mathscr{E}^{\mathrm{u}}\big(L_{(P_{+a})_{+b}}\big) \cap v\big((\widehat{a} - a) - H\big) \ = \ \mathscr{E}^{\mathrm{u}}(L_P) \cap v(\widehat{a} - H) \ \leqslant \ \gamma \ < \ v\big((\widehat{a} - a) - b\big).$$

Thus $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is ultimate. □

Remark 4.4.15. Proposition 4.4.14 goes through if instead of assuming that $(P, \mathfrak{m}, \widehat{a})$ is normal, we assume that $(P, \mathfrak{m}, \widehat{a})$ is linear. (Same argument, using Lemma 4.4.13 in place of Lemma 4.4.12.)

Finally, here is a consequence of Corollaries 2.5.13, 4.4.2, and Lemma 4.4.12, where we recall that $\operatorname{order}(L_{P_{\times \mathfrak{m}}}) = \operatorname{order}(L_P \mathfrak{m}) = \operatorname{order}(L_P)$:

Corollary 4.4.16. *Suppose* $\mathrm{I}(K) \subseteq K^\dagger$ *and* $(P, \mathfrak{m}, \widehat{a})$ *is normal of order* $r = 1$. *Then* $L_P = f(\partial - g)$ *with* $f \in H^\times$, $g \in H$, *and for* $\mathfrak{g} \in H^\times$ *with* $\mathfrak{g}^\dagger = g$ *we have:*

$$(P, \mathfrak{m}, \widehat{a}) \text{ is ultimate} \quad\Longleftrightarrow\quad (P, \mathfrak{m}, \widehat{a}) \text{ is isolated} \quad\Longleftrightarrow\quad .\mathfrak{g} \succcurlyeq \mathfrak{m} \text{ or } \mathfrak{g} \prec \widehat{a} - H.$$

(*In particular, if* $g \in \mathrm{I}(H)$ *and* $\mathfrak{m} \preccurlyeq 1$, *then* $(P, \mathfrak{m}, \widehat{a})$ *is ultimate.*)

Ultimate slots in K**.** *In this subsection,* a, b *range over* $K = H[\mathrm{i}]$. Also $(P, \mathfrak{m}, \widehat{a})$ is a slot in K of order $r \geqslant 1$, where $\widehat{a} \in \widehat{K} \setminus K$. Lemma 4.4.8 goes through in this setting, with H in the proof replaced by K:

Lemma 4.4.17. *Suppose* $(P, \mathfrak{m}, \widehat{a})$ *is normal, and* $a \prec \mathfrak{m}$. *Then* L_P *and* $L_{P_{+a}}$ *have order* r, *and if* $\mathrm{I}(K) \subseteq K^\dagger$, *then* $\mathscr{E}^{\mathrm{u}}(L_P) = \mathscr{E}^{\mathrm{u}}(L_{P_{+a}})$.

We adapt Definition 4.4.9 to slots in K: call $(P, \mathfrak{m}, \widehat{a})$ **ultimate** if for all $a \prec \mathfrak{m}$ we have $\operatorname{order}(L_{P_{+a}}) = r$ and $\mathscr{E}^{\mathrm{u}}(L_{P_{+a}}) \cap v(\widehat{a} - K) < v(\widehat{a} - a)$. If $(P, \mathfrak{m}, \widehat{a})$ is ultimate, then it is isolated. Moreover, if $(P, \mathfrak{m}, \widehat{a})$ is ultimate, then so is $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$ as well as the slot $(P^\phi, \mathfrak{m}, \widehat{a})$ in K^ϕ. Lemmas 4.4.10 and 4.4.11 go through in the present context, and so do Lemmas 4.4.12 and 4.4.13 with H replaced by K. The analogue of Proposition 4.4.14 follows likewise:

Proposition 4.4.18. *If* $\mathrm{I}(K) \subseteq K^\dagger$ *and* $(P, \mathfrak{m}, \widehat{a})$ *is normal, then* $(P, \mathfrak{m}, \widehat{a})$ *has an ultimate refinement.*

Remark 4.4.19. Proposition 4.4.18 also holds if instead of assuming that $(P, \mathfrak{m}, \widehat{a})$ is normal, we assume that $(P, \mathfrak{m}, \widehat{a})$ is linear.

Corollary 4.4.2 and the K-versions of Lemmas 4.4.12 and 4.4.13 yield:

Corollary 4.4.20. *Suppose* $\mathrm{I}(K) \subseteq K^\dagger$, $r = 1$, *and* $(P, \mathfrak{m}, \widehat{a})$ *is normal or linear. Then* $L_P = f(\partial - g)$ *with* $f \in K^\times, g \in K$, *and for* $\mathfrak{g} \in H^\times$ *with* $\mathfrak{g}^\dagger = \operatorname{Re} g$ *we have:*

$$(P, \mathfrak{m}, \widehat{a}) \text{ is ultimate} \quad\Longleftrightarrow\quad \mathfrak{g} \succcurlyeq \mathfrak{m} \text{ or } \mathfrak{g} \prec \widehat{a} - K.$$

(*In particular, if* $\operatorname{Re} g \in \mathrm{I}(H)$ *and* $\mathfrak{m} \preccurlyeq 1$, *then* $(P, \mathfrak{m}, \widehat{a})$ *is ultimate.*)

Using the norm to characterize being ultimate. We use here the "norm" $\|\cdot\|$ on U and the gaussian extension v_{g} of the valuation of K from Section 2.1.

Lemma 4.4.21. *For* $u \in \mathrm{U}^\times$ *we have* $\|u\|^\dagger = \operatorname{Re} u^\dagger$.

Proof. For $u = f\,\mathrm{e}(\lambda)$, $f \in K^\times$ we have $\|u\| = |f|$ and $u^\dagger = f^\dagger + \lambda$, so

$$\|u\|^\dagger \;=\; |f|^\dagger \;=\; \operatorname{Re} f^\dagger \;=\; \operatorname{Re} u^\dagger,$$

using Corollary 1.2.5 for the second equality. □

Using Corollary 2.1.10, Lemma 4.4.21, and [ADH, 10.5.2(i)] we obtain:

Lemma 4.4.22. *Let* $\mathfrak{W} \subseteq H^\times$ *be* $\preccurlyeq$*-closed. Then for all* $u \in \mathrm{U}^\times$,

$$\|u\| \in \mathfrak{W} \quad\Longleftrightarrow\quad v_{\mathrm{g}} u \in v(\mathfrak{W}) \quad\Longleftrightarrow\quad \operatorname{Re} u^\dagger < \mathfrak{n}^\dagger \text{ for all } \mathfrak{n} \notin \mathfrak{W}.$$

Let $(P, \mathfrak{m}, \widehat{a})$ be a slot in H of order $r \geqslant 1$. Applying Lemma 4.4.22 to the set $\mathfrak{W} = \{\mathfrak{w} : \mathfrak{w} \prec \widehat{a} - H\}$— so $v(\mathfrak{W}) = \Gamma \setminus v(\widehat{a} - H)$—we obtain a reformulation of the condition "$(P, \mathfrak{m}, \widehat{a})$ is ultimate" in terms of the "norm" $\|\cdot\|$ on U:

Corollary 4.4.23. *The following are equivalent* (*with a ranging over H*)*:*

(i) $(P, \mathfrak{m}, \widehat{a})$ *is ultimate;*
(ii) *for all $a \prec \mathfrak{m}$:* $\operatorname{order}(L_{P_{+a}}) = r$ *and whenever $u \in \mathrm{U}^\times$, $v_{\mathrm{g}} u \in \mathscr{E}^{\mathrm{u}}(L_{P_{+a}})$, and $\|u\| \preccurlyeq \widehat{a} - a$, then $\|u\| \prec \widehat{a} - H$;*
(iii) *for all $a \prec \mathfrak{m}$:* $\operatorname{order}(L_{P_{+a}}) = r$ *and whenever $u \in \mathrm{U}^\times$, $v_{\mathrm{g}} u \in \mathscr{E}^{\mathrm{u}}(L_{P_{+a}})$, and $\|u\| \preccurlyeq \widehat{a} - a$, then* $\operatorname{Re} u^\dagger < \mathfrak{n}^\dagger$ *for all $\mathfrak{n}$ with $v(\mathfrak{n}) \in v(\widehat{a} - H)$.*

A counterexample. Suppose $\mathrm{I}(K) \subseteq K^\dagger$ and H is not ω-free. (Example 1.3.16 provides such H.) Let (λ_ρ) and (ω_ρ) be as in Lemma 3.2.10 with H in the role of K there. That lemma yields a minimal hole $(P, \mathfrak{m}, \lambda)$ in H with $P = 2Y' + Y^2 + \omega$ ($\omega \in H$). This is a good source of counterexamples:

Lemma 4.4.24. *The minimal hole $(P, \mathfrak{m}, \lambda)$ in H is ultimate, and none of its refinements is quasilinear or normal.*

Proof. Let $a \in H$. Then $P_{+a} = 2Y' + 2aY + Y^2 + P(a)$ and thus $L_{P_{+a}} = 2(\partial + a)$, so for $b \in H^\times$ with $b^\dagger = -a$ we have $\mathscr{E}^{\mathrm{u}}(L_{P_{+a}}) = \{vb\}$, by Corollary 4.4.2. Thus $(P, \mathfrak{m}, \lambda)$ is ultimate iff $\lambda - a \prec b$ for all $a \prec \mathfrak{m}$ in H and $b \in H^\times$ with $b^\dagger = -a$ and $vb \in v(\lambda - H)$; the latter holds by [ADH, 11.5.6] since $v(\lambda - H) = \Psi$. Hence $(P, \mathfrak{m}, \lambda)$ is ultimate. No refinement of $(P, \mathfrak{m}, \lambda)$ is quasilinear by Corollary 3.2.25 and [ADH, 11.7.9], and so by Corollary 3.3.21, no refinement of $(P, \mathfrak{m}, \lambda)$ is normal. $\square$

4.5. Repulsive-Normal Slots

In this section H is a real closed H-field with small derivation and asymptotic integration, with $\Gamma := v(H^\times)$. Also $K := H[\mathrm{i}]$ with $\mathrm{i}^2 = -1$ is an algebraic closure of H. We study here the concept of a repulsive-normal slot in H, which strengthens that of a split-normal slot in H. Despite their name, repulsive-normal slots will turn out to have attractive analytic properties in the realm of Hardy fields.

Attraction and repulsion. In this subsection a, b range over H, $\mathfrak{m}$, $\mathfrak{n}$ over $H^\times$, f, g, h (possibly with subscripts) over K, and γ, δ over Γ. We say that f is **attractive** if $\operatorname{Re} f \succcurlyeq 1$ and $\operatorname{Re} f < 0$, and **repulsive** if $\operatorname{Re} f \succcurlyeq 1$ and $\operatorname{Re} f > 0$. If $\operatorname{Re} f \sim \operatorname{Re} g$, then f is attractive iff g is attractive, and likewise with "repulsive" in place of "attractive". Moreover, if $a > 0$, $a \succcurlyeq 1$, and f is attractive (repulsive), then af is attractive (repulsive, respectively).

Definition 4.5.1. Let $\gamma > 0$; we say f is **γ-repulsive** if $v(\operatorname{Re} f) < \gamma^\dagger$ or $\operatorname{Re} f > 0$. Given $S \subseteq \Gamma$, we say f is **S-repulsive** if f is γ-repulsive for all $\gamma \in S \cap \Gamma^>$, equivalently, $\operatorname{Re} f > 0$, or $v(\operatorname{Re} f) < \gamma^\dagger$ for all $\gamma \in S \cap \Gamma^>$.

Note the following implications for $\gamma > 0$:

$$\begin{aligned} f \text{ is } \gamma\text{-repulsive} \quad &\Longrightarrow \quad \operatorname{Re} f \neq 0, \\ f \text{ is } \gamma\text{-repulsive}, \operatorname{Re} g \sim \operatorname{Re} f \quad &\Longrightarrow \quad g \text{ is } \gamma\text{-repulsive}. \end{aligned}$$

The following is easy to show:

Lemma 4.5.2. *Suppose $\gamma > 0$ and $\operatorname{Re} f \succcurlyeq 1$. Then f is γ-repulsive iff $v(\operatorname{Re} f) < \gamma^\dagger$ or f is repulsive. Hence, if f is repulsive, then f is Γ-repulsive; the converse of this implication holds if Ψ is not bounded from below in Γ.*

Let $\gamma, \delta > 0$. If f is γ-repulsive and $a > 0$, $a \succcurlyeq 1$, then af is γ-repulsive. If f is γ-repulsive and δ-repulsive, then f is $(\gamma+\delta)$-repulsive. If f is γ-repulsive and $\gamma > \delta$, then f is $(\gamma - \delta)$-repulsive. Moreover:

Lemma 4.5.3. *Suppose $\gamma \geqslant \delta = v\mathfrak{n} > 0$. Set $g := f - \mathfrak{n}^\dagger$. Then:*

$$f \text{ is } \gamma\text{-repulsive} \iff f \text{ is } \delta\text{-repulsive and } g \text{ is } \gamma\text{-repulsive.}$$

Proof. Note that $\gamma \geqslant \delta > 0$ gives $\gamma^\dagger \leqslant \delta^\dagger$. Suppose f is γ-repulsive; by our remark, f is δ-repulsive. Now if $v(\operatorname{Re} f) < \gamma^\dagger$, then $\operatorname{Re} g \sim \operatorname{Re} f$, whereas if $\operatorname{Re} f > 0$, then $\operatorname{Re} g = \operatorname{Re} f - \mathfrak{n}^\dagger > \operatorname{Re} f > 0$; in both cases, g is γ-repulsive. Conversely, suppose f is δ-repulsive and g is γ-repulsive. If $\operatorname{Re} f > 0$, then clearly f is γ-repulsive. Otherwise, $v(\operatorname{Re} f) < \delta^\dagger$, hence $\operatorname{Re} g \sim \operatorname{Re} f$, so f is also γ-repulsive. □

In a similar way we deduce a useful characterization of repulsiveness:

Lemma 4.5.4. *Suppose $\gamma = v\mathfrak{m} > 0$. Set $g := f - \mathfrak{m}^\dagger$. Then:*

$$f \text{ is repulsive} \iff \operatorname{Re} f \succcurlyeq 1,\ f \text{ is } \gamma\text{-repulsive, and } g \text{ is repulsive.}$$

Proof. Suppose f is repulsive; then by Lemma 4.5.2, f is γ-repulsive. Moreover, $\operatorname{Re} g = \operatorname{Re} f - \mathfrak{m}^\dagger > \operatorname{Re} f > 0$, hence $\operatorname{Re} g \succcurlyeq 1$ and $\operatorname{Re} g > 0$, that is, g is repulsive. Conversely, suppose $\operatorname{Re} f \succcurlyeq 1$, f is γ-repulsive, and g is repulsive. If $v(\operatorname{Re} f) < \gamma^\dagger$, then $\operatorname{Re} f \sim \operatorname{Re} g$; otherwise $\operatorname{Re} f > 0$. In both cases, f is repulsive. □

Corollary 4.5.5. *Suppose f is γ-repulsive where $\gamma = v\mathfrak{m} > 0$, and $\operatorname{Re} f \succcurlyeq 1$. Then f is repulsive iff $f - \mathfrak{m}^\dagger$ is repulsive, and f is attractive iff $f - \mathfrak{m}^\dagger$ is attractive.*

Proof. The first equivalence is immediate from Lemma 4.5.4; this equivalence yields

$$\begin{aligned} f \text{ is attractive} &\iff f \text{ is not repulsive} \iff f - \mathfrak{m}^\dagger \text{ is not repulsive} \\ &\iff \operatorname{Re} f - \mathfrak{m}^\dagger \prec 1 \text{ or } f - \mathfrak{m}^\dagger \text{ is attractive.} \end{aligned}$$

Thus if $f - \mathfrak{m}^\dagger$ is attractive, so is f. Now assume towards a contradiction that f is attractive and $f - \mathfrak{m}^\dagger$ is not. Then $\operatorname{Re} f < 0$ and $\operatorname{Re} f - \mathfrak{m}^\dagger \prec 1$ by the above equivalence, so $\operatorname{Re} f \sim \mathfrak{m}^\dagger$ thanks to $\operatorname{Re} f \succcurlyeq 1$. But f is γ-repulsive, that is, $\operatorname{Re} f \succ \mathfrak{m}^\dagger$ or $\operatorname{Re} f > 0$, a contradiction. □

Lemma 4.5.6. *Suppose $\gamma = v\mathfrak{m} > 0$ and $v(\operatorname{Re} g) \geqslant \gamma^\dagger$. Then for all sufficiently large $c \in C^>$ we have $\operatorname{Re} g - c\mathfrak{m}^\dagger > 0$ (and hence $g - c\mathfrak{m}^\dagger$ is Γ-repulsive).*

Proof. If $v(\operatorname{Re} g) > \gamma^\dagger$, then $\operatorname{Re} g - c\mathfrak{m}^\dagger \sim -c\mathfrak{m}^\dagger > 0$ for all $c \in C^>$. Suppose $v(\operatorname{Re} g) = \gamma^\dagger$. Take $c_0 \in C^\times$ with $\operatorname{Re} g \sim c_0\mathfrak{m}^\dagger$; then $\operatorname{Re} g - c\mathfrak{m}^\dagger > 0$ for $c > c_0$. □

In the rest of this subsection we assume that $S \subseteq \Gamma$. If f is S-repulsive, then so is af for $a > 0$, $a \succcurlyeq 1$. If $S > 0$, $\delta > 0$, and f is S-repulsive and δ-repulsive, then f is $(S + \delta)$-repulsive.

Lemma 4.5.7. *Suppose f is S-repulsive and $0 < \delta = v\mathfrak{n} \in S$. Then*

(i) *f is $(S - \delta)$-repulsive;*
(ii) *$g := f - \mathfrak{n}^\dagger$ is S-repulsive.*

Proof. Let $\gamma \in (S - \delta)$, $\gamma > 0$. Then $\gamma + \delta \in S$, so f is $(\gamma + \delta)$-repulsive, hence γ-repulsive. This shows (i). For (ii), suppose $\gamma \in S$, $\gamma > 0$; we need to show that g is γ-repulsive. If $\gamma \geqslant \delta$, then g is γ-repulsive by Lemma 4.5.3. Taking $\gamma = \delta$ we see that g is δ-repulsive, hence if $\gamma < \delta$, then g is also γ-repulsive. □

Let $A \in K[\partial]^{\neq}$ have order $r \geqslant 1$. An **S-repulsive splitting** of A over K is a splitting $(g_1, \dots, g_r)$ of A over K where $g_1, \dots, g_r$ are S-repulsive. An S-repulsive splitting of A over K remains an S-repulsive splitting of hA over K for $h \neq 0$. We say that A **splits S-repulsively** over K if there is an S-repulsive splitting of A over K. From Lemmas 1.1.1 and 4.5.7 we obtain:

Lemma 4.5.8. *Suppose $(g_1, \dots, g_r)$ is an S-repulsive splitting of A over K and $0 < \delta = v\mathfrak{n} \in S$. Then $(g_1, \dots, g_r)$ is an $(S-\delta)$-repulsive splitting of A over K, and $(h_1, \dots, h_r) := (g_1 - \mathfrak{n}^\dagger, \dots, g_r - \mathfrak{n}^\dagger)$ is an S-repulsive splitting of $A\mathfrak{n}$ over K. (Hence $(h_1, \dots, h_r)$ is also an $(S - \delta)$-repulsive splitting of $A\mathfrak{n}$ over K.)*

Note that if ϕ is active in H with $0 < \phi \preccurlyeq 1$, and f is γ-repulsive (in K), then $\phi^{-1} f$ is γ-repulsive in $K^\phi = H^\phi[i]$.

Lemma 4.5.9. *Suppose $(g_1, \dots, g_r)$ is an S-repulsive splitting of A over K and $S \cap \Gamma^> \not\subseteq \Gamma^\flat$. Let ϕ be active in H with $0 < \phi \prec 1$, and set $h_j := g_j - (r-j)\phi^\dagger$ for $j = 1, \dots, r$. Then $(\phi^{-1}h_1, \dots, \phi^{-1}h_r)$ is an S-repulsive splitting of A^ϕ over K^ϕ.*

Proof. By Lemma 1.1.2, $(\phi^{-1}h_1, \dots, \phi^{-1}h_r)$ is splitting of A^ϕ over K^ϕ. Let $j \in \{1, \dots, r\}$. If $\operatorname{Re} g_j > 0$, then $\phi^\dagger < 0$ yields $\operatorname{Re} h_j \geqslant \operatorname{Re} g_j > 0$. Otherwise, $v(\operatorname{Re} g_j) < \gamma^\dagger$ whenever $0 < \gamma \in S$; in particular, $\operatorname{Re} g_j \succ 1 \succ \phi^\dagger$, so $\operatorname{Re} h_j \sim \operatorname{Re} g_j$. In both cases h_j is S-repulsive, so $\phi^{-1}h_j$ is S-repulsive in K^ϕ. □

Proposition 4.5.10. *Suppose $S \cap \Gamma^> \neq \emptyset$, $nS \subseteq S$ for all $n \geqslant 1$, the ordered constant field C of H is archimedean, and $(g_1, \dots, g_r)$ is a splitting of A over K. Then there exists $\gamma \in S \cap \Gamma^>$ such that for any $\mathfrak{m}$ with $\gamma = v\mathfrak{m}$: $(g_1 - n\mathfrak{m}^\dagger, \dots, g_r - n\mathfrak{m}^\dagger)$ is an S-repulsive splitting of $A\mathfrak{m}^n$ over K, for all big enough n.*

Proof. Let J be the set of $j \in \{1, \dots, r\}$ such that g_j is not S-repulsive. If $\gamma > 0$ and g is not γ-repulsive, then g is not δ-repulsive, for all $\delta \geqslant \gamma$. Hence we can take $\gamma \in S \cap \Gamma^>$ such that g_j is not γ-repulsive, for all $j \in J$. Suppose $\gamma = v\mathfrak{m}$. Lemma 4.5.6 yields $m \geqslant 1$ such that for all $n \geqslant m$, setting $\mathfrak{n} := \mathfrak{m}^n$, $g_j - \mathfrak{n}^\dagger$ is Γ-repulsive for all $j \in J$. For such $\mathfrak{n}$ we have $v\mathfrak{n} \in S$, so by Lemma 4.5.7(ii), $g_j - \mathfrak{n}^\dagger$ is also S-repulsive for $j \notin J$. □

Corollary 4.5.11. *If C is archimedean and $(g_1, \dots, g_r)$ is a splitting of A over K, then there exists $\gamma > 0$ such that for all $\mathfrak{m}$ with $\gamma = v\mathfrak{m}$: $(g_1 - n\mathfrak{m}^\dagger, \dots, g_r - n\mathfrak{m}^\dagger)$ is a Γ-repulsive splitting of $A\mathfrak{m}^n$ over K, for all big enough n. If $\Gamma \neq \Gamma^\flat$ then we can choose such $\gamma > \Gamma^\flat$.*

Proof. Taking $S = \Gamma$ this follows from Proposition 4.5.10 and its proof. □

In logical jargon, the condition that C is archimedean is not *first-order*. But it is satisfied when H is a Hardy field, the case where the results of this section will be applied. For other possible uses we indicate here a first-order variant of Proposition 4.5.10 with essentially the same proof:

Corollary 4.5.12. *Suppose $(g_1, \dots, g_r)$ is a splitting of A over K. Then there exists $\mathfrak{m} \prec 1$ such that for all sufficiently large $c \in C^>$ and all $\mathfrak{n}$, if $\mathfrak{n}^\dagger = c\mathfrak{m}^\dagger$, then $(g_1 - \mathfrak{n}^\dagger, \dots, g_r - \mathfrak{n}^\dagger)$ is a Γ-repulsive splitting of $A\mathfrak{n}$ over K.*

In connection with this corollary we recall from [1, p. 105] that H is said to be *closed under powers* if for all $c \in C$ and $\mathfrak{m}$ there is an $\mathfrak{n}$ with $c\mathfrak{m}^\dagger = \mathfrak{n}^\dagger$.

In the rest of this section $\widehat{H}$ is an immediate asymptotic extension of H and i with $\mathrm{i}^2 = -1$ lies in an asymptotic extension of $\widehat{H}$. Also $K := H[\mathrm{i}]$ and $\widehat{K} := \widehat{H}[\mathrm{i}]$.

Let $\widehat{a} \in \widehat{H} \setminus H$, so $v(\widehat{a} - H)$ is a downward closed subset of Γ. We say that f is **$\widehat{a}$-repulsive** if f is $v(\widehat{a} - H)$-repulsive; that is, $\operatorname{Re} f > 0$, or $\operatorname{Re} f \succ \mathfrak{m}^\dagger$ for all a, $\mathfrak{m}$ with $\mathfrak{m} \asymp \widehat{a} - a \prec 1$. (Of course, this notion is only interesting if $v(\widehat{a} - H) \cap \Gamma^> \neq \emptyset$, since otherwise every f is $\widehat{a}$-repulsive.) Various earlier results give:

Lemma 4.5.13. *Suppose f is $\widehat{a}$-repulsive. Then*

(i) $b > 0,\ b \succcurlyeq 1 \implies bf$ *is $\widehat{a}$-repulsive;*
(ii) f *is $(\widehat{a} - a)$-repulsive;*
(iii) $\mathfrak{m} \asymp 1 \implies f$ *is $\widehat{a}\mathfrak{m}$-repulsive;*
(iv) $\mathfrak{n} \asymp \widehat{a} - a \prec 1 \implies f$ *is $\widehat{a}/\mathfrak{n}$ repulsive and $f - \mathfrak{n}^\dagger$ is $\widehat{a}$-repulsive.*

For (iv), use Lemma 4.5.7. An **$\widehat{a}$-repulsive splitting** of A over K is a $v(\widehat{a} - H)$-repulsive splitting $(g_1, \dots, g_r)$ of A over K:

$$A \ = \ f(\partial - g_1) \cdots (\partial - g_r) \qquad \text{where } f \neq 0 \text{ and } g_1, \dots, g_r \text{ are } \widehat{a}\text{-repulsive.}$$

We say that A **splits $\widehat{a}$-repulsively** over K if it splits $v(\widehat{a} - H)$-repulsively over K. Thus if A splits $\widehat{a}$-repulsively over K, then so does hA ($h \neq 0$), and A splits $(\widehat{a} - a)$-repulsively over K, and splits $\widehat{a}\mathfrak{m}$-repulsively over K for $\mathfrak{m} \asymp 1$. Moreover, from Lemma 4.5.8 we obtain:

Corollary 4.5.14. *Suppose $(g_1, \dots, g_r)$ is an $\widehat{a}$-repulsive splitting of A over K and $\mathfrak{n} \asymp \widehat{a} - a \prec 1$. Then $(g_1, \dots, g_r)$ is an $\widehat{a}/\mathfrak{n}$-repulsive splitting of A over K and $(g_1 - \mathfrak{n}^\dagger, \dots, g_r - \mathfrak{n}^\dagger)$ is an $\widehat{a}$-repulsive splitting of $A\mathfrak{n}$ over K.*

Taking $S := v(\widehat{a} - H)$ in Proposition 4.5.10 we obtain:

Corollary 4.5.15. *If $\widehat{a} \preccurlyeq 1$ is special over H, C is archimedean, and A splits over K, then $A\mathfrak{n}$ splits $\widehat{a}$-repulsively over K for some a and $\mathfrak{n} \asymp \widehat{a} - a \prec 1$.*

Recall that in Section 4.2 we defined a splitting $(g_1, \dots, g_r)$ of A over K to be *strong* if $\operatorname{Re} g_j \succcurlyeq \mathfrak{v}(A)^\dagger$ for $j = 1, \dots, r$.

Lemma 4.5.16. *Suppose $\widehat{a} - a \prec^\flat 1$ for some a. Let $(g_1, \dots, g_r)$ be an $\widehat{a}$-repulsive splitting of A over K, let ϕ be active in H with $0 < \phi \prec 1$, and set*

$$h_j \ := \ \phi^{-1}\big(g_j - (r-j)\phi^\dagger\big) \qquad (j = 1, \dots, r).$$

Then $(h_1, \dots, h_r)$ is an $\widehat{a}$-repulsive splitting of A^ϕ over $K^\phi = H^\phi[\mathrm{i}]$. If $\mathfrak{v}(A) \prec^\flat 1$ and $(g_1, \dots, g_r)$ is strong, then $(h_1, \dots, h_r)$ is strong.

This follows from Lemmas 4.2.12 and 4.5.9.

Lemma 4.5.17. *Suppose $\mathfrak{v} := \mathfrak{v}(A) \prec 1$ and $\widehat{a} \prec_{\Delta(\mathfrak{v})} 1$. Let $(g_1, \dots, g_r)$ be an $\widehat{a}$-repulsive splitting of A over K. Then for all sufficiently small $q \in \mathbb{Q}^>$ and any $\mathfrak{n} \asymp |\mathfrak{v}|^q$, $(g_1 - \mathfrak{n}^\dagger, \dots, g_r - \mathfrak{n}^\dagger)$ is a strong $\widehat{a}/\mathfrak{n}$-repulsive splitting of $A\mathfrak{n}$ over K.*

Proof. Take $q_0 \in \mathbb{Q}^>$ with $\widehat{a} \prec |\mathfrak{v}|^{q_0} \prec 1$. Then for any $q \in \mathbb{Q}$ with $0 < q \leqslant q_0$ and any $\mathfrak{n} \asymp |\mathfrak{v}|^q$, $(g_1 - \mathfrak{n}^\dagger, \dots, g_r - \mathfrak{n}^\dagger)$ is an $\widehat{a}/\mathfrak{n}$-repulsive splitting of $A\mathfrak{n}$ over K, by Corollary 4.5.14. Using Lemmas 4.2.13 and 4.2.10 (in that order) we can decrease q_0 so that for all $q \in \mathbb{Q}$ with $0 < q \leqslant q_0$ and $\mathfrak{n} \asymp |\mathfrak{v}|^q$, $(g_1 - \mathfrak{n}^\dagger, \dots, g_r - \mathfrak{n}^\dagger)$ is also a strong splitting of $A\mathfrak{n}$ over K. $\square$

In the rest of this subsection we assume that H is Liouville closed with $\mathrm{I}(K) \subseteq K^\dagger$. We choose a complement $\Lambda \subseteq Hi$ of $K^\dagger$ in K as in Section 4.4 and set $\mathrm{U} := K\big[\mathrm{e}(\Lambda)\big]$. We then have the set $\mathscr{E}^{\mathrm{u}}(A) \subseteq \Gamma$ of ultimate exceptional values of A (which doesn't depend on Λ by Corollary 4.4.1). Recall from Corollary 1.2.28 that H is of Hardy type iff C is archimedean. *We now assume $r = 1$ and $\widehat{a} \prec 1$ is special over H, and let Δ be the nontrivial convex subgroup of Γ that is cofinal in $v(\widehat{a} - H)$.*

Lemma 4.5.18. *Suppose C is archimedean and $\mathscr{E}^{\mathrm{u}}(A) \cap v(\widehat{a} - H) < 0$. Then A splits $\widehat{a}$-repulsively over K.*

Proof. We may arrange $A = \partial - f$. Take $u \in \mathrm{U}^\times$ with $u^\dagger = f$, and $b := \|u\| \in H^>$. Then $\mathscr{E}^{\mathrm{u}}(A) = \{vb\}$ by Lemma 2.5.12 and its proof, hence

$$\mathscr{E}^{\mathrm{u}}(A) \cap v(\widehat{a} - H) < 0 \quad\Longleftrightarrow\quad b \succ 1 \text{ or } vb > \Delta,$$

and $\operatorname{Re} f = b^\dagger$ by Lemma 4.4.21. If $b \succ 1$, then $\operatorname{Re} f > 0$, and if $vb > \Delta$, then for all $\delta \in \Delta^{\neq}$ we have $\psi(vb) < \psi(\delta)$ by Lemma 1.2.24, so $\operatorname{Re} f \succ \mathfrak{m}^\dagger$ for all a, $\mathfrak{m}$ with $\widehat{a} - a \asymp \mathfrak{m} \prec 1$. In both cases A splits $\widehat{a}$-repulsively over K. $\square$

Lemma 4.5.19. *Suppose $A \in H[\partial]$ and $\mathfrak{v}(A) \prec 1$. Then $0 \notin \mathscr{E}^{\mathrm{u}}(A)$, and if A splits $\widehat{a}$-repulsively over K, then $\mathscr{E}^{\mathrm{u}}(A) \cap v(\widehat{a} - H) < 0$.*

Proof. We again arrange $A = \partial - f$ and take u, b as in the proof of Lemma 4.5.18. Then $f \in H$ and $b^\dagger = f = -1/\mathfrak{v}(A) \succ 1$, so $b \not\asymp 1$, and thus $0 \notin \{vb\} = \mathscr{E}^{\mathrm{u}}(A)$. Now suppose A splits $\widehat{a}$-repulsively over K, that is, $f > 0$ or $f \succ \mathfrak{m}^\dagger$ for all a, $\mathfrak{m}$ with $\widehat{a} - a \asymp \mathfrak{m} \prec 1$. In the first case $f = b^\dagger$ and $b \not\asymp 1$ yield $b \succ 1$. In the second case $\psi(vb) = vf < \psi(\delta)$ for all $\delta \in \Delta^{\neq}$, hence $vb \notin \Delta$. $\square$

Combining Lemma 4.2.11 with the previous two lemmas yields:

Corollary 4.5.20. *Suppose $A \in H[\partial]$ and $\mathfrak{v}(A) \prec 1$, and H is of Hardy type. Then A splits strongly over K, and we have the equivalence*

$$A \text{ splits } \widehat{a}\text{-repulsively over } K \iff \mathscr{E}^{\mathrm{u}}(A) \cap v(\widehat{a} - H) \leqslant 0.$$

Defining repulsive-normality. *In this subsection $(P, \mathfrak{m}, \widehat{a})$ is a slot in H of order $r \geqslant 1$ with $\widehat{a} \in \widehat{H} \setminus H$ and linear part $L := L_{P_{\times\mathfrak{m}}}$.* Set $w := \operatorname{wt}(P)$; if order $L = r$, set $\mathfrak{v} := \mathfrak{v}(L)$. We let a, b range over H and $\mathfrak{n}$ over $H^\times$.

Definition 4.5.21. Call $(P, \mathfrak{m}, \widehat{a})$ **repulsive-normal** if order $L = r$, and

(RN1) $\mathfrak{v} \prec^\flat 1$;
(RN2) $(P_{\times\mathfrak{m}})_{\geqslant 1} = Q + R$ where $Q, R \in H\{Y\}$, Q is homogeneous of degree 1 and order r, L_Q splits $\widehat{a}/\mathfrak{m}$-repulsively over K, and $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$.

Compare this with "split-normality" from Definition 4.3.3: clearly repulsive-normal implies split-normal, and hence normal. If $(P, \mathfrak{m}, \widehat{a})$ is normal and L splits $\widehat{a}/\mathfrak{m}$-repulsively over K, then $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal. If $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal, then so are $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$ and $(P_{\times\mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{a}/\mathfrak{n})$.

Lemma 4.5.22. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal and ϕ is active in H such that $0 < \phi \prec 1$, and $\widehat{a} - a \prec^\flat \mathfrak{m}$ for some a. Then the slot $(P^\phi, \mathfrak{m}, \widehat{a})$ in H^ϕ is repulsive-normal.*

Proof. First arrange $\mathfrak{m} = 1$, and let Q, R be as in (RN2) for $\mathfrak{m} = 1$. Now $(P^\phi, 1, \widehat{a})$ is split-normal by Lemma 4.3.5. In fact, $P^\phi_{\geqslant 1} = Q^\phi + R^\phi$, and the proof of this lemma shows that $R^{\phi} \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1} P_1^{\phi}$ where $\mathfrak{w} := \mathfrak{v}(L_{P^{\phi}})$. By Lemma 4.5.16, $L_{Q^{\phi}} = L_Q^{\phi}$ splits $\widehat{a}$-repulsively over K^{ϕ}. So $(P^{\phi}, 1, \widehat{a})$ is repulsive-normal. $\square$

If order $L = r$, $\mathfrak{v} \prec^{\flat} 1$, and $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$, then $\widehat{a} - a \prec^{\flat} \mathfrak{m}$. Thus we obtain from Lemmas 3.3.13 and 4.5.22 the following result:

Corollary 4.5.23. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, deep, and repulsive-normal. Let ϕ be active in H with $0 < \phi \prec 1$. Then the slot $(P^{\phi}, \mathfrak{m}, \widehat{a})$ in H^{ϕ} is repulsive-normal.*

Before we turn to the task of obtaining repulsive-normal slots, we deal with the preservation of repulsive-normality under refinements.

Lemma 4.5.24. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal, and let Q, R be as in* (RN2)*. Let $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ be a steep refinement of $(P, \mathfrak{m}, \widehat{a})$ where $\mathfrak{n} \prec \mathfrak{m}$ or $\mathfrak{n} = \mathfrak{m}$. Suppose*

$$(P_{+a,\times\mathfrak{n}})_{\geqslant 1} - Q_{\times\mathfrak{n}/\mathfrak{m}} \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1} (P_{+a,\times\mathfrak{n}})_1 \qquad \text{where } \mathfrak{w} := \mathfrak{v}(L_{P_{+a,\times\mathfrak{n}}}).$$

Then $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is repulsive-normal.

Proof. By (RN2), L_Q splits $\widehat{a}/\mathfrak{m}$-repulsively over K, so L_Q also splits $(\widehat{a} - a)/\mathfrak{m}$-repulsively over K. We have $(\widehat{a} - a)/\mathfrak{m} \prec \mathfrak{n}/\mathfrak{m} \prec 1$ or $(\widehat{a} - a)/\mathfrak{m} \prec 1 = \mathfrak{n}/\mathfrak{m}$, so L_Q splits $(\widehat{a} - a)/\mathfrak{n}$-repulsively over K by the first part of Corollary 4.5.14, and hence $L_{Q_{\times\mathfrak{n}/\mathfrak{m}}} = L_Q \cdot (\mathfrak{n}/\mathfrak{m})$ splits $(\widehat{a} - a)/\mathfrak{n}$-repulsively over K by the second part of that Corollary 4.5.14. Thus $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ is repulsive-normal. $\square$

The proofs of Lemmas 4.3.18, 4.3.19, 4.3.20 give the following repulsive-normal analogues of these lemmas, using also Lemma 4.5.24; for Lemma 4.5.27 below we adopt the notational conventions about $\mathfrak{n}^q$ $(q \in \mathbb{Q}^{>})$ stated before Lemma 4.3.20.

Lemma 4.5.25. *If $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal and $(P_{+a}, \mathfrak{m}, \widehat{a}-a)$ is a refinement of $(P, \mathfrak{m}, \widehat{a})$, then $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ is also repulsive-normal.*

Lemma 4.5.26. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal, $\widehat{a} \prec \mathfrak{n} \prec \mathfrak{m}$, and $[\mathfrak{n}/\mathfrak{m}] \leqslant [\mathfrak{v}]$. Then the refinement $(P, \mathfrak{n}, \widehat{a})$ of $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal: if $\mathfrak{m}$, P, Q, $\mathfrak{v}$ are as in* (RN2)*, then* (RN2) *holds with $\mathfrak{n}$, $Q_{\times\mathfrak{n}/\mathfrak{m}}$, $R_{\times\mathfrak{n}/\mathfrak{m}}$, $\mathfrak{v}(L_{P_{\times\mathfrak{n}}})$ in place of $\mathfrak{m}$, Q, R, $\mathfrak{v}$.*

Lemma 4.5.27. *Suppose $\mathfrak{m} = 1$, $(P, 1, \widehat{a})$ is repulsive-normal, $\widehat{a} \prec \mathfrak{n} \prec 1$, and for $\mathfrak{v} := \mathfrak{v}(L_P)$ we have $[\mathfrak{n}^{\dagger}] < [\mathfrak{v}] < [\mathfrak{n}]$; then $(P, \mathfrak{n}^q, \widehat{a})$ is a repulsive-normal refinement of $(P, 1, \widehat{a})$ for all but finitely many $q \in \mathbb{Q}$ with $0 < q < 1$.*

Achieving repulsive-normality. In this subsection we adopt the setting of the subsection *Achieving split-normality* of Section 4.3: *H is $\upomega$-free and $(P, \mathfrak{m}, \widehat{a})$ is a minimal hole in K of order $r \geqslant 1$, $\mathfrak{m} \in H^{\times}$, and $\widehat{a} \in \widehat{K} \setminus K$, with $\widehat{a} = \widehat{b} + \widehat{c}i$, $\widehat{b}, \widehat{c} \in \widehat{H}$.* We let a range over K, b, c over H, and $\mathfrak{n}$ over $H^{\times}$. We prove here the following variant of Theorem 4.3.9:

Theorem 4.5.28. *Suppose the constant field C of H is archimedean and* $\deg P > 1$. *Then one of the following conditions is satisfied:*

(i) *$\widehat{b} \notin H$ and some Z-minimal slot $(Q, \mathfrak{m}, \widehat{b})$ in H has a special refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ such that $(Q_{+b}^{\phi}, \mathfrak{n}, \widehat{b}-b)$ is eventually deep and repulsive-normal;*

(ii) *$\widehat{c} \notin H$ and some Z-minimal slot $(R, \mathfrak{m}, \widehat{c})$ in H has a special refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R_{+c}^{\phi}, \mathfrak{n}, \widehat{c}-c)$ is eventually deep and repulsive-normal.*

To establish this theorem we need to take up the approximation arguments in the proof of Theorem 4.3.9 once again. While in that proof we treated the cases $\widehat{b} \in H$ and $\widehat{c} \in H$ separately to obtain stronger results in those cases (Lemmas 4.3.10, 4.3.11), here we proceed differently and first show a repulsive-normal version of Proposition 4.3.12 which also applies to those cases. *In the rest of this subsection we assume that C is archimedean.*

Proposition 4.5.29. *Suppose the hole $(P, \mathfrak{m}, \widehat{a})$ in K is special and $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$ (so $\widehat{b} \notin H$). Let $(Q, \mathfrak{m}, \widehat{b})$ be a Z-minimal deep normal slot in H. Then $(Q, \mathfrak{m}, \widehat{b})$ has a repulsive-normal refinement.*

Proof. As in the proof of Proposition 4.3.12 we first arrange $\mathfrak{m} = 1$, and set

$$\Delta \;:=\; \big\{\delta \in \Gamma :\; |\delta| \in v\big(\widehat{a} - K\big)\big\},$$

a convex subgroup of Γ which is cofinal in $v(\widehat{a} - K) = v(\widehat{b} - H)$, so $\widehat{b}$ is special over H. Lemma 3.3.13 applied to $(Q, 1, \widehat{b})$ and $\mathfrak{v}(L_Q) \prec^\flat 1$ gives that $\Gamma^\flat$ is strictly contained in Δ. To show that $(Q, 1, \widehat{b})$ has a repulsive-normal refinement, we follow the proof of Proposition 4.3.12, skipping the initial compositional conjugation, and arranging first that $P, Q \asymp 1$. Recall from that proof that $\dot{\widehat{a}} \in \dot{K}^{\mathrm{c}} = \dot{H}^{\mathrm{c}}[\mathrm{i}]$ and $\operatorname{Re} \dot{\widehat{a}} = \dot{\widehat{b}} \in \dot{H}^{\mathrm{c}} \setminus \dot{H}$, with $\dot{\widehat{b}} \prec 1$, $\dot{Q} \in \dot{H}\{Y\}$, and so $\dot{Q}_{+\dot{\widehat{b}}} \in \dot{H}^{\mathrm{c}}\{Y\}$. Let $A \in \dot{H}^{\mathrm{c}}[\partial]$ be the linear part of $\dot{Q}_{+\dot{\widehat{b}}}$. Recall from that proof that $1 \leqslant s := \operatorname{order} Q = \operatorname{order} A \leqslant 2r$ and that A splits over $\dot{K}^{\mathrm{c}}$. Then Lemma 1.1.4 gives a *real* splitting $(g_1, \dots, g_s)$ of A over $\dot{K}^{\mathrm{c}}$:

$$A \;=\; f(\partial - g_1)\cdots(\partial - g_s), \qquad 0 \neq f \in \dot{H}^{\mathrm{c}},\; g_1, \dots, g_s \in \dot{K}^{\mathrm{c}}.$$

It follows easily from [ADH, 10.1.8] that the real closed d-valued field $\dot{H}$ is an H-field, and so its completion $\dot{H}^{\mathrm{c}}$ is also a real closed H-field by [ADH, 10.5.9]. Recall also that $\Delta = v(\dot{H}^\times)$ is the value group of $\dot{H}^{\mathrm{c}}$ and properly contains $\Gamma^\flat$. Thus we can apply Corollary 4.5.11 with $\dot{H}^{\mathrm{c}}$ in the role of H to get $\mathfrak{n} \in \dot{\mathcal{O}}$ with $0 \neq \dot{\mathfrak{n}} \prec^\flat 1$ and m such that for all $n > m$, $(h_1, \dots, h_s) := (g_1 - n\dot{\mathfrak{n}}^\dagger, \dots, g_s - n\dot{\mathfrak{n}}^\dagger)$ is a Δ-repulsive splitting of $A\dot{\mathfrak{n}}^n$ over $\dot{K}^{\mathrm{c}}$, so $\operatorname{Re} h_1, \dots, \operatorname{Re} h_s \neq 0$. For any n, $A\dot{\mathfrak{n}}^n$ is the linear part of $\dot{Q}_{+\dot{\widehat{b}}, \times\dot{\mathfrak{n}}^n} \in \dot{H}^{\mathrm{c}}\{Y\}$, and $(h_1, \dots, h_s)$ is also a real splitting of $A\dot{\mathfrak{n}}^n$ over $\dot{K}^{\mathrm{c}}$:

$$A\dot{\mathfrak{n}}^n \;=\; \dot{\mathfrak{n}}^n f(\partial - h_1)\cdots(\partial - h_s).$$

By increasing m we arrange that for all $n > m$ we have $g_j \not\sim n\dot{\mathfrak{n}}^\dagger$ ($j = 1, \dots, s$), and also $\mathfrak{v}(A\dot{\mathfrak{n}}^n) \preccurlyeq \mathfrak{v}(A)$ provided $\big[\mathfrak{v}(A)\big] < [\dot{\mathfrak{n}}]$; for the latter part use Lemma 3.1.16. Below we assume $n > m$. Then $\mathfrak{v}(A\dot{\mathfrak{n}}^n) \prec 1$: to see this use Corollary 3.1.4, $\mathfrak{v}(A) \prec 1$, and $g_j \preccurlyeq h_j$ ($j = 1, \dots, s$). Note that $h_1, \dots, h_s \succcurlyeq 1$. We now apply Corollary 4.2.9 to $\dot{H}$, $\dot{K}$, $\dot{Q}$, s, $\dot{\mathfrak{n}}^n$, $\dot{\widehat{b}}$, $\dot{\mathfrak{n}}^n f$, $h_1, \dots, h_s$ in place of H, K, P, r, $\mathfrak{m}$, f, a, $b_1, \dots, b_r$, respectively, and any $\gamma \in \Delta$ with $\gamma > v(\dot{\mathfrak{n}}^n), v(\operatorname{Re} h_1), \dots, v(\operatorname{Re} h_s)$. This gives $a, b \in \dot{\mathcal{O}}$ and $b_1, \dots, b_s \in \dot{\mathcal{O}}_K$ such that $\dot{a}, \dot{b} \neq 0$ in $\dot{H}$ and such that for the linear part $\widetilde{A} \in \dot{H}[\partial]$ of $\dot{Q}_{+\dot{b}, \times\dot{\mathfrak{n}}^n}$ we have

$$\dot{b} - \dot{\widehat{b}} \;\prec\; \dot{\mathfrak{n}}^n, \qquad \widetilde{A} \;\sim\; A\dot{\mathfrak{n}}^n, \qquad \operatorname{order} \widetilde{A} \;=\; s, \qquad \mathfrak{w} \;:=\; \mathfrak{v}(\widetilde{A}) \;\sim\; \mathfrak{v}(A\dot{\mathfrak{n}}^n),$$

and such that for $w := \mathrm{wt}(Q)$ and with $\Delta(\mathfrak{w}) \subseteq \Delta$:

$$\widetilde{A} \ = \ \widetilde{B} + \widetilde{E}, \quad \widetilde{B} \ = \ \dot{a}(\partial - \dot{b}_1)\cdots(\partial - \dot{b}_s) \in \dot{H}[\partial], \quad \widetilde{E} \in \dot{H}[\partial],$$
$$v(\dot{b}_1 - h_1), \dots, v(\dot{b}_s - h_s) \ > \ \gamma, \quad \widetilde{E} \ \prec_{\Delta(\mathfrak{w})} \ \mathfrak{w}^{w+1}\widetilde{A},$$

and $(\dot{b}_1, \dots, \dot{b}_s)$ is a real splitting of $\widetilde{B}$ over $\dot{K}$. This real splitting over $\dot{K}$ has a consequence that will be crucial at the end of the proof: by changing $b_1, \dots, b_s$ if necessary, without changing $\dot{b}_1, \dots, \dot{b}_s$ we arrange that $B := a(\partial - b_1)\cdots(\partial - b_s)$ lies in $\dot{\mathcal{O}}[\partial] \subseteq H[\partial]$ and that $(b_1, \dots, b_s)$ is a real splitting of B over K. (Lemma 1.1.6.)

Since $\operatorname{Re}\dot{b}_1 \sim \operatorname{Re} h_1, \dots, \operatorname{Re}\dot{b}_s \sim \operatorname{Re} h_s$, the implication just before Lemma 4.5.2 gives that $(\dot{b}_1, \dots, \dot{b}_s)$ is a Δ-repulsive splitting of $\widetilde{B}$ over $\dot{K}$. Now $\widehat{b} - b \prec \mathfrak{n}^n \prec 1$, so $(Q_{+b}, 1, \widehat{b}-b)$ is a refinement of the normal slot $(Q, 1, \widehat{b})$ in H, hence $(Q_{+b}, 1, \widehat{b}-b)$ is normal by Proposition 3.3.25. We claim that the refinement $(Q_{+b}, \mathfrak{n}^n, \widehat{b} - b)$ of $(Q_{+b}, 1, \widehat{b} - b)$ is also normal. If $[\mathfrak{n}] \leqslant \big[\mathfrak{v}(L_{Q_{+b}})\big]$, this claim holds by Corollary 3.3.27. From Lemma 3.1.28 and 3.1.7 we obtain:

$$\operatorname{order} L_{Q_{+b}} \ = \ \operatorname{order} L_Q \ = \ \operatorname{order} L_{Q_{+\widehat{b}}} \ = \ s,$$
$$\mathfrak{v}(L_{Q_{+b}}) \ \sim \ \mathfrak{v}(L_Q) \ \sim \ \mathfrak{v}(L_{Q_{+\widehat{b}}}), \qquad v\big(\mathfrak{v}(L_{Q_{+\widehat{b}}})\big) \ = \ v\big(\mathfrak{v}(A)\big),$$

so $v\big(\mathfrak{v}(L_{Q_{+b}})\big) = v\big(\mathfrak{v}(A)\big)$. Moreover, by Lemma 3.1.7 and the facts about $\widetilde{A}$,

$$v\big(\mathfrak{v}(L_{Q_{+b,\times\mathfrak{n}^n}})\big) \ = \ v\big(\mathfrak{v}(\widetilde{A})\big) \ = \ v\big(\mathfrak{v}(A\dot{\mathfrak{n}}^n)\big) \ = \ v(\mathfrak{w}).$$

Suppose $\big[\mathfrak{v}(L_{Q_{+b}})\big] < [\mathfrak{n}]$. Then $[\mathfrak{v}(A)] < [\dot{\mathfrak{n}}]$, so $\mathfrak{v}(A\dot{\mathfrak{n}}^n) \preccurlyeq \mathfrak{v}(A)$ using $n > m$. Now the asymptotic relations among the various $\mathfrak{v}(\dots)$ above give

$$\mathfrak{v}(L_{Q_{+b,\times\mathfrak{n}^n}}) \ \preccurlyeq \ \mathfrak{v}(L_{Q_{+b}}),$$

hence $(Q_{+b}, \mathfrak{n}^n, \widehat{b} - b)$ is normal by Corollary 3.3.29 applied to H and the normal slot $(Q_{+b}, 1, \widehat{b} - b)$ in H in the role of K and $(P, 1, \widehat{a})$, respectively. Put $\mathfrak{v} := \mathfrak{v}(L_{Q_{+b,\times\mathfrak{n}^n}})$, so $\mathfrak{v} \asymp \mathfrak{w}$. Note that $Q_{+b,\times\mathfrak{n}^n} \in \dot{\mathcal{O}}\{Y\}$, so the image of $L_{Q_{+b,\times\mathfrak{n}^n}} \in \dot{\mathcal{O}}[\partial]$ in $\dot{H}[\partial]$ is $\widetilde{A}$. Thus in $H[\partial]$ we have:

$$L_{Q_{+b,\times\mathfrak{n}^n}} \ = \ B + E \qquad \text{where } E \in \dot{\mathcal{O}}[\partial], \ \ E \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1} L_{Q_{+b,\times\mathfrak{n}^n}}.$$

Now $\dot{b}_1, \dots, \dot{b}_s$ are Δ-repulsive, so $b_1, \dots, b_s$ are Δ-repulsive, hence

$$B \ = \ a(\partial - b_1)\cdots(\partial - b_s)$$

splits Δ-repulsively, and thus $(\widehat{b} - b)/\mathfrak{n}^n$-repulsively. Therefore $(Q_{+b}, \mathfrak{n}^n, \widehat{b} - b)$ is repulsive-normal. □

Instead of assuming in the above proposition that $(P, \mathfrak{m}, \widehat{a})$ is special and $(Q, \mathfrak{m}, \widehat{b})$ is deep and normal, we can assume, as with Corollary 4.3.13, that $\deg P > 1$:

Corollary 4.5.30. *Suppose* $\deg P > 1$ *and* $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$. *Let* $Q \in Z(H, \widehat{b})$ *have minimal complexity. Then the Z-minimal slot* $(Q, \mathfrak{m}, \widehat{b})$ *in H has a special refinement* $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ *such that* $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ *is eventually deep and repulsive-normal.*

Proof. The beginning of the subsection *Achieving split-normality* of Section 4.3 and $\deg P > 1$ give that K is r-linearly newtonian. Lemmas 3.2.26 and 3.3.23 yield a quasilinear refinement $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ of our hole $(P, \mathfrak{m}, \widehat{a})$ in K. Set $b := \operatorname{Re} a$. By Lemma 4.1.3 we have

$$v\big((\widehat{a}-a)-K\big) \;=\; v(\widehat{a}-K) \;=\; v\big(\widehat{b}-H\big) \;=\; v\big((\widehat{b}-b)-H\big).$$

Replacing $(P, \mathfrak{m}, \widehat{a})$ and $(Q, \mathfrak{m}, \widehat{b})$ by $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ and $(Q_{+b}, \mathfrak{n}, \widehat{b}-b)$, respectively, we arrange that $(P, \mathfrak{m}, \widehat{a})$ is quasilinear. Then by Proposition 1.5.12 and K being r-linearly newtonian, $(P, \mathfrak{m}, \widehat{a})$ is special; hence so is $(Q, \mathfrak{m}, \widehat{b})$. Proposition 3.3.36 gives a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b}-b)$ of $(Q, \mathfrak{m}, \widehat{b})$ and an active $\phi_0 \in H^{>}$ such that $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b}-b)$ is deep and normal. Refinements of $(P, \mathfrak{m}, \widehat{a})$ remain quasilinear by Corollary 3.2.23. Since $v(\widehat{b}-H) \subseteq v(\widehat{c}-H)$, Lemma 4.1.3(ii) gives a refinement $(P_{+a}, \mathfrak{n}, \widehat{a}-a)$ of $(P, \mathfrak{m}, \widehat{a})$ with $\operatorname{Re} a = b$. By Lemma 3.2.35 the minimal hole $(P^{\phi_0}_{+a}, \mathfrak{n}, \widehat{a}-a)$ in K^{ϕ_0} is special. Proposition 4.5.29 applied to $(P^{\phi_0}_{+a}, \mathfrak{n}, \widehat{a}-a)$, $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b}-b)$ in place of $(P, \mathfrak{m}, \widehat{a})$, $(Q, \mathfrak{m}, \widehat{b})$, respectively, gives us $b_0 \in H$, $\mathfrak{n}_0 \in H^{\times}$ and a repulsive-normal refinement $\big(Q^{\phi_0}_{+(b+b_0)}, \mathfrak{n}_0, \widehat{b}-(b+b_0)\big)$ of $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b}-b)$. This refinement is steep and hence deep by Corollary 3.3.6, since $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b}-b)$ is deep. Thus by Corollary 4.5.23, $\big(Q_{+(b+b_0)}, \mathfrak{n}_0, \widehat{b}-(b+b_0)\big)$ is a refinement of $(Q, \mathfrak{m}, \widehat{b})$ such that that $\big(Q^{\phi}_{+(b+b_0)}, \mathfrak{n}_0, \widehat{b}-(b+b_0)\big)$ is eventually deep and repulsive-normal. As a refinement of $(Q, \mathfrak{m}, \widehat{b})$, it is special. $\square$

In the same way that Corollary 4.3.13 gave rise to Corollary 4.3.14, Corollary 4.5.30 gives rise to the following:

Corollary 4.5.31. *If $\deg P > 1$, $v(\widehat{c}-H) \subseteq v(\widehat{b}-H)$, and $R \in Z(H, \widehat{c})$ has minimal complexity, then the Z-minimal slot $(R, \mathfrak{m}, \widehat{c})$ in H has a special refinement $(R_{+c}, \mathfrak{n}, \widehat{c}-c)$ such that $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c}-c)$ is eventually deep and repulsive-normal.*

By Lemma 4.1.3 we have $v(\widehat{b}-H) \subseteq v(\widehat{c}-H)$ or $v(\widehat{c}-H) \subseteq v(\widehat{b}-H)$, hence the two corollaries above yield Theorem 4.5.28, completing its proof. $\square$

Strengthening repulsive-normality. In this subsection we adopt the setting of the subsection *Strengthening split-normality* of Section 4.3. Thus $(P, \mathfrak{m}, \widehat{a})$ is a slot in H of order $r \geqslant 1$ and weight $w := \operatorname{wt}(P)$, and $L := L_{P_{\times\mathfrak{m}}}$. If $\operatorname{order} L = r$, we set $\mathfrak{v} := \mathfrak{v}(L)$. We let a, b range over H and $\mathfrak{m}$, $\mathfrak{n}$ over $H^{\times}$.

Definition 4.5.32. We say that $(P, \mathfrak{m}, \widehat{a})$ is **almost strongly repulsive-normal** if $\operatorname{order} L = r$, $\mathfrak{v} \prec^{\flat} 1$, and there are $Q, R \in H\{Y\}$ such that

(RN2as) $(P_{\times\mathfrak{m}})_{\geqslant 1} = Q + R$, Q is homogeneous of degree 1 and order r, L_Q has a strong $\widehat{a}/\mathfrak{m}$-repulsive splitting over K, and $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$.

We say that $(P, \mathfrak{m}, \widehat{a})$ is **strongly repulsive-normal** if $\operatorname{order} L = r$, $\mathfrak{v} \prec^{\flat} 1$, and there are $Q, R \in H\{Y\}$ such that:

(RN2s) $P_{\times\mathfrak{m}} = Q + R$, Q is homogeneous of degree 1 and order r, L_Q has a strong $\widehat{a}/\mathfrak{m}$-repulsive splitting over K, and $R \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$.

If $(P, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal, then $(P, \mathfrak{m}, \widehat{a})$ is almost strongly split-normal; likewise without "almost". Thus we can augment our diagram from Section 4.3 as follows, the implications holding for slots of order $\geqslant 1$ in real closed

H-fields with small derivation and asymptotic integration:

$$\begin{tikzcd}
\text{strongly repulsive-normal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{almost strongly repulsive-normal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{repulsive-normal} \arrow[d, Rightarrow] \\
\text{strongly split-normal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{almost strongly split-normal} \arrow[r, Rightarrow] & \text{split-normal} \arrow[d, Rightarrow] \\
\text{strictly normal} \arrow[rr, Rightarrow] & & \text{normal}
\end{tikzcd}$$

Adapting the proof of Lemma 4.3.23 gives:

Lemma 4.5.33. *The following are equivalent:*

(i) $(P, \mathfrak{m}, \widehat{a})$ *is strongly repulsive-normal;*
(ii) $(P, \mathfrak{m}, \widehat{a})$ *is almost strongly repulsive-normal and strictly normal;*
(iii) $(P, \mathfrak{m}, \widehat{a})$ *is almost strongly repulsive-normal and* $P(0) \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_1)_{\times\mathfrak{m}}$.

Corollary 4.5.34. *If L has a strong $\widehat{a}/\mathfrak{m}$-repulsive splitting over K, then:*

$$\begin{aligned}
(P, \mathfrak{m}, \widehat{a}) \text{ is almost strongly repulsive-normal} &\iff (P, \mathfrak{m}, \widehat{a}) \text{ is normal,} \\
(P, \mathfrak{m}, \widehat{a}) \text{ is strongly repulsive-normal} &\iff (P, \mathfrak{m}, \widehat{a}) \text{ is strictly normal.}
\end{aligned}$$

If $(P, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal, then so are $(bP, \mathfrak{m}, \widehat{a})$ for $b \neq 0$ and $(P_{\times\mathfrak{n}}, \mathfrak{m}/\mathfrak{n}, \widehat{a}/\mathfrak{n})$, and likewise with "strongly" in place of "almost strongly". The proof of the next lemma is like that of Lemma 4.3.25, using Lemmas 4.5.25 and 4.5.33 in place of Lemmas 4.3.18 and 4.3.23, respectively.

Lemma 4.5.35. *Suppose $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ refines $(P, \mathfrak{m}, \widehat{a})$. If $(P, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal, then so is $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$. If $(P, \mathfrak{m}, \widehat{a})$ is strongly repulsive-normal, Z-minimal, and $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+w+1}\mathfrak{m}$, then $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ is strongly repulsive-normal.*

Here is the key to achieving almost strong repulsive-normality; its proof is similar to that of Lemma 4.3.26:

Lemma 4.5.36. *Suppose that $(P, \mathfrak{m}, \widehat{a})$ is repulsive-normal and $\widehat{a} \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$. Then for all sufficiently small $q \in \mathbb{Q}^{>}$, any $\mathfrak{n} \asymp \mathfrak{v}^q\mathfrak{m}$ yields an almost strongly repulsive-normal refinement $(P, \mathfrak{n}, \widehat{a})$ of $(P, \mathfrak{m}, \widehat{a})$.*

Proof. First arrange $\mathfrak{m} = 1$. Take Q, R as in (RN2) for $\mathfrak{m} = 1$. Then Lemma 4.5.17 gives $q_0 \in \mathbb{Q}^{>}$ such that $\widehat{a} \prec \mathfrak{v}^{q_0}$ and for all $q \in \mathbb{Q}$ with $0 < q \leqslant q_0$ and $\mathfrak{n} \asymp \mathfrak{v}^q$, $L_{Q_{\times\mathfrak{n}}} = L_Q\mathfrak{n}$ has a strong $\widehat{a}/\mathfrak{n}$-repulsive splitting over K. Now Lemma 4.5.26 yields that $(P, \mathfrak{n}, \widehat{a})$ is almost strongly repulsive-normal for such $\mathfrak{n}$. $\square$

Using this lemma we now adapt the proof of Corollary 4.3.27 to obtain:

Corollary 4.5.37. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, deep, and repulsive-normal. Then $(P, \mathfrak{m}, \widehat{a})$ has a deep and almost strongly repulsive-normal refinement.*

Proof. Lemma 3.3.13 gives a such that $\widehat{a} - a \prec_{\Delta(\mathfrak{v})} \mathfrak{m}$. By Corollary 3.3.8, the refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ is deep with $\mathfrak{v}(L_{P_{+a,\times\mathfrak{m}}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$, and by Lemma 4.5.25 it is also repulsive-normal. Now apply Lemma 4.5.36 to $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ in place of $(P, \mathfrak{m}, \widehat{a})$ and again use Corollary 3.3.8 to preserve being deep. $\square$

Next we adapt the proof of Lemma 4.3.28 to obtain a result about the behavior of (almost) repulsive-normality under compositional conjugation:

Lemma 4.5.38. *Suppose ϕ is active in H with $0 < \phi \prec 1$, and there exists a with $\widehat{a} - a \prec^{\flat} \mathfrak{m}$. If $(P, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal, then so is the slot $(P^{\phi}, \mathfrak{m}, \widehat{a})$ in H^{ϕ}. Likewise with "strongly" in place of "almost strongly".*

Proof. We arrange $\mathfrak{m} = 1$, assume $(P, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal, and take Q, R as in (RN2as). The proof of Lemma 4.3.5 shows that with $\mathfrak{w} := \mathfrak{v}(L_{P^{\phi}})$ we have $\mathfrak{w} \prec^{\flat}_{\phi} 1$ and $(P^{\phi})_{\geqslant 1} = Q^{\phi} + R^{\phi}$ where $Q^{\phi} \in H^{\phi}\{Y\}$ is homogeneous of degree 1 and order r, $L_{Q^{\phi}}$ splits over K^{ϕ}, and $R^{\phi} \prec_{\Delta(\mathfrak{w})} \mathfrak{w}^{w+1}(P^{\phi})_1$. By Lemma 4.5.16, $L_{Q^{\phi}} = L_Q^{\phi}$ has even a strong $\widehat{a}$-repulsive splitting over K. Hence $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal. For the rest we use Lemma 4.5.33 and the fact that if $(P, \mathfrak{m}, \widehat{a})$ is strictly normal, then so is $(P^{\phi}, \mathfrak{m}, \widehat{a})$. $\square$

Lemma 3.3.13, the remark preceding Corollary 4.5.23, and Lemma 4.5.38 yield:

Corollary 4.5.39. *Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal and deep, and ϕ is active in H with $0 < \phi \prec 1$. If $(P, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal, then so is the slot $(P^{\phi}, \mathfrak{m}, \widehat{a})$ in H^{ϕ}. Likewise with "strongly" in place of "almost strongly".*

In the case $r = 1$, ultimateness yields almost strong repulsive-normality, under suitable assumptions; more precisely:

Lemma 4.5.40. *Suppose H is Liouville closed and of Hardy type, and $\mathrm{I}(K) \subseteq K^{\dagger}$. Assume also that $(P, \mathfrak{m}, \widehat{a})$ is normal and special, of order $r = 1$. Then*

$$(P, \mathfrak{m}, \widehat{a}) \text{ is ultimate} \iff L \text{ has a strong } \widehat{a}/\mathfrak{m}\text{-repulsive splitting over } K,$$

in which case $(P, \mathfrak{m}, \widehat{a})$ is almost strongly repulsive-normal.

Proof. By Lemma 4.4.12, $(P, \mathfrak{m}, \widehat{a})$ is ultimate iff $\mathscr{E}^{\mathrm{u}}(L) \cap v\big((\widehat{a}/\mathfrak{m}) - H\big) \leqslant 0$, and the latter is equivalent to L having a strong $\widehat{a}/\mathfrak{m}$-repulsive splitting over K, by Corollary 4.5.20. For the rest use Corollary 4.5.34. $\square$

Liouville closed H-fields are 1-linearly newtonian by Corollary 1.7.29, so in view of Lemma 3.2.36 and Corollary 3.3.21 we may replace the hypothesis "$(P, \mathfrak{m}, \widehat{a})$ is special" in the previous lemma by "$(P, \mathfrak{m}, \widehat{a})$ is Z-minimal or a hole in H". This leads to repulsive-normal analogues of Lemma 4.3.29 and Corollary 4.3.30 for $r = 1$:

Lemma 4.5.41. *Assume H is Liouville closed and of Hardy type, and $\mathrm{I}(K) \subseteq K^{\dagger}$. Suppose $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal and quasilinear of order $r = 1$. Then there is a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ and an active ϕ in H with $0 < \phi \preccurlyeq 1$ such that $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a} - a)$ is deep, strictly normal, and ultimate (so $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a} - a)$ is strongly repulsive-normal by Lemmas 4.5.40 and 4.5.33).*

Proof. For any active ϕ in H with $0 < \phi \preccurlyeq 1$ we may replace H, $(P, \mathfrak{m}, \widehat{a})$ by H^{ϕ}, $(P^{\phi}, \mathfrak{m}, \widehat{a})$. We may also replace $(P, \mathfrak{m}, \widehat{a})$ by any of its refinements. Since H is 1-linearly newtonian, Corollary 3.3.35 gives a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$

and an active ϕ in H such that $0 < \phi \preccurlyeq 1$ and $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a} - a)$ is normal. Replacing H, $(P, \mathfrak{m}, \widehat{a})$ by H^{ϕ}, $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a} - a)$, we arrange that $(P, \mathfrak{m}, \widehat{a})$ itself is normal. Then $(P, \mathfrak{m}, \widehat{a})$ has an ultimate refinement by Proposition 4.4.14, and applying Corollary 3.3.35 to this refinement and using Lemma 4.4.10, we obtain an ultimate refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ and an active ϕ in H with $0 < \phi \preccurlyeq 1$ such that the Z-minimal slot $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a} - a)$ in H^{ϕ} is deep, normal, and ultimate. Again replacing H, $(P, \mathfrak{m}, \widehat{a})$ by H^{ϕ}, $(P^{\phi}_{+a}, \mathfrak{n}, \widehat{a} - a)$, we arrange that $(P, \mathfrak{m}, \widehat{a})$ is deep, normal, and ultimate. Corollary 3.3.47 yields a deep and strictly normal refinement $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$; this refinement is still ultimate by Lemma 4.4.10. Hence $(P_{+a}, \mathfrak{m}, \widehat{a} - a)$ is a refinement of $(P, \mathfrak{m}, \widehat{a})$ as required, with $\phi = 1$. $\square$

Combining Lemmas 3.2.26 and 4.5.41 with Corollary 4.5.39 yields:

Corollary 4.5.42. *Assume H is Liouville closed, ω-free, and of Hardy type, and $\mathrm{I}(K) \subseteq K^{\dagger}$. Then every Z-minimal slot in H of order $r = 1$ has a refinement $(P, \mathfrak{m}, \widehat{a})$ such that $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is eventually deep, ultimate, and strongly repulsive-normal.*

In the next subsection we show how minimal holes of degree > 1 in K give rise to deep, ultimate, strongly repulsive-normal, Z-minimal slots in H.

Achieving strong repulsive-normality. Let H be an ω-free Liouville closed H-field with small derivation and constant field C, and $(P, \mathfrak{m}, \widehat{a})$ a minimal hole of order $r \geqslant 1$ in $K := H[\mathrm{i}]$. Other conventions are as in the subsection *Achieving repulsive-normality.* Our goal is to prove a version of Theorem 4.5.28 with "repulsive-normal" improved to "strongly repulsive-normal + ultimate":

Theorem 4.5.43. *Suppose C is archimedean, $\mathrm{I}(K) \subseteq K^{\dagger}$, and $\deg P > 1$. Then one of the following conditions is satisfied:*

- (i) *$\widehat{b} \notin H$ and some Z-minimal slot $(Q, \mathfrak{m}, \widehat{b})$ in H has a special refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ such that $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ is eventually deep, strongly repulsive-normal, and ultimate;*
- (ii) *$\widehat{c} \notin H$ and some Z-minimal slot $(R, \mathfrak{m}, \widehat{c})$ in H has a special refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ is eventually deep, strongly repulsive-normal, and ultimate.*

The proof of this theorem rests on the following two lemmas, where the standing assumption that H is Liouville closed can be dropped.

Lemma 4.5.44. *Suppose $\widehat{b} \notin H$ and $(Q, \mathfrak{m}, \widehat{b})$ is a Z-minimal slot in H with a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ such that $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ is eventually deep and repulsive-normal. Then $(Q, \mathfrak{m}, \widehat{b})$ has a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ such that $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ is eventually deep and almost strongly repulsive-normal.*

Proof. We adapt the proof of Lemma 4.3.34. Let $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ be a refinement of $(Q, \mathfrak{m}, \widehat{b})$ and let ϕ_0 be active in H such that $0 < \phi_0 \preccurlyeq 1$ and $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$ is deep and repulsive-normal. Then Corollary 4.5.37 yields a refinement

$$\big((Q^{\phi_0}_{+b})_{+b_0}, \mathfrak{n}_0, (\widehat{b} - b) - b_0\big)$$

of $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$ which is deep and almost strongly repulsive-normal. Hence

$$\big((Q_{+b})_{+b_0}, \mathfrak{n}_0, (\widehat{b} - b) - b_0\big) \;=\; \big(Q_{+(b+b_0)}, \mathfrak{n}_0, \widehat{b} - (b + b_0)\big)$$

is a refinement of $(Q, \mathfrak{m}, \widehat{b})$, and $\big(Q^{\phi}_{+(b+b_0)}, \mathfrak{n}_0, \widehat{b} - (b + b_0)\big)$ is eventually deep and almost strongly repulsive-normal by Corollary 4.5.39. $\square$

In the same way we obtain:

Lemma 4.5.45. *Suppose $\widehat{c} \notin H$ and $(R, \mathfrak{m}, \widehat{c})$ is a Z-minimal slot in H with a refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ is eventually deep and repulsive-normal. Then $(R, \mathfrak{m}, \widehat{c})$ has a refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ is eventually deep and almost strongly repulsive-normal.*

Theorem 4.5.28 and the two lemmas above give Theorem 4.5.28 with "repulsive-normal" improved to "almost strongly repulsive-normal". We now upgrade this further to "strongly repulsive-normal + ultimate" (under an extra assumption).

Recall from Lemma 4.1.3 that $v(\widehat{b} - H) \subseteq v(\widehat{c} - H)$ or $v(\widehat{c} - H) \subseteq v(\widehat{b} - H)$. Thus the next two lemmas finish the proof of Theorem 4.5.43.

Lemma 4.5.46. *Suppose C is archimedean, $\mathrm{I}(K) \subseteq K^{\dagger}$, $\deg P > 1$, and*

$$v(\widehat{b} - H) \ \subseteq \ v(\widehat{c} - H).$$

Let $Q \in Z(H, \widehat{b})$ have minimal complexity. Then the Z-minimal slot $(Q, \mathfrak{m}, \widehat{b})$ in H has a special refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ such that $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ is eventually deep, strongly repulsive-normal, and ultimate.

Proof. Here are two ways of modifying $(Q, \mathfrak{m}, \widehat{b})$. First, let $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ be a refinement of $(Q, \mathfrak{m}, \widehat{b})$. Lemma 4.1.3 gives $c \in H$ with $v(\widehat{a} - a) = v(\widehat{b} - b)$ with $a := b + c i$, and so the minimal hole $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ in K is a refinement of $(P, \mathfrak{m}, \widehat{a})$ that relates to $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ as $(P, \mathfrak{m}, \widehat{a})$ relates to $(Q, \mathfrak{m}, \widehat{b})$. So we can replace $(P, \mathfrak{m}, \widehat{a})$ and $(Q, \mathfrak{m}, \widehat{b})$ by $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ and $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$, whenever convenient. Second, let ϕ be active in H with $0 < \phi \preccurlyeq 1$. Then we can likewise replace H, K, $(P, \mathfrak{m}, \widehat{a})$, $(Q, \mathfrak{m}, \widehat{b})$ by H^{ϕ}, K^{ϕ}, $(P^{\phi}, \mathfrak{m}, \widehat{a})$, $(Q^{\phi}, \mathfrak{m}, \widehat{b})$.

In this way we first arrange as in the proof of Corollary 4.5.30 that $(Q, \mathfrak{m}, \widehat{b})$ is special. Next, we use Proposition 3.3.36 likewise to arrange that $(Q, \mathfrak{m}, \widehat{b})$ is also normal. By Propositions 4.4.14 (where the assumption $\mathrm{I}(K) \subseteq K^{\dagger}$ comes into play) and 3.3.25 we arrange that $(Q, \mathfrak{m}, \widehat{b})$ is ultimate as well. The properties "special" and "ultimate" persist under further refinements and compositional conjugations.

Now Corollary 4.5.30 and Lemma 4.5.44 give a refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ of the slot $(Q, \mathfrak{m}, \widehat{b})$ in H and an active ϕ_0 in H with $0 < \phi_0 \preccurlyeq 1$ such that the slot $(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b)$ in H^{ϕ_0} is deep and almost strongly repulsive-normal. Corollary 3.3.47 then yields a deep and strictly normal refinement

$$\big((Q^{\phi_0}_{+b})_{+b_0}, \mathfrak{n}, (\widehat{b} - b) - b_0\big)$$

of $\big(Q^{\phi_0}_{+b}, \mathfrak{n}, \widehat{b} - b\big)$. This refinement is still almost strongly repulsive-normal by Lemma 4.5.35, and therefore strongly repulsive-normal by Lemma 4.5.33. Corollary 4.5.39 then gives that $\big(Q_{+(b+b_0)}, \mathfrak{n}, \widehat{b} - (b + b_0)\big)$ is a special refinement of our slot $(Q, \mathfrak{m}, \widehat{b})$ such that $\big(Q^{\phi}_{+(b+b_0)}, \mathfrak{n}, \widehat{b} - (b + b_0)\big)$ is eventually deep and strongly repulsive-normal. $\square$

Likewise:

Lemma 4.5.47. *Suppose C is archimedean,* $\mathrm{I}(K) \subseteq K^{\dagger}$, $\deg P > 1$*, and*

$$v(\widehat{c} - H) \ \subseteq \ v(\widehat{b} - H).$$

Let $R \in Z(H, \widehat{c})$ have minimal complexity. Then the Z-minimal slot $(R, \mathfrak{m}, \widehat{c})$ in H has a special refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ is eventually deep, strongly repulsive-normal, and ultimate.

4.6. The Main Theorem

We prove here the Normalization Theorem from the Introduction, as a corollary of Theorem 4.5.43. It is accordingly less detailed than the latter, but more user-friendly. It is what will get used at a key stage in [6].

Corollary 4.6.1. *Let H be an ω-free Liouville closed H-field with small derivation, archimedean ordered constant field C, and 1-linearly newtonian algebraic closure $H[\mathrm{i}]$. Suppose H is not newtonian. Then for some Z-minimal special hole $(Q, 1, \widehat{b})$ in H with* $\operatorname{order} Q \geqslant 1$ *and some active $\phi > 0$ in H with $\phi \preccurlyeq 1$, the hole $(Q^{\phi}, 1, \widehat{b})$ in H^{ϕ} is deep, strongly repulsive-normal, and ultimate.*

Proof. By Proposition 1.7.28, $K := H[\mathrm{i}]$ is 1-linearly surjective and $\mathrm{I}(K) \subseteq K^{\dagger}$. As H is not newtonian, neither is K, by (0.11), so Lemma 3.2.1 and subsequent remarks give a minimal hole $(P, \mathfrak{m}, \widehat{a})$ in K of order $r \geqslant 1$, where $\mathfrak{m} \in H^{\times}$. Then $\deg P > 1$ by Corollary 3.2.8. By Lemma 4.2.15 we arrange that $\widehat{a} \in \widehat{K} := \widehat{H}[\mathrm{i}]$ where $\widehat{H}$ is an immediate ω-free newtonian H-field extension of H. Now $\widehat{a} = \widehat{b} + \widehat{c}\mathrm{i}$ with $\widehat{b}, \widehat{c} \in \widehat{H}$. By Theorem 4.5.43 there are two cases:

(1) $\widehat{b} \notin H$ and some Z-minimal slot $(Q, \mathfrak{m}, \widehat{b})$ in H has a special refinement $(Q_{+b}, \mathfrak{n}, \widehat{b} - b)$ such that $(Q^{\phi}_{+b}, \mathfrak{n}, \widehat{b} - b)$ is eventually deep, strongly repulsive-normal, and ultimate;

(2) $\widehat{c} \notin H$ and some Z-minimal slot $(R, \mathfrak{m}, \widehat{c})$ in H has a special refinement $(R_{+c}, \mathfrak{n}, \widehat{c} - c)$ such that $(R^{\phi}_{+c}, \mathfrak{n}, \widehat{c} - c)$ is eventually deep, strongly repulsive-normal, and ultimate.

Assume Q, $\mathfrak{m}$, $\widehat{b}$ are as in Case (1). (Case (2) goes the same way.) Lemma 4.3.7 gives $1 \leqslant \operatorname{order} Q \leqslant 2r$. Multiplicatively conjugating by $\mathfrak{n}$ and renaming $Q_{+b, \times\mathfrak{n}}$ and $\frac{\widehat{b} - b}{\mathfrak{n}}$ as Q and $\widehat{b}$ we arrange that $(Q, 1, \widehat{b})$ is a Z-minimal special slot such that $(Q^{\phi}, 1, \widehat{b})$ is eventually deep, strongly repulsive-normal, and ultimate (using Lemma 4.4.11 to preserve *ultimate*). With Lemma 3.2.14, changing $\widehat{b}$ if necessary, we arrange that $(Q, 1, \widehat{b})$ is a hole in H, not just a slot in H. $\square$

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List of Symbols

Errata and Comments to [ADH]

The changes below apply to the edition published by Princeton University Press, and are already reflected in the versions posted on the arXiv and on our personal web pages (as of February 2024). We thank Allen Gehret for pointing out most of the errors left in that edition. Linguistic slips like missing commas or articles are not listed below unless they might mislead. Citations are to the bibliography of [ADH].

Acknowledgments:

The date of September 2015 on p. xiv indicates when the manuscript was first submitted to Princeton University Press. The published version incorporates some changes and additions made since then.

Dramatis Personae:

In the item for "$\boldsymbol{\omega}$-free" under the heading "Asymptotic Fields", $f-\omega(g^{\dagger\dagger})\succcurlyeq g^{\dagger}$ should be $f-\omega(g^{\dagger\dagger})\succcurlyeq (g^{\dagger})^2$.

Chapter 1:

(1) The first sentence of the subsection **Irreducibility** in 1.1 should be: *Let X and Y be topological spaces.*
(2) In the subsection **Localization of modules** in 1.4 the formula for addition should have $s_2x_1+s_1x_2$ in the numerator.
(3) In the subsection **Tensor products** in 1.7, the H in the 4th line should be a B.
(4) In the subsection **Rational rank** in 1.7, in the line following the display: $\mathbb{Q}\otimes_{\mathbb{Z}} N$ should be $\mathbb{Q}\otimes_{\mathbb{Z}} M$.
(5) In the 4th line of the proof of 1.8.12, the second ":=" should be "=".
(6) In the 6th line of the proof of 1.8.13, "$(a,b)\to$" should be "$(a,b)\mapsto$".
(7) In 1.9.6 one should add the assumption that L is separably generated over K, that is, L is separably algebraic over an intermediate field $K(B)$ with $B\subseteq L$ algebraically independent over K. This assumption is satisfied if $\operatorname{char} K=0$. Corollary 1.9.7 is still correct as stated, but its proof requires for positive characteristic a variant of 1.9.6, namely: *L is separably algebraic over K iff every derivation on L extending the trivial derivation on K is trivial.* (This variant with a proof, as in [249, pp. 370–371] is now included in the arXiv version.) Lemma 1.9.8 should be restricted to the case $\operatorname{char} K=0$.

Chapter 2:

(1) In the fourth paragraph of 2.3, replace "valued subgroup of (G,S,v)" by "valued subgroup of (G',S',v')".

Chapter 3:

(1) In the second sentence of the proof of 3.1.21, one can omit "with $\mathfrak{q}\cap A=\mathfrak{q}'\cap A=\mathfrak{m}$" since this condition is automatically satisfied.
(2) F.-V. Kuhlmann pointed out that in the "Notes and comments" to 3.2 we misattribute 3.2.26 to Krull [229]. An early source for a result of this kind is Theorem 11 in O. Schilling's book,

The Theory of Valuations, Mathematical Surveys, no. 4, American Mathematical Society, New York, 1950.

This book refers for this theorem to I. Kaplansky's unpublished Ph.D. thesis *Maximal Fields with Valuations,* Harvard University, 1941.

(3) Replace "theorem" by "proposition" in the sentence following the statement of 3.4.22.

(4) Marcus Tressl alerted us to an error in the proof of 3.6.11: replace the condition $\boldsymbol{K} \preccurlyeq \boldsymbol{F}$ in the first sentence of the proof by $\boldsymbol{K} \subseteq \boldsymbol{F}$, so that Zorn's lemma can be applied as indicated in the next sentence.

(5) Right after 3.7.6, replace
"open ball of the form $\{y : v(y - f) > vf\}$ where $f \in K^\times$" by
"open ball of the form $\{y : v(y - f) > vg\}$ where $f, g \in K^\times$, $f \succcurlyeq g$".

Chapter 4:

(1) In the first sentence of the proof of 4.1.10, omit *be.*

(2) The last three sentences of the proof of 4.6.12 can be shortened to: *Then by Lemma 1.3.10, a is algebraic over K, so a is algebraic over C by Lemma 4.1.2.*

Chapter 5:

(1) In line 5 of 5.5, replace $K[\partial]$ by $R[\partial]$.

(2) In 5.7.3, replace
"$\mathbb{Q}[\phi, \dots, \partial^n(\phi)] = \mathbb{Q}[\phi, \dots, \delta^n(\phi)]$" by
"$\mathbb{Q}[\phi, \dots, \partial^n(\phi), \phi^{-1}] = \mathbb{Q}[\phi, \dots, \delta^n(\phi), \phi^{-1}]$".

Chapter 6:

(1) In the second to last line of the proof of 6.1.9, replace C by D_0.

(2) In the second line before the first display in the proof of 6.3.2 there is a misplaced parenthesis in $K[Y, \dots, Y^{(r-1)}]$.

(3) In the last line of the proof of 6.6.5, replace (ii) by (iii).

Chapter 7:

(1) In the third line of the proof of 7.5.6, replace E by $E^\times$.

Chapter 8:

(1) In the proof of 8.3.2, $(E, \Gamma, \boldsymbol{k}_E)$ should be $(E, \boldsymbol{k}_E, \Gamma)$.

(2) A few lines before 8.3.3, $\theta_{\mathrm{v}}(v_1, \dots, v_k, y)$ should be $\theta_{\mathrm{v}}(v_1, \dots, v_k, z)$.

(3) In the proof of 8.4.12, third line from the bottom, "$\Gamma_{K_3} = \Gamma_{K_3}$" should be "$\Gamma_{K_2} = \Gamma_{K_3}$".

Chapter 9:

(1) Two lines before 9.1.10, (3) should be (2).

(2) Replace "Lemma" in the last line of the proof of 9.2.17 by "Corollary".

(3) The correction following 3.7.6 leads to a corresponding correction in describing the condition $z \in G_i$ when $s_i \neq 0$, in the proof of 9.7.3.

(4) Verifying (AC3) in proof of 9.8.2 can be shortened using

$$\max\left\{\psi^\alpha(\gamma + k\alpha) : \ \gamma \in \Gamma,\ k \in \mathbb{Z},\ \gamma + k\alpha \neq 0\right\} \ = \ \beta - \alpha.$$

(5) In proof of 9.9.3, insert right after "v-slow on the right" the phrase ", where v is the standard valuation of Γ".

Chapter 10:

(1) In 10.5.12, add "If K is an H-field, then so is $K(y)$ with that ordering, and $C_{K(y)} = C$" and in its proof refer to the remarks after 10.2.3.
(2) In the last sentence of the third paragraph in the "Notes and comments" to 10.6, "not not" should be "not".

Chapter 11:

(1) In the last display before 11.1.4, the expression $\{\gamma : \gamma < (\Gamma^{>})'\}$ should be replaced by $\{\gamma \in \Gamma : \gamma < (\Gamma^{>})'\}$.
(2) In 11.2.3(ii), complete to "$\operatorname{nmul} P = \operatorname{nmul} P_{+a}$" at the end.
(3) In proof of 11.6.3, replace $v(s - a^{\dagger}) \in (\Gamma_F^{>})'$ by $v(s - a^{\dagger}) \in (\Gamma_F^{>})' \cup \{\infty\}$.
(4) In last sentence of proof of 11.6.14, replace $\sim sf$ by $\sim -sf$.
(5) In proof of 11.6.17, end of the fourth paragraph, replace λ by $\boldsymbol{\lambda}$.
(6) In second part of 11.8.5, omit the assumption that K has asymptotic integration and replace $=$ at end of proof by $\subseteq$.
(7) Omit the proof of 11.8.13; it has an erroneous forward reference.

Chapter 12:

(1) In the statement of 12.6.3, the last part should be $[g]' = [g']$.

Chapter 13:

(1) In the *Notes and comments* to 13.3, replace "$n_0 = 2\operatorname{dwm}(P)$" by "$n_0 = 2\operatorname{dwm}(P) + m + 1$ where m is such that $P{\uparrow}^m \in \mathbb{T}_{\exp}\{Y\}$". (We thank Julian Ziegler-Hunts for pointing this out.)

Chapter 14:

(1) In the line following the statement of 14.0.1, it would be better to refer to 11.7.13 than to 11.7.10.
(2) In the third line of the proof of 14.1.8, replace K by $K^{\times}$.
(3) In the last line of the last display preceding 14.2.18, replace Y'' by $Y''Y$.
(4) In 14.3.2(iii), replace *at newton position* by *in newton position*.
(5) In line 6 of the proof of 14.3.2, after "$\operatorname{nmul} P_{+b} = \operatorname{nmul} P_{+a} = 1$" add "by Lemma 11.2.3" (referring to the addition to 11.2.3(ii) made above).
(6) After 14.5.11, replace *In Section 16.1* by *In Section 16.2*.

Appendix A:

(1) In the sixth line before the subsection "Representing $\mathbb{T}$ ..." on p. 719, replace $[v(\ell_{n-1}]$ by $[v(\ell_{n-1})]$.

Appendix B:

(1) In the example after B.5.15, replace "Then $V \setminus W$ is infinite ..." by "If $V \neq W$, then $V \setminus W$ is infinite ..."
(2) In B.6.1(4) add the axiom $\forall x \forall y (x \leqslant y \vee y \leqslant x)$ to Or.
(3) In the remark following the definition of "proper filter on Λ" in B.7 omit "either".
(4) In the displayed equivalences in the proof of B.7.7 replace $\mathcal{F}$ by $\mathcal{U}$.
(5) In B.10.4(ii) replace "some $\kappa > |N|$" by "some infinite $\kappa > |N|$", and in the proof of that corollary replace "$|N|$-saturated" by "κ-saturated where $\kappa > |N|$ is infinite".

(6) In B.10.15 replace "embeddings $f_i \colon A \to M_i$ $(i = 1, 2)$ of abelian groups" by "embeddings $f_i \colon A \to M_i$ $(i = 1, 2)$ of torsion-free abelian groups".
(7) Replace the last sentence in the remark before B.12.14 by "Then RCF$'$ is model complete, but does not have QE: $\{a \in \mathbb{R} : \ a \geqslant 0\}$ is neither finite nor cofinite and hence is not definable in the field $\mathbb{R}$ by a quantifier-free $\mathcal{L}_{\mathrm{R}}$-formula."
(8) In B.12.15, replace "singletons" by "singletons $\{a\}$ where $a \in K$".
(9) Add to the "Notes and comments" of B.12 that B.12.9 and B.12.11, with different proofs, are from:

    A. H. Lightstone, A. Robinson, *On the representation of Herbrand functions in algebraically closed fields,* J. Symb. Logic **22** (1957), 187–204.

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