THE THEORY OF MAXIMAL HARDY FIELDS

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Abstract. We show that all maximal Hardy fields are elementarily equivalent as differential fields to the differential field $\mathbb{T}$ of transseries, and give various applications of this result and its proof.

Contents

Introduction

Hardy [49] made sense of Du Bois-Reymond's "orders of infinity" [17]–[20]. This led to the notion of a Hardy field (Bourbaki [27]). A *Hardy field* is a field H of germs at $+\infty$ of differentiable real-valued functions on intervals $(a, +\infty)$ such that for any differentiable function whose germ is in H the germ of its derivative is also in H. (See Section 2 for more precision.) Every Hardy field is naturally a differential field, and also an ordered field with the germ of f being > 0 iff $f(t) > 0$, eventually. *Hardy fields are the natural domain of asymptotic analysis, where all rules hold, without qualifying conditions* [72, p. 297]. Many basic facts about Hardy fields can be found in Bourbaki [27], Boshernitzan [21]–[24], and Rosenlicht [72]–[75].

Hardy [47] focused on the Hardy field consisting of the germs of logarithmico-exponential functions (LE-functions, for short): these functions are the real-valued functions obtainable from real constants and the identity function x using addition, multiplication, division, taking logarithms, and exponentiating. Examples include

Date: August, 2024.

the germs of the function given for large positive x by x^r ($r \in \mathbb{R}$), e^{x^2}, and $\log\log x$. Besides the germs of LE-functions, the germs of many other naturally occurring non-oscillating differentially algebraic functions lie in Hardy fields. This includes in particular several special functions like the error function erf, the exponential integral Ei, the Airy functions Ai and Bi, etc. There are also Hardy fields which contain (germs of) differentially transcendental functions, such as the Riemann ζ-function and Euler's Γ-function [72], and even functions ultimately growing faster than each LE-function [23].

Germs of functions in Hardy fields are non-oscillating in a strong sense. In certain applications, this kind of tameness alone is crucial: for example, Écalle's proof [35] of Dulac's Conjecture (a weakened version of Hilbert's 16th Problem) essentially amounts to showing that the germ of the Poincaré return map at a cross section of a limit cycle lies in a Hardy field (at $0+$ instead of $+\infty$). A stronger form of tameness is o-minimality: indeed, every o-minimal structure on the real field naturally gives rise to a Hardy field (of germs of definable functions). This yields a wealth of examples such as those obtained from quasi-analytic Denjoy-Carleman classes [70], or containing certain transition maps of plane analytic vector fields [56], and explains the role of Hardy fields in model theory and its applications to real analytic geometry and dynamical systems [2, 14, 64].

Hardy fields have also found other applications: for effective counterparts to Hardy's theory of LE-functions, see [43, 46, 52, 67, 79]. Hardy fields have provided an analytic setting for extensions of this work beyond LE-functions [50, 77, 78, 82, 83]. They have also been useful in ergodic theory (see, e.g., [13, 26, 42, 57]), and other areas of mathematics [12, 28, 30, 37, 39, 45].

In the remainder of this introduction, H is a Hardy field. Then $H(\mathbb{R})$ (obtained by adjoining the germs of the constant functions) is also a Hardy field, and for any $h \in H$, the germ e^h generates a Hardy field $H(\mathrm{e}^h)$ over H, and so does any differentiable germ with derivative h. Moreover, H has a unique Hardy field extension that is algebraic over H and real closed. (See Section 2 for these facts, especially Proposition 2.1.) Our main result is Theorem 11.19, and it yields what appears to be the ultimate fact about differentially algebraic Hardy field extensions:

Theorem A. *Let $P(Y)$ be a differential polynomial in a single differential indeterminate Y over H, and let $f < g$ in H be such that $P(f) < 0 < P(g)$. Then there is a y in a Hardy field extension of H such that $f < y < g$ and $P(y) = 0$.*

By Zorn, every Hardy field extends to a maximal Hardy field, so by the theorem above, maximal Hardy fields have the intermediate value property for differential polynomials. (In [7] we show there are very many maximal Hardy fields, namely $2^{\mathfrak{c}}$ many, where $\mathfrak{c}$ is the cardinality of the continuum.) By the results mentioned earlier, maximal Hardy fields are also Liouville closed H-fields in the sense of [1]; thus they contain the germs of all LE-functions. Hiding behind the intermediate value property of Theorem A are two more fundamental properties, ω-*freeness* and *newtonianity,* which are central in our book [ADH]. (Roughly speaking, ω-freeness concerns second-order homogeneous differential equations, and newtonianity is a strong version of differential-henselianity.) In [9] we show that any Hardy field has an ω-free Hardy field extension, and in this paper we prove the much harder result that any ω-free Hardy field extends to a newtonian ω-free Hardy field: Theorem 11.19, which is really the main result of this paper. It follows that every maximal Hardy field is, in the terminology of [4], an *H-closed field* with small derivation. Now the elementary theory T_H of H-closed fields with small derivation (denoted by $T^{\text{nl}}_{\text{small}}$ in [ADH]) is *complete*, by [ADH, 16.6.3]. This means in particular that, as advertised in the abstract of this paper, any two maximal Hardy fields are indistinguishable by their elementary properties:

Corollary 1. *If H_1 and H_2 are maximal Hardy fields, then H_1 and H_2 are elementarily equivalent as ordered differential fields.*

To derive Theorem A we use also key results from the book [54] to the effect that $\mathbb{T}_{\text{g}}$, the ordered differential field of grid-based transseries, is H-closed with small derivation and the intermediate value property for differential polynomials. In particular, it is a model of the complete theory T_H. Thus maximal Hardy fields have the intermediate value property for differential polynomials as well, and this amounts to Theorem A, obtained here as a byproduct of more fundamental results. (A detailed account of the differential intermediate value property for H-fields is in [5].) We sketch the proof of our main result (Theorem 11.19) later in this introduction, after describing further consequences.

Further consequences of our main result. In [ADH] we prove more than completeness of T_H: a certain natural extension by definitions of T_H has quantifier elimination. This leads to a strengthening of Corollary 1 by allowing parameters from a common Hardy subfield of H_1 and H_2. To fully appreciate this statement requires more knowledge of model theory, as in [ADH, Appendix B], which we do not assume for this introduction. However, we can explain a special case in a direct way, in terms of solvability of systems of algebraic differential equations, inequalities, and asymptotic inequalities. Here we find it convenient to use the notation for asymptotic relations introduced by du Bois-Reymond and Hardy instead of Bachmann-Landau's O-notation: for germs f, g in a Hardy field set

$$\begin{aligned} f \preccurlyeq g \quad &:\Longleftrightarrow \quad f = O(g) \quad :\Longleftrightarrow \quad |f| \leqslant c|g| \text{ for some real } c > 0, \\ f \prec g \quad &:\Longleftrightarrow \quad f = o(g) \quad :\Longleftrightarrow \quad |f| < c|g| \text{ for all real } c > 0. \end{aligned}$$

Let now $Y = (Y_1, \dots, Y_n)$ be a tuple of distinct (differential) indeterminates, and consider a system of the following form:

$$(*) \qquad \left\{ \begin{array}{ccc} P_1(Y) & \varrho_1 & Q_1(Y) \\ \vdots & \vdots & \vdots \\ P_k(Y) & \varrho_k & Q_k(Y) \end{array} \right.$$

Here the P_i and Q_i are differential polynomials in Y (that is, polynomials in the indeterminates Y_j and their formal derivatives $Y_j', Y_j'', \dots$) with coefficients in our Hardy field H, and each ϱ_i is one of the symbols $=$, $\neq$, $\leqslant$, $<$, $\preccurlyeq$, $\prec$. Given a Hardy field $E \supseteq H$, a *solution* of $(*)$ in E is an n-tuple $y = (y_1, \dots, y_n) \in E^n$ such that for $i = 1, \dots, k$, the relation $P_i(y)\, \varrho_i\, Q_i(y)$ holds in E. Here is a Hardy field analogue of the "Tarski Principle" of real algebraic geometry [ADH, B.12.14]:

Corollary 2. *If the system $(*)$ has a solution in* some *Hardy field extension of H, then $(*)$ has a solution in* every *maximal Hardy field extension of H.*

(The symbols $\neq$, $\leqslant$, $<$, $\preccurlyeq$ in $(*)$ are for convenience only: their occurrences can be eliminated at the cost of increasing m, n. But $\prec$ is essential; see [ADH, 16.2.6].)

Besides the quantifier elimination alluded to, Corollary 2 depends on Lemma 12.1, which says that for any Hardy field H all maximal Hardy field extensions of H induce the same $\Lambda\Omega$-cut on H, as defined in [ADH, 16.3].

In particular, taking for H the smallest Hardy field $\mathbb{Q}$, we see that a system $(*)$ with a solution in some Hardy field has a solution in *every* maximal Hardy field, thus recovering a special case of our Corollary 1. Call such a system $(*)$ over $\mathbb{Q}$ *consistent.* For example, with X, Y, Z denoting here single distinct differential indeterminates, the system

$$Y'Z \ \preccurlyeq\ Z', \qquad Y \preccurlyeq 1, \qquad 1 \prec Z$$

is inconsistent, whereas for any $Q \in \mathbb{Q}\{Y\}$ and $n \geqslant 2$ the system

$$X^n Y' \ =\ Q(Y), \qquad X' = 1, \quad Y \prec 1$$

is consistent. As a consequence of the completeness of T_H we obtain the existence of an algorithm (albeit a very impractical one) for deciding whether a system $(*)$ over $\mathbb{Q}$ is consistent, and this opens up the possibility of automating a substantial part of asymptotic analysis in Hardy fields. We remark that Singer [84] proved the existence of an algorithm for deciding whether a given system $(*)$ over $\mathbb{Q}$ without occurrences of $\preccurlyeq$ or $\prec$ has a solution in *some* ordered differential field (and then it will have a solution in the ordered differential field of germs of real meromorphic functions at 0); but there are such systems, like

$$X' \ =\ 1, \qquad XY^2 \ =\ 1 - X,$$

which are solvable in an ordered differential field, but not in a Hardy field. Also, algorithmically deciding the solvability of a system $(*)$ over $\mathbb{Q}$ in a *given* Hardy field H may be impossible when H is "too small": e.g., if $H = \mathbb{R}(x)$, by [32].

As these results suggest, the aforementioned quantifier elimination for T_H yields a kind of "resultant" for systems $(*)$ that allows one to make explicit within H itself for which choices of coefficients of the differential polynomials P_i, Q_i the system $(*)$ has a solution in a Hardy field extension of H. Without going into details, we only mention here some attractive consequences for systems $(*)$ depending on parameters. For this, let $X_1, \dots, X_m, Y_1, \dots, Y_n$ be distinct indeterminates and $X = (X_1, \dots, X_m)$, $Y = (Y_1, \dots, Y_n)$, and consider a system

$$(**) \qquad \left\{ \begin{array}{ccc} P_1(X,Y) & \varrho_1 & Q_1(X,Y) \\ \vdots & \vdots & \vdots \\ P_k(X,Y) & \varrho_k & Q_k(X,Y) \end{array} \right.$$

where P_i, Q_i are now differential polynomials in (X, Y) over H, and the ϱ_i are as before. Specializing X to $c \in \mathbb{R}^m$ then yields a system

$$(*c) \qquad \left\{ \begin{array}{ccc} P_1(c,Y) & \varrho_1 & Q_1(c,Y) \\ \vdots & \vdots & \vdots \\ P_k(c,Y) & \varrho_k & Q_k(c,Y) \end{array} \right.$$

where $P_i(c, Y)$, $Q_i(c, Y)$ are differential polynomials in Y with coefficients in the Hardy field $H(\mathbb{R})$. (We only substitute real constants, so may assume that the P_i, Q_i are *polynomial* in X, that is, none of the derivatives $X_j', X_j'', \dots$ occur in the P_i, Q_i.) Using [ADH, 16.0.2(ii)] we obtain:

Corollary 3. *The set of all $c \in \mathbb{R}^m$ such that the system $(*c)$ has a solution in some Hardy field extension of H is semialgebraic.*

Recall: a subset of $\mathbb{R}^m$ is said to be *semialgebraic* if it is a finite union of sets

$$\left\{c \in \mathbb{R}^m : \ p(c) = 0, \ q_1(c) > 0, \dots, q_l(c) > 0\right\}$$

where $p, q_1, \dots, q_l \in \mathbb{R}[X]$ are ordinary polynomials. (The topological and geometric properties of semialgebraic sets have been studied extensively [15]. For example, it is well-known that a semialgebraic set can have only have finitely many connected components, and that each such component is itself semialgebraic.)

In connection with Corollary 3 we mention that the asymptotics of Hardy field solutions to algebraic differential equations $Q(Y) = 0$, where Q is a differential polynomial with constant real coefficients, has been investigated by Hardy [48] and Fowler [41] in cases where $\operatorname{order} Q \leqslant 2$ (see [11, Chapter 5]), and later by Shackell [76, 80, 81] in general. Special case of our corollary: for any differential polynomial $P(X,Y)$ with constant real coefficients, the set of parameters $c \in \mathbb{R}^m$ such that the differential equation $P(c, Y) = 0$ has a solution y in some Hardy field, in addition possibly also satisfying given asymptotic side conditions (such as $y \prec 1$), is semialgebraic. Example: the set of real parameters $(c_1, \dots, c_m) \in \mathbb{R}^m$ for which the homogeneous linear differential equation

$$y^{(m)} + c_1 y^{(m-1)} + \cdots + c_m y \ = \ 0$$

has a nonzero solution $y \prec 1$ in a Hardy field is semialgebraic; in fact, it is the set of all $(c_1, \dots, c_m) \in \mathbb{R}^m$ such that the polynomial $Y^m + c_1 Y^{m-1} + \cdots + c_m \in \mathbb{R}[Y]$ has a negative real zero. Nonlinear example: for $g_2, g_3 \in \mathbb{R}$ the differential equation

$$(Y')^2 \ = \ 4Y^3 - g_2 Y - g_3$$

has a nonconstant solution in a Hardy field iff $g_2^3 = 27 g_3^2$ and $g_3 \leqslant 0$. In both cases, the Hardy field solutions are germs of logarithmico-exponential functions. But the class of differentially algebraic germs in Hardy fields is much more extensive; for example, the antiderivatives of e^{x^2} are not logarithmico-exponential (Liouville).

Instead of $c \in \mathbb{R}^m$, substitute $h \in H^m$ for X in $(**)$, resulting in a system

$$(*h) \qquad \begin{cases} P_1(h,Y) & \varrho_1 & Q_1(h,Y) \\ \vdots & \vdots & \vdots \\ P_k(h,Y) & \varrho_k & Q_k(h,Y) \end{cases}$$

where $P_i(h,Y)$, $Q_i(h,Y)$ are now differential polynomials in Y with coefficients in H. It is well-known that for any semialgebraic set $S \subseteq \mathbb{R}^{m+1}$ there is a natural number $B = B(S)$ such that for every $c \in \mathbb{R}^m$, if the section $\left\{y \in \mathbb{R} : (c, y) \in S\right\}$ has $> B$ elements, then this section has nonempty interior in $\mathbb{R}$. In contrast, the set of solutions of $(*h)$ for $n = 1$ in a maximal H can be simultaneously infinite and discrete in the order topology of H: this happens precisely if some nonzero one-variable differential polynomial over H vanishes on this solution set [ADH, 16.6.11]. (Consider the example of the single algebraic differential equation $Y' = 0$, which has solution set $\mathbb{R}$ in each maximal Hardy field.) Nevertheless, we have the following uniform finiteness principle for solutions of $(*h)$; its proof is considerably deeper than Corollary 3 and also draws on results from [3].

Corollary 4. *There is a natural number $B = B(**)$ such that for all $h \in H^m$: if the system $(*h)$ has $> B$ solutions in some Hardy field extension of H, then $(*h)$ has continuum many solutions in every maximal Hardy field extension of H.*

Next we turn to issues of smoothness and analyticity in Corollary 2. By definition, a Hardy field is a differential subfield of the differential ring $\mathcal{C}^{<\infty}$ consisting of the germs of functions $(a, +\infty) \to \mathbb{R}$ ($a \in \mathbb{R}$) which are, for each n, eventually n-times continuously differentiable. Now $\mathcal{C}^{<\infty}$ has the differential subring $\mathcal{C}^{\infty}$ whose elements are the germs that are eventually $\mathcal{C}^{\infty}$. A *$\mathcal{C}^{\infty}$-Hardy field* is a Hardy field $H \subseteq \mathcal{C}^{\infty}$. (See [44] for an example of a Hardy field $H \not\subseteq \mathcal{C}^{\infty}$.) A $\mathcal{C}^{\infty}$-Hardy field is said to be *$\mathcal{C}^{\infty}$-maximal* if it has no proper $\mathcal{C}^{\infty}$-Hardy field extension. Now $\mathcal{C}^{\infty}$ in turn has the differential subring $\mathcal{C}^{\omega}$ whose elements are the germs that are eventually real analytic, and so we define likewise $\mathcal{C}^{\omega}$-Hardy fields ($\mathcal{C}^{\omega}$-maximal Hardy fields, respectively). Our main theorems go through in the $\mathcal{C}^{\infty}$- and $\mathcal{C}^{\omega}$-settings; combined with model completeness of T_H shown in [ADH, 16.2] this ensures the existence of solutions with appropriate smoothness in Corollary 2:

Corollary 5. *If $H \subseteq \mathcal{C}^{\infty}$ and the system $(*)$ has a solution in some Hardy field extension of H, then $(*)$ has a solution in every $\mathcal{C}^{\infty}$-maximal Hardy field extension of H. In particular, if H is $\mathcal{C}^{\infty}$-maximal and $(*)$ has a solution in a Hardy field extension of H, then it has a solution in H. (Likewise with $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{\infty}$.)*

We already mentioned $\mathbb{T}_{\mathrm{g}}$ as a quintessential example of an H-closed field. Its cousin $\mathbb{T}$, the ordered differential field of transseries, extends $\mathbb{T}_{\mathrm{g}}$ and is also H-closed with constant field $\mathbb{R}$ [ADH, 15.0.2]. The elements of $\mathbb{T}$ are certain generalized series (in the sense of Hahn) in an indeterminate $x > \mathbb{R}$ with real coefficients, involving exponential and logarithmic terms, such as

$$f \ = \ \mathrm{e}^{\frac{1}{2}\mathrm{e}^{x}} - 5\,\mathrm{e}^{x^2} + \mathrm{e}^{x^{-1}+2x^{-2}+\cdots} + \sqrt[3]{2}\log x - x^{-1} + \mathrm{e}^{-x} + \mathrm{e}^{-2x} + \cdots + 5\,\mathrm{e}^{-x^{3/2}}\,.$$

Mathematically significant examples are the more simply structured transseries

$$\mathrm{Ai} \ = \ \frac{\mathrm{e}^{-\xi}}{2\sqrt{\pi}x^{1/4}}\sum_n (-1)^n c_n \xi^{-n}, \quad \mathrm{Bi} \ = \ \frac{\mathrm{e}^{\xi}}{\sqrt{\pi}x^{-1/4}}\sum_n c_n \xi^{-n},$$

$$\text{where } \ c_n \ = \ \frac{(2n+1)(2n+3)\cdots(6n-1)}{(216)^n n!} \ \text{ and } \ \xi \ = \ \frac{2}{3}x^{3/2},$$

which are $\mathbb{R}$-linearly independent solutions of the Airy equation $Y'' = xY$ [65, Chapter 11, (1.07)]. For information about $\mathbb{T}$ see [ADH, Appendix A] or [36, 54]. We just mention here that like each H-field, $\mathbb{T}$ comes equipped with its own versions of the asymptotic relations $\preccurlyeq$, $\prec$, defined as for H above. The asymptotic rules valid in all Hardy fields, such as

$$f \preccurlyeq 1 \ \Rightarrow \ f' \prec 1, \qquad f \preccurlyeq g \prec 1 \ \Rightarrow \ f' \preccurlyeq g', \qquad f' = f \neq 0 \ \Rightarrow \ f \succ x^n$$

also hold in $\mathbb{T}$. Here x denotes, depending on the context, the germ of the identity function on $\mathbb{R}$, as well as the element $x \in \mathbb{T}$. Section 13 gives a finite axiomatization of these rules, thus completing an investigation initiated by A. Robinson [69].

Now suppose that we are given an embedding $\iota\colon H \to \mathbb{T}$ of ordered differential fields. We may view such an embedding as a *formal expansion operator* and its inverse as a *summation operator.* (See Section 13 below for an example of a Hardy field, arising from a fairly rich o-minimal structure, which admits such an embedding.) From $(*)$ we obtain a system

$$(\iota *) \qquad \begin{cases} \iota(P_1)(Y) & \varrho_1 & \iota(Q_1)(Y) \\ \vdots & \vdots & \vdots \\ \iota(P_k)(Y) & \varrho_m & \iota(Q_k)(Y) \end{cases}$$

of algebraic differential equations and (asymptotic) inequalities over $\mathbb{T}$, where $\iota(P_i)$, $\iota(Q_i)$ denote the differential polynomials over $\mathbb{T}$ obtained by applying ι to the coefficients of P_i, Q_i, respectively. A *solution* of $(\iota *)$ is a tuple $y = (y_1, \dots, y_n) \in \mathbb{T}^n$ such that $\iota(P_i)(y)\, \varrho_i\, \iota(Q_i)(y)$ holds in $\mathbb{T}$, for $i = 1, \dots, m$. Differential-difference equations in $\mathbb{T}$ are sometimes amenable to functional-analytic techniques like fixed point theorems or small (compact-like) operators [53], and the formal nature of transseries also makes it possible to solve algebraic differential equations in $\mathbb{T}$ by quasi-algorithmic methods [51, 54]. The simple example of the Euler equation

$$Y' + Y \;=\; x^{-1}$$

is instructive: its solutions in $\mathcal{C}^{<\infty}$ are given by the germs of

$$t \mapsto \mathrm{e}^{-t} \int_1^t \frac{\mathrm{e}^s}{s}\, ds + c\,\mathrm{e}^{-t} \colon (1, +\infty) \to \mathbb{R} \qquad (c \in \mathbb{R}),$$

all contained in a common Hardy field extension of $\mathbb{R}(x)$. The solutions of this differential equation in $\mathbb{T}$ are

$$\sum_n n!\, x^{-(n+1)} + c\,\mathrm{e}^{-x} \qquad (c \in \mathbb{R}),$$

where the particular solution $\sum_n n!\, x^{-(n+1)}$ is obtained as the unique fixed point of the operator $f \mapsto x^{-1} - f'$ on the differential subfield $\mathbb{R}(\!(x^{-1})\!)$ of $\mathbb{T}$ (cf. [ADH, 2.2.13]). (Note: $\sum_n n!\, t^{-(n+1)}$ diverges for each $t > 0$.) In general, the existence of a solution of $(\iota *)$ in $\mathbb{T}$ entails the existence of a solution of $(*)$ in some Hardy field extension of H and vice versa; more precisely:

Corollary 6. *The system $(\iota *)$ has a solution in $\mathbb{T}$ iff $(*)$ has a solution in some Hardy field extension of H. In this case, we can choose a solution of $(*)$ in a Hardy field extension E of H for which ι extends to an embedding of ordered differential fields $E \to \mathbb{T}$.*

In particular, a system $(*)$ over $\mathbb{Q}$ is consistent if and only if it has a solution in $\mathbb{T}$. (The "if" direction already follows from [ADH, Chapter 16] and [55]; the latter constructs a summation operator on the ordered differential subfield $\mathbb{T}^{\mathrm{da}} \subseteq \mathbb{T}$ of differentially algebraic transseries.)

It is worth noting that a result about differential equations in one variable, like Theorem A (or Theorem B below), yields similar facts about *systems* of algebraic differential equations and asymptotic inequalities in several variables over Hardy fields as in the corollaries above; we owe this to the strength of the model-theoretic methods employed in [ADH]. But our theorem in combination with [ADH] already has interesting consequences for one-variable differential polynomials over H and over its "complexification" $K := H[\mathrm{i}]$ (where $\mathrm{i}^2 = -1$), which is a differential subfield of the differential ring $\mathcal{C}^{<\infty}[\mathrm{i}]$. Some of these facts are analogous to familiar properties of ordinary one-variable polynomials over the real or complex numbers.

First, by Theorem A, every differential polynomial in one variable over H of odd degree has a zero in a Hardy field extension of H. (See Corollary 12.13.) For example, a differential polynomial like

$$(Y'')^5 + \sqrt{2}\,\mathrm{e}^x (Y'')^4 Y''' - x^{-1} \log x\, Y^2 Y'' + YY' - \Gamma$$

has a zero in every maximal Hardy field extension of the Hardy field $\mathbb{R}\langle \mathrm{e}^x, \log x, \Gamma\rangle$. Passing to $K = H[\mathrm{i}]$ we have:

Corollary 7. *For each differential polynomial $P \notin K$ in a single differential indeterminate with coefficients in K there are f, g in a Hardy field extension of H such that $P(f + g\mathrm{i}) = 0$.*

In particular, each linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y \;=\; b \qquad (a_1, \dots, a_n, b \in K)$$

has a solution $y = f + gi$ where f, g lie in some Hardy field extension of H. (Of course, if $b = 0$, then we may take here the trivial solution $y = 0$.) Although this special case of Corollary 7 concerns differential polynomials of degree 1, it seems hard to obtain this result without recourse to our more general extension theorems: a solution y of a linear differential equation of order n over K as above may simultaneously be a zero of a non-linear differential polynomial P over K of order $< n$, and the structure of the differential field extension of K generated by y is governed by P (when taken of minimal complexity in the sense of [ADH, 4.3]).

The main theorem of this paper has new consequences even for linear differential equations over Hardy fields. These are the subject of the follow-up paper [10], where we also prove Conjecture 4 of Boshernitzan [22]. Here we just mention one immediate consequence of Corollary 7 for linear differential operators, as it helps to motivate our interest in the "universal exponential extension" of K, explained later in this introduction. For this, consider the ring $R[\partial]$ of linear differential operators over a differential ring R: this ring is a free left R-module with basis ∂^n ($n \in \mathbb{N}$) such that $\partial^0 = 1$ and $\partial \cdot f = f\partial + f'$ for $f \in R$, where $\partial := \partial^1$. (See [ADH, 5.1] or [66, 2.1].) Any $A \in R[\partial]$ gives rise to an additive map $y \mapsto A(y)\colon R \to R$, with $\partial^n(y) = y^{(n)}$ (the nth derivative of y in R) and $r(y) = ry$ for $r = r \cdot 1 \in R \subseteq R[\partial]$. The elements of $\partial^n + R\partial^{n-1} + \cdots + R \subseteq R[\partial]$ are said to be *monic* of order n. It is well-known [29, 62, 63] that for $R = \mathcal{C}^{<\infty}[\mathrm{i}]$, each monic $A \in R[\partial]$ factors as a product of monic operators of order 1 in $R[\partial]$; if $A \in K[\partial]$, then such a factorization already happens over the complexification of some Hardy field extension of H:

Corollary 8. *If H is maximal, then each monic operator in $K[\partial]$ is a product of monic operators of order* 1 *in $K[\partial]$.*

This follows quite easily from Corollary 7 using the Riccati transform [ADH, 5.8]. Let now $A \in K[\partial]$ be monic of order n, and fix a maximal Hardy field extension E of H. The factorization result in Corollary 8 gives rise to a description of a fundamental system of solutions for the homogeneous linear differential equation $A(y) = 0$ in terms of Hardy field germs. Here, of course, complex exponential terms naturally appear, but only in a controlled way: the $\mathbb{C}$-linear space consisting of all $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ with $A(y) = 0$ has a basis of the form

$$f_1\, \mathrm{e}^{\phi_1 i},\ \dots,\ f_n\, \mathrm{e}^{\phi_n i}$$

where $f_j \in E[\mathrm{i}]$ and $\phi_j \in E$ with $\phi_j = 0$ or $|\phi_j| > \mathbb{R}$ for $j = 1, \dots, n$. (This will be refined in [10]; e.g., the basis elements $f_i \, \mathrm{e}^{\phi_i \mathrm{i}}$ for distinct frequencies ϕ_i can be arranged to be pairwise orthogonal in a certain sense.) A special case: if $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ is *holonomic,* that is, $L(y) = 0$ for some monic $L \in \mathbb{C}(x)[\partial]$, then y is a $\mathbb{C}$-linear combination of germs $f \, \mathrm{e}^{\phi \mathrm{i}}$ where $f \in E[\mathrm{i}]$, $\phi \in E$, and $\phi = 0$ or $|\phi| > \mathbb{R}$. Here, more information about the f, ϕ is available (see, e.g., [40, VIII.7], [88, §19.1]). Many special functions are holonomic [40, B.4].

Synopsis of the proof of our main theorem. In the rest of the paper we assume familiarity with the terminology and concepts of asymptotic differential algebra from our book [ADH]. We refer to the section *Concepts and Results from* [ADH] in the introduction of [8] for a concise exposition of the background from [ADH] required to read [8] and the present paper. The differential-algebraic and valuation-theoretic tools from [ADH] were further developed in [8], and we review some of this in Section 1. However, the proof of our main result also requires analytic arguments in an essential way. Some of this analysis adapts [55] to a more general setting. As mentioned earlier, Theorem A is a consequence of the following extension theorem:

Theorem B. *Every ω-free Hardy field has a newtonian Hardy field extension.*

The proof of this is long and intertwined with the normalization procedures in [8], so it may be useful to outline the strategy behind it.

Holes and slots. For now, let K be an H-asymptotic field with rational asymptotic integration. (Cases to keep in mind are $K = H$ and $K = H[\mathrm{i}]$ where H is a Liouville closed Hardy field.) In [8] we introduced the apparatus of *holes* in K as a means to systematize the study of solutions of algebraic differential equations over K in immediate asymptotic extensions of K: such a hole in K is a triple $(P, \mathfrak{m}, \widehat{f})$ where P is a nonzero differential polynomial in a single differential indeterminate Y with coefficients in K, $\mathfrak{m} \in K^{\times}$, and $\widehat{f} \notin K$ lies an immediate asymptotic extension of K with $P(\widehat{f}) = 0$ and $\widehat{f} \prec \mathfrak{m}$. It is sometimes technically convenient to work with the more flexible concept of a *slot* in K, where instead of $P(\widehat{f}) = 0$ we only require P to vanish at $(K, \widehat{f})$ in the sense of [ADH, 11.4]. The *complexity* of a slot $(P, \mathfrak{m}, \widehat{f})$ is the complexity of the differential polynomial P as in [ADH, p. 216]. (See also Section 1 below.) Now if K is ω-free, then by [8, Lemma 3.2.1],

$$K \text{ is newtonian} \quad \Longleftrightarrow \quad K \text{ has no hole.}$$

This equivalence suggests an opening move for proving Theorem B by induction on complexity as follows: Let $H \supseteq \mathbb{R}$ be an ω-free Liouville closed Hardy field, and suppose H is not newtonian; it is enough to show that then H has a proper Hardy field extension. By the above equivalence, H has a hole $(P, \mathfrak{m}, \widehat{f})$, and we can take here $(P, \mathfrak{m}, \widehat{f})$ to be of minimal complexity among holes in H. This minimality has consequences that are important for us; for example $r := \operatorname{order} P \geqslant 1$, P is a minimal annihilator of $\widehat{f}$ over H, and H is $(r-1)$-newtonian as defined in [ADH, 14.2]. We arrange $\mathfrak{m} = 1$ by replacing $(P, \mathfrak{m}, \widehat{f})$ with the hole $(P_{\times \mathfrak{m}}, 1, \widehat{f}/\mathfrak{m})$ in H.

Solving algebraic differential equations over Hardy fields. For Theorem B it is enough to show that under these conditions P is a minimal annihilator of some germ $f \in \mathcal{C}^{<\infty}$ that generates a (necessarily proper) Hardy field extension $H\langle f \rangle$ of H. So at a minimum, we need to find a solution in $\mathcal{C}^{<\infty}$ to the algebraic differential equation $P(Y) = 0$. For this, it is natural to use fixed point techniques as in [55]. Notation: for $a \in \mathbb{R}$, $\mathcal{C}^n_a$ is the $\mathbb{R}$-linear space of n-times continuously differentiable functions $[a, +\infty) \to \mathbb{R}$; each $f \in \mathcal{C}^{<\infty}$ has a representative in $\mathcal{C}^n_a$.

A fixed point theorem. Let $L := L_P \in H[\partial]$ be the linear part of P. We can replace $(P, 1, \widehat{f})$ with another minimal hole in H to arrange order $L = r$. Representing the coefficients of P (and thus of L) by functions in $\mathcal{C}^0_a$ we obtain an $\mathbb{R}$-linear operator $y \mapsto L(y)\colon \mathcal{C}^r_a \to \mathcal{C}^0_a$. For now we make the bold assumption that $L \in H[\partial]$ splits over H. Using such a splitting and increasing a if necessary, r-fold integration yields an $\mathbb{R}$-linear operator $L^{-1}\colon \mathcal{C}^0_a \to \mathcal{C}^r_a$ which is a *right-inverse* of $L\colon \mathcal{C}^r_a \to \mathcal{C}^0_a$, that is, $L\big(L^{-1}(y)\big) = y$ for all $y \in \mathcal{C}^0_a$. Consider the (generally non-linear) operator

$$f \mapsto \Phi(f) := L^{-1}\big(R(f)\big)$$

on $\mathcal{C}^r_a$; here $P = P_1 - R$ where P_1 is the homogeneous part of degree 1 of P. We try to show that Φ restricts to a contractive operator on a closed ball of an appropriate subspace of $\mathcal{C}^r_a$ equipped with a suitable complete norm, whose fixed points are then solutions to $P(Y) = 0$; this may also involve increasing a again and replacing the coefficient functions of P by their corresponding restrictions. We can obtain such contractivity if R is asymptotically small compared to P_1 in a certain sense. The latter can indeed be achieved by transforming $(P, 1, \widehat{f})$ into a certain normal form through successive *refinements* and (*additive*, *multiplicative*, and *compositional*) *conjugations* of the hole $(P, 1, \widehat{f})$. This normalization is done under more general algebraic assumptions in [8, Section 3.3]. The analytic arguments leading to fixed points are in Sections 5–7. Developments below involve the algebraic closure $K := H[i]$ of H and we work more generally with a decomposition $P = \widetilde{P}_1 - R$ where $\widetilde{P}_1 \in K\{Y\}$ is homogeneous of degree 1, not necessarily $\widetilde{P}_1 = P_1$, such that $L_{\widetilde{P}_1}$ splits over K and R is "small" compared to $\widetilde{P}_1$.

Passing to the complex realm. In general we are not so lucky that L splits over H. The minimality of our hole $(P, 1, \widehat{f})$ does not even ensure that L splits over K. At this point we recall from [ADH, 11.7.23] that K is ω-free because H is. We can also draw hope from the fact that every nonzero linear differential operator over K would split over K if H were newtonian [ADH, 14.5.8]. Although H is not newtonian, it is $(r-1)$-newtonian, and L is only of order r, so we optimistically restart our attempt, and instead of a hole of minimal complexity in H, we now let $(P, \mathfrak{m}, \widehat{f})$ be a hole of minimal complexity in K. Again it follows that $r := \operatorname{order} P \geqslant 1$, P is a minimal annihilator of $\widehat{f}$ over K, and K is $(r-1)$-newtonian. As before we arrange that $\mathfrak{m} = 1$ and the linear part $L_P \in K[\partial]$ of P has order r. We can also arrange $\widehat{f} \in \widehat{K} = \widehat{H}[i]$ where $\widehat{H}$ is an immediate asymptotic extension of H. So $\widehat{f} = \widehat{g} + \widehat{h}i$ where $\widehat{g}, \widehat{h} \in \widehat{H}$ satisfy $\widehat{g}, \widehat{h} \prec 1$, and $\widehat{g} \notin H$ or $\widehat{h} \notin H$, say $\widehat{g} \notin H$. Now minimality of $(P, 1, \widehat{f})$ and algebraic closedness of K give that K is r-linearly closed, that is, every nonzero $A \in K[\partial]$ of order $\leqslant r$ splits over K [8, Corollary 3.2.4]. Then L_P splits over K as desired, and a version of the above fixed point construction with $\mathcal{C}^r_a[i]$ in place of $\mathcal{C}^r_a$ can be carried out successfully to solve $P(Y) = 0$ in the differential ring extension $\mathcal{C}^{<\infty}[i]$ of $\mathcal{C}^{<\infty}$.

Return to the real world. But at this point we face another obstacle: even once we have our hands on a zero $f \in \mathcal{C}^{<\infty}[i]$ of P, it is not clear why $g := \operatorname{Re} f$ should generate a proper Hardy field extension of H: Let Q be a minimal annihilator of $\widehat{g}$ over H; we cannot expect that $Q(g) = 0$. If $L_Q \in H[\partial]$ splits over K, then we can try to apply fixed point arguments like the ones above, with $(P, 1, \widehat{f})$ replaced by the hole $(Q, 1, \widehat{g})$ in H, to find a zero $y \in \mathcal{C}^{<\infty}$ of Q. (We do need to take care that constructed zero is real.) Unfortunately we can only ascertain that $1 \leqslant s \leqslant 2r$ for $s := \operatorname{order} Q$, and since we may have $s > r$, we cannot leverage the minimality of $(P, 1, \widehat{f})$ anymore to ensure that L_Q splits over K, or to normalize $(Q, 1, \widehat{g})$ in the same way as indicated above for $(P, 1, \widehat{f})$. This situation seems hopeless, but now a purely differential-algebraic observation comes to the rescue: although the linear part $L_{Q_{+\widehat{g}}} \in \widehat{H}[\partial]$ of the differential polynomial $Q_{+\widehat{g}} \in \widehat{H}\{Y\}$ also has order s (which may be $> r$), *if $\widehat{K}$ is r-linearly closed, then $L_{Q_{+\widehat{g}}}$ does split over $\widehat{K}$*; see [ADH, 5.1.37]. If moreover $g \in H$ is sufficiently close to $\widehat{g}$, then the linear part $L_{Q_{+g}} \in H[\partial]$ of $Q_{+g} \in H\{Y\}$ is close to an operator in $H[\partial]$ that does split over $K = H[\mathrm{i}]$, and so using $(Q_{+g}, 1, \widehat{g} - g)$ instead of $(Q, 1, \widehat{g})$ may offer a way out of this impasse.

Approximating $\widehat{g}$. Suppose for a moment that H is (valuation) dense in $\widehat{H}$. Then by extending $\widehat{H}$ we arrange that $\widehat{H}$ is the completion of H, and $\widehat{K}$ of K (as in [ADH, 4.4]). In this case $\widehat{K}$ inherits from K the property of being r-linearly closed, by results in [8, Section 1.7], and the desired approximation of $\widehat{g}$ by $g \in H$ can be achieved. We cannot in general expect H to be dense in $\widehat{H}$. But we are saved by [8, Section 1.5] to the effect that $\widehat{g}$ can be made *special* over H in the sense of [ADH, 3.4], that is, some nontrivial convex subgroup Δ of the value group of H is cofinal in $v(\widehat{g} - H)$. Then passing to the Δ-specializations of the various valued differential fields encountered above (see [ADH, 9.4]) we regain density and this allows us to implement the desired approximation. The technical details are carried out in the first three sections in Part 4 of [8]. A minor obstacle to obtain the necessary specialness of $\widehat{g}$ is that the hole $(Q, 1, \widehat{g})$ in H may not be of minimal complexity. This can be ameliorated by using a differential polynomial of minimal complexity vanishing at $(H, \widehat{g})$ instead of Q, in the process replacing the hole $(Q, 1, \widehat{g})$ in H by a slot in H, which we then aim to approximate by a *strongly split-normal* slot in H; see [8, Definition 4.3.21] (or Section 1 below). Another caveat: to carry out our approximation scheme we require $\deg P > 1$. Fortunately, if $\deg P = 1$, then necessarily $r = \operatorname{order} P = 1$, and this case can be dealt with through separate (non-trivial) arguments: see Section 11 where we finish the proof of Theorem B.

Enlarging the Hardy field. Now suppose we have finally arranged things so that our Fixed Point Theorem applies: it delivers $g \in \mathcal{C}^{<\infty}$ such that $Q(g) = 0$ and $g \prec 1$. (Notation: for a germ $\phi \in \mathcal{C}^{<\infty}[\mathrm{i}]$ and $0 \neq \mathfrak{n} \in H$ we write $\phi \prec \mathfrak{n}$ if $\phi(t)/\mathfrak{n}(t) \to 0$ as $t \to +\infty$.) However, in order that g generates a proper Hardy field extension $H\langle g \rangle$ of H isomorphic to $H\langle \widehat{g} \rangle$ by an isomorphism over H sending g to $\widehat{g}$ requires that g and $\widehat{g}$ have similar asymptotic properties with respect to the elements of H. For example, suppose $h, \mathfrak{n} \in H$ and $\widehat{g} - h \prec \mathfrak{n} \preccurlyeq 1$; then we must show $g - h \prec \mathfrak{n}$. (Of course, we need to show much more about the asymptotic behavior of g, and this is expressed using the notion of *asymptotic similarity*: see Sections 10 and 11.) Now the germ $(g - h)/\mathfrak{n} \in \mathcal{C}^{<\infty}$ is a zero of the conjugated differential polynomial $Q_{+h, \times \mathfrak{n}} \in H\{Y\}$, as is the element $(\widehat{g} - h)/\mathfrak{n} \prec 1$ of $\widehat{H}$. The Fixed Point Theorem can also be used to produce a zero $y \prec 1$ of $Q_{+h, \times \mathfrak{n}}$ in $\mathcal{C}^{<\infty}$. Set $g_1 := y\mathfrak{n} + h$; then $Q(g) = Q(g_1) = 0$ and $g, g_1 \prec 1$. We are thus naturally lead to consider the difference $g - g_1$ between the solutions $g, g_1 \in \mathcal{C}^{<\infty}$ of the differential equation (with asymptotic side condition)

$$\text{(E)} \qquad Q(Y) = 0, \qquad Y \prec 1.$$

If we manage to show $g - g_1 \prec \mathfrak{n}$, then $g - h = (g - g_1) - y\mathfrak{n} \prec \mathfrak{n}$ as required. Simple estimates coming out of the proof of the Fixed Point Theorem are not good enough for this (cf. Lemma 6.4). We need a generalization of the Fixed Point Theorem for *weighted norms* with weight function given by a representative of $\mathfrak{n}$; this is what Section 9 is about. To render this generalized version useful, we also have to construct a right-inverse A^{-1} of the linear differential operator $A \in H[\partial]$ that depends in some sense uniformly on $\mathfrak{n}$. This is also carried out in Section 9, by refining our approximation arguments, in the process improving strong split-normality to *strong repulsive-normality* as defined in [8, 4.5.32] (see also Section 1).

Exponential sums. Just for this discussion, call $\phi \in \mathcal{C}^{<\infty}[i]$ *small* if $\phi \prec \mathfrak{n}$ for all $\mathfrak{n} \in H$ with $v\mathfrak{n} \in v(\widehat{g} - H)$. Thus our aim is to show that differences between solutions of (E) in $\mathcal{C}^{<\infty}$ are small in this sense. We show that each such difference gives rise to a zero $z \in \mathcal{C}^{<\infty}[i]$ of A with $z \prec 1$ whose smallness would imply the smallness of the difference under consideration. To ensure that every zero $z \prec 1$ of A is indeed small requires us to have performed beforehand yet another (rather unproblematic) normalization procedure on our slot, transforming it into *ultimate* shape. (See [8, Section 4.4] or Section 1 below.) Recall the special fundamental systems of solutions to linear differential equations over maximal Hardy fields explained after Corollary 8: since A splits over K, our zero z of A is a $\mathbb{C}$-linear combination of exponential terms. In [8] we introduced a formalism to deal with such exponential sums over K, based on the concept of the *universal exponential extension* of a differential field. This is interpreted analytically in Section 4. From an asymptotic condition like $z \prec 1$ we need to obtain asymptotic information about the summands of z when expressed as an exponential sum. For this we exploit facts about uniform distribution mod 1 for germs in Hardy fields due to Boshernitzan [25]; see Sections 3 and 4.

This finishes the sketch of the proof of our main result in Sections 1–11. The remaining Sections 12 and 13 contain the applications of this theorem discussed earlier in this introduction: In Section 12 we show how to transfer first-order logical properties of the differential field $\mathbb{T}$ of transseries to maximal Hardy fields, proving in particular Theorem A and Corollaries 1–5, 7, and 8, as well as the first part of Corollary 6. In Section 13 we then investigate embeddings of Hardy fields into $\mathbb{T}$, finish the proof of Corollary 6, and determine the universal theory of Hardy fields, in the course establishing a conjecture from [1].

Previous work. This paper depends essentially on [8, 9] (and on our book [ADH]). Here are some earlier special cases of our results: Theorem A for P of order 1 is in [34]. By [55] there exists a Hardy field $H \supseteq \mathbb{R}$ isomorphic as an ordered differential field to $\mathbb{T}_{\mathrm{g}}$, so by [54] this H has the intermediate value property for all differential polynomials over it. Boshernitzan [24, Remark on p. 117] states a consequence of Corollary 7: if a_1, a_2, b are elements of a Hardy field H, then some y in a Hardy field extension of H satisfies $y'' + a_1y' + a_2y = b$.

Notations and conventions. We follow the conventions from [ADH]. Thus m, n range over the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. Given an additively written abelian group A we set $A^{\neq} := A \setminus \{0\}$, and given a commutative ring R (always with identity 1) we let $R^\times$ be the multiplicative group of units of R. (So if K is a field, then $K^{\neq} = K^\times$.) If R is a differential ring (by convention containing $\mathbb{Q}$ as a subring) and $y \in R^\times$, then $y^\dagger = y'/y$ denotes the logarithmic derivative of y, and $R^\dagger := \{y^\dagger : y \in R^\times\}$, an additive subgroup of R. The prefix "d" abbreviates "differentially"; for example, "d-algebraic" means "differentially algebraic". A differential polynomial $P(Y) \in R\{Y\} = R[Y, Y', Y'', \dots]$ of order $\leqslant r \in \mathbb{N}$ is often expressed as

$$P \;=\; \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}, \qquad (Y^{\boldsymbol{i}} \;:=\; Y^{i_0}(Y')^{i_1}\cdots(Y^{(r)})^{i_r})$$

where $\boldsymbol{i} = (i_0, \dots, i_r)$ ranges over $\mathbb{N}^{1+r}$, with coefficients $P_{\boldsymbol{i}} \in R$, and $P_{\boldsymbol{i}} \neq 0$ for only finitely many $\boldsymbol{i}$. Also $|\boldsymbol{i}| := i_0 + \cdots + i_r$, $\|\boldsymbol{i}\| := i_1 + 2i_2 + \cdots + ri_r$ for such $\boldsymbol{i}$.

1. Preliminaries

For ease of reference and convenience of the reader we summarize in this section much of the asymptotic differential algebra from [8] that we need later.

The universal exponential extension. *In this subsection K is a differential field with algebraically closed constant field C and divisible group $K^\dagger$ of logarithmic derivatives.* (These conditions are satisfied if K is an algebraically closed differential field.) An *exponential extension* of K is a differential ring extension R of K such that $R = K[E]$ for some $E \subseteq R^\times$ with $E^\dagger \subseteq K$. By [8, Section 2.2, especially Corollary 2.2.11]:

Proposition 1.1. *There is an exponential extension U of K with $C_{\mathrm{U}} = C$ such that every exponential extension R of K with $C_R = C$ embeds into U over K; any two such exponential extensions of K are isomorphic over K.*

We call U the *universal exponential extension* of K, denoted by U_K if we want to stress the dependence on K. Here is how we constructed U in [8, Section 2.2]: First, take a *complement* Λ of $K^\dagger$, that is, a $\mathbb{Q}$-linear subspace of K such that $K = K^\dagger \oplus \Lambda$ (internal direct sum of $\mathbb{Q}$-linear subspaces of K). Below λ ranges over Λ. Next, let $\mathrm{e}(\Lambda)$ be a multiplicatively written abelian group, isomorphic to the additive subgroup Λ of K, with isomorphism $\lambda \mapsto \mathrm{e}(\lambda) \colon \Lambda \to \mathrm{e}(\Lambda)$, and let $\mathrm{U} := K\big[\mathrm{e}(\Lambda)\big]$ be the group ring of $\mathrm{e}(\Lambda)$ over K, an integral domain. As K-linear space, $\mathrm{U} = \bigoplus_\lambda K\,\mathrm{e}(\lambda)$ (an internal direct sum of K-linear subspaces), so for every $f \in \mathrm{U}$ we have a unique family (f_λ) in K with $f_\lambda = 0$ for all but finitely many λ and

$$f \;=\; \sum_\lambda f_\lambda\,\mathrm{e}(\lambda). \tag{1.1}$$

We turn U into a differential ring extension of K such that $\mathrm{e}(\lambda)' = \lambda\,\mathrm{e}(\lambda)$ for all λ. Then U is the universal exponential extension of K; in fact, by [8, Lemma 2.2.9]:

Lemma 1.2. *Let R be an exponential extension of K with $C_R = C$, and set $\Lambda_0 := \Lambda \cap (R^\times)^\dagger$, a subgroup of Λ. Then there is a morphism $K\big[\mathrm{e}(\Lambda_0)\big] \to R$ of differential rings over K, and any such morphism is an isomorphism.*

We denote the differential fraction field of U by $\Omega = \Omega_K$; then $C_\Omega = C$ by [8, remark before Lemma 2.2.7]. In particular, $C_{\mathrm{U}} = C$; moreover, $\mathrm{U}^\times = K^\times\,\mathrm{e}(\Lambda)$ and thus $(\mathrm{U}^\times)^\dagger = K^\dagger + \Lambda = K$, by [8, remark before Example 2.2.4]. These properties also characterize U up to isomorphism over K, by [8, Corollary 2.2.10]:

Corollary 1.3. *Every exponential extension U of K with $C_U = C$ and $K \subseteq (U^\times)^\dagger$ is isomorphic to* U *over K.*

Our interest in U has to do with factoring linear differential operators in $K[\partial]$: Let $A \in K[\partial]^{\neq}$ and $r := \operatorname{order}(A)$, and consider $\ker_{\mathrm{U}} A = \{f \in \mathrm{U} : A(f) = 0\}$, a C-linear subspace of U. By [8, Lemma 2.4.1], $\ker_{\mathrm{U}} A$ has a basis contained in $\mathrm{U}^\times$. Moreover, $\dim_C \ker_{\mathrm{U}} A \leqslant r$, and assuming that K is 1-linearly surjective when $r \geqslant 2$, we have by [8, Corollary 2.4.8]:

$$\dim_C \ker_{\mathrm{U}} A = r \iff A \text{ splits over } K. \tag{1.2}$$

Let $v\colon K^\times \to \Gamma$ be a valuation on K. Call the valuation $v_{\mathrm{g}}\colon \mathrm{U}^{\neq} \to \Gamma$ on U such that $v_{\mathrm{g}} f = \min_\lambda v f_\lambda$ for $f \in \mathrm{U}^{\neq}$ as in (1.1) the *gaussian extension* of v to U; cf. [8, Proposition 2.1.3]. We denote by $\preccurlyeq_{\mathrm{g}}$ the dominance relation on Ω associated to the extension of v_{g} to a valuation on Ω, with corresponding asymptotic relations $\asymp_{\mathrm{g}}$ and $\prec_{\mathrm{g}}$; cf. [ADH, (3.1.1)]. The valued differential field K may be asymptotic with small derivation, while Ω with the above valuation has neither one of these properties [8, example before Lemma 2.5.1]. If K is d-valued of H-type with small derivation and asymptotic integration, then $|v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A)| \leqslant \dim_C \ker_{\mathrm{U}} A$ by [8, Lemma 2.5.11], where $|S|$ is the cardinality of a set S. Next a variant of (1.2) which follows from [8, Corollary 2.4.18]:

Lemma 1.4. *If H is a Liouville closed H-field, $r = 2$, and $A \in H[\partial]$ splits over $K := H[\mathrm{i}]$, $\mathrm{i}^2 = -1$, then $\dim_C \ker_{\mathrm{U}} A = r$.*

The span of a linear differential operator. Let K be a valued differential field with small derivation, and $A = a_0 + a_1\partial + \cdots + a_r\partial^r \in K[\partial]$ where $a_0, \dots, a_r \in K$, $a_r \neq 0$. The *span* $\mathfrak{v}(A)$ of A is defined as

$$\mathfrak{v}(A) \ := \ a_r/a_m \in \ K^\times \qquad \text{where } m := \operatorname{dwt}(A).$$

Note that $\mathfrak{v}(A) \preccurlyeq 1$. The next result is [8, Lemma 3.1.1] and says that $\mathfrak{v}(A)$ does not change much under small additive pertubations of A:

Lemma 1.5. *If $B \in K[\partial]$, $\operatorname{order}(B) \leqslant r$ and $B \prec \mathfrak{v}(A)A$, then:*

- (i) $A + B \sim A$, $\operatorname{dwm}(A+B) = \operatorname{dwm}(A)$, *and* $\operatorname{dwt}(A+B) = \operatorname{dwt}(A)$*;*
- (ii) $\operatorname{order}(A+B) = r$ *and* $\mathfrak{v}(A+B) \sim \mathfrak{v}(A)$.

The span of a linear differential operator serves as a "yardstick" for approximation arguments in [8]. Another important use of $\mathfrak{v}(A)$ is in bounding the factors in a splitting of A [8, Corollary 3.1.6]. To state this, let $(g_1, \dots, g_r) \in K^r$ be a splitting of A over K, that is, $A = f(\partial - g_1)\cdots(\partial - g_r)$ where $f \in K^\times$. Then

$$g_1, \dots, g_r \ \preccurlyeq \ \mathfrak{v}(A)^{-1}. \tag{1.3}$$

Suppose $K = H[\mathrm{i}]$ where H is a real closed H-field and $\mathrm{i}^2 = -1$. Then we call the splitting $(g_1, \dots, g_r)$ of A over K *strong* if $\operatorname{Re} g_1, \dots, \operatorname{Re} g_r \succcurlyeq \mathfrak{v}(A)^\dagger$. We say that A *splits strongly* over K if it has a strong splitting over K; see [8, Section 4.2].

Holes and slots. *In this subsection K is an H-asymptotic field with small derivation and rational asymptotic integration, and $\Gamma := v(K^\times)$.* We let a, b range over K and $\mathfrak{m}$, $\mathfrak{n}$ over $K^\times$. A *hole* in K is a triple $(P, \mathfrak{m}, \widehat{a})$ with $P \in K\{Y\} \setminus K$ and $\widehat{a} \in \widehat{K} \setminus K$ for some immediate asymptotic extension $\widehat{K}$ of K, such that $\widehat{a} \prec \mathfrak{m}$ and $P(\widehat{a}) = 0$. A *slot* in K is a triple $(P, \mathfrak{m}, \widehat{a})$ where $P \in K\{Y\} \setminus K$ and $\widehat{a}$ is an element of $\widehat{K} \setminus K$, for some immediate asymptotic extension $\widehat{K}$ of K, such that $\widehat{a} \prec \mathfrak{m}$

and $P \in Z(K, \widehat{a})$. The *order*, *degree*, and *complexity* of a slot $(P, \mathfrak{m}, \widehat{a})$ in K are defined to be the order, degree, and complexity of the differential polynomial P, respectively, and its *linear part* is the linear part $L_{P_{\times\mathfrak{m}}} \in K[\partial]$ of $P_{\times\mathfrak{m}}$. Every hole in K is a slot in K, by [8, Lemma 3.2.2]. If ϕ is active in K and $(P, \mathfrak{m}, \widehat{a})$ is a slot in K (respectively, a hole in K), then $(P^{\phi}, \mathfrak{m}, \widehat{a})$ is a slot in K^{ϕ} (respectively a hole in K^{ϕ}) of the same complexity as $(P, \mathfrak{m}, \widehat{a})$.

A hole in K is *minimal* if no hole in K has smaller complexity. A slot $(P, \mathfrak{m}, \widehat{a})$ in K is Z*-minimal* if P is of minimal complexity among elements of $Z(K, \widehat{a})$. By [8, Lemma 3.2.2], minimal holes in K are Z-minimal slots in K. Slots $(P, \mathfrak{m}, \widehat{a})$ and $(Q, \mathfrak{n}, \widehat{b})$ in K are said to be *equivalent* if $P = Q$, $\mathfrak{m} = \mathfrak{n}$, and $v(\widehat{a}-a) = v(\widehat{b}-a)$ for all a. This is an equivalence relation on the class of slots in K. Each Z-minimal slot in K is equivalent to a Z-minimal hole in K, by [8, Lemma 3.2.14].

In the rest of this subsection $(P, \mathfrak{m}, \widehat{a})$ is a slot in K of order $r \geqslant 1$. For a, $\mathfrak{n}$ such that $\widehat{a} - a \prec \mathfrak{n} \preccurlyeq \mathfrak{m}$ we obtain a slot $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ in K of the same complexity as $(P, \mathfrak{m}, \widehat{a})$, and slots of this form are said to *refine* $(P, \mathfrak{m}, \widehat{a})$ and are called *refinements* of $(P, \mathfrak{m}, \widehat{a})$. Here is [8, Corollary 3.2.29]:

Lemma 1.6. *Suppose K is* d*-valued and* ω*-free, Γ is divisible, L is a newtonian H-asymptotic extension of K, and $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal. Then there exists $\widehat{b} \in L$ such that $K\langle\widehat{b}\rangle$ is an immediate extension of K and $(P, \mathfrak{m}, \widehat{b})$ is a hole in K equivalent to $(P, \mathfrak{m}, \widehat{a})$. If $(P, \mathfrak{m}, \widehat{a})$ is also a hole in K, then there is an embedding $K\langle\widehat{a}\rangle \to L$ of valued differential fields over K.*

Set $w := \operatorname{wt}(P)$, and if $\operatorname{order}(L_{P_{\times\mathfrak{m}}}) = r$, set $\mathfrak{v} := \mathfrak{v}(L_{P_{\times\mathfrak{m}}})$. We call $(P, \mathfrak{m}, \widehat{a})$

(1) *quasilinear* if $\operatorname{ndeg} P_{\times\mathfrak{m}} = 1$;
(2) *special* if some nontrivial convex subgroup of Γ is cofinal in $v\big(\frac{\widehat{a}}{\mathfrak{m}} - K\big)$;
(3) *steep* if $\operatorname{order}(L_{P_{\times\mathfrak{m}}}) = r$ and $\mathfrak{v} \prec^{\flat} 1$; and
(4) *deep* if it is steep and for all active $\phi \preccurlyeq 1$ in K, we have $\operatorname{ddeg} S_{P^{\phi}_{\times\mathfrak{m}}} = 0$ (hence $\operatorname{ndeg} S_{P_{\times\mathfrak{m}}} = 0$) and $\operatorname{ddeg} P^{\phi}_{\times\mathfrak{m}} = 1$ (hence $\operatorname{ndeg} P_{\times\mathfrak{m}} = 1$).

(Here, $S_Q \in K\{Y\}$ denotes the separant of a differential polynomial $Q \in K\{Y\}$.) From [8, Lemma 3.2.36] we recall a way to obtain special holes in K:

Lemma 1.7. *Suppose K is r-linearly newtonian, and* ω*-free if $r > 1$. If $(P, \mathfrak{m}, \widehat{a})$ is quasilinear, and Z-minimal or a hole in K, then $(P, \mathfrak{m}, \widehat{a})$ is special.*

Next some important approximation properties of special Z-minimal slots in K (cf. [8, Lemma 3.2.37 and Corollary 3.3.15]):

Proposition 1.8. *With $\mathfrak{m} = 1$, if $(P, 1, \widehat{a})$ is special and Z-minimal, and $\widehat{a} - a \preccurlyeq \mathfrak{n} \prec 1$ for some a, then $\widehat{a} - b \prec \mathfrak{n}^{r+1}$ for some b, and $P(b) \prec \mathfrak{n}P$ for any such b.*

Proposition 1.9. *If $(P, \mathfrak{m}, \widehat{a})$ is deep, special, and Z-minimal, then for all $n \geqslant 1$ there is an a with $\widehat{a} - a \prec \mathfrak{v}^n\mathfrak{m}$, where $\mathfrak{v} := \mathfrak{v}(L_{P_{\times\mathfrak{m}}})$.*

For $Q \in K\{Y\}$ we denote by Q_d the homogeneous part of degree $d \in \mathbb{N}$ of Q, and we set $Q_{>1} := \sum_{d>1} Q_d$, and likewise with $\geqslant$ or $\neq$ in place of $>$. We say that $(P, \mathfrak{m}, \widehat{a})$ is *normal* if it is steep and $(P_{\times\mathfrak{m}})_{>1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$, and that $(P, \mathfrak{m}, \widehat{a})$ is *strictly normal* if it is steep and $(P_{\times\mathfrak{m}})_{\neq 1} \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$. Here and below, $\Delta(\mathfrak{v}) := \big\{\gamma \in \Gamma : \gamma = o(v\mathfrak{v})\big\}$, a convex subgroup of Γ (cf. [8, Part 3]).

Normalizing minimal holes. *In the rest of this section H is a real closed H-field with small derivation and asymptotic integration. Let $C := C_H$ be its constant field and $\Gamma := v(H^\times)$ its value group, with γ ranging over Γ.* We also let $\widehat{H}$ be an immediate asymptotic extension of H and i an element of an asymptotic extension of $\widehat{H}$ with $i^2 = -1$. Then $\widehat{H}$ is also an H-field and $i \notin \widehat{H}$. Moreover, the d-valued field $K := H[i]$ is an algebraic closure of H with $C_K = C[i]$, and $\widehat{K} := \widehat{H}[i]$ is an immediate asymptotic extension of K.

Lemma 4.2.15 in [8] complements Lemma 1.6: if H is ω-free, then every Z-minimal slot in K of positive order is equivalent to a hole $(P, \mathfrak{m}, \widehat{a})$ in K such that $\widehat{a} \in \widehat{K} \setminus K$ for a suitable choice of ω-free $\widehat{H}$ as above.

For the next two results (Lemmas 4.3.31 and 4.3.32 in [8]) we assume that H is ω-free and $(P, \mathfrak{m}, \widehat{a})$ is a minimal hole in K of positive order, with $\mathfrak{m} \in H^\times$ and $\widehat{a} \in \widehat{K} \setminus K$. We let a range over K and $\mathfrak{n}$ over $H^\times$.

Lemma 1.10. *For some refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ and active ϕ in H with $0 < \phi \preccurlyeq 1$, the hole $(P^\phi_{+a}, \mathfrak{n}, \widehat{a} - a)$ in K^ϕ is deep and normal, its linear part splits strongly over K^ϕ, and it is moreover strictly normal if* $\deg P > 1$.

With further hypotheses on K we can also achieve strict normality when $\deg P = 1$:

Lemma 1.11. *If $\partial K = K$ and $\mathrm{I}(K) \subseteq K^\dagger$, then there is a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ and an active ϕ in H with $0 < \phi \preccurlyeq 1$ such that the hole $(P^\phi_{+a}, \mathfrak{n}, \widehat{a} - a)$ in K^ϕ is deep and strictly normal, and its linear part splits strongly over K^ϕ.*

Remark 1.12. Instead of assuming that H is ω-free and $(P, \mathfrak{m}, \widehat{a})$ is a minimal hole in K of positive order, assume that H is λ-free and $(P, \mathfrak{m}, \widehat{a})$ is a slot in K of order and degree 1. Then Lemma 1.11 goes through with "hole" replaced by "slot"; cf. [8, remark after Lemma 4.3.32].

Ultimate exceptional values, ultimate slots. *Let H be a Liouville closed H-field with small derivation.* By the discussion at the beginning of Section 4.4 of [8], the subspace $K^\dagger$ of the $\mathbb{Q}$-linear space K has a complement Λ such that $\Lambda \subseteq Hi$. We fix such Λ and let $\mathrm{U} := K\big[\mathrm{e}(\Lambda)\big]$ be the universal exponential extension of K as above, with differential fraction field $\Omega := \mathrm{Frac}(\mathrm{U})$. If $\mathrm{I}(K) \subseteq K^\dagger$, then we may additionally choose $\Lambda = \Lambda_H i$ where Λ_H is a complement of the subspace $\mathrm{I}(H)$ of the C-linear space H; see [8, remarks before Lemma 4.4.5]. Let $A \in K[\partial]^{\neq}$ and $r := \mathrm{order}(A)$. For each λ the operator

$$A_\lambda \ := \ A_{\ltimes \mathrm{e}(\lambda)} \ = \ \mathrm{e}(-\lambda) A\, \mathrm{e}(\lambda) \in \Omega[\partial]$$

has coefficients in K, by [ADH, 5.8.8]. We call the elements of the set

$$\mathscr{E}^{\mathrm{u}}(A) \ = \ \mathscr{E}^{\mathrm{u}}_K(A) \ := \ \bigcup_\lambda \mathscr{E}^{\mathrm{e}}(A_\lambda) \ \subseteq \ \Gamma$$

the *ultimate exceptional values of* A with respect to Λ. Thus $\mathscr{E}^{\mathrm{e}}(A) \subseteq \mathscr{E}^{\mathrm{u}}(A)$, and by [8, (2.5.2)], we have $v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) \subseteq \mathscr{E}^{\mathrm{u}}(A)$.

In the rest of this subsection we assume $\mathrm{I}(K) \subseteq K^\dagger$. (By [8, Proposition 1.7.28], this holds if K is 1-linearly newtonian.) Then $\mathscr{E}^{\mathrm{u}}(A)$ does not depend on our choice of Λ [8, Corollary 4.4.1]. Moreover, by [8, Lemma 4.4.4] we have $|\mathscr{E}^{\mathrm{u}}(A)| \leqslant r$, and

$$\dim_{C[i]} \ker_{\mathrm{U}} A = r \implies v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) = \mathscr{E}^{\mathrm{u}}(A). \tag{1.4}$$

Let $(P,\mathfrak{m},\widehat{a})$ be a slot in H of order $r\geqslant 1$ in H, where $\widehat{a}\in\widehat{H}\setminus H$. We call $(P,\mathfrak{m},\widehat{a})$ *ultimate* if for all $a\prec\mathfrak{m}$ in H,

$$\operatorname{order}(L_{P_{+a}})=r \ \text{ and } \ \mathscr{E}^{\mathrm{u}}(L_{P_{+a}})\cap v(\widehat{a}-H) \ < \ v(\widehat{a}-a).$$

Sometimes, this ultimate condition can be simplified: by [8, Lemmas 4.4.12, 4.4.13], if $(P,\mathfrak{m},\widehat{a})$ is normal or $\deg P=1$, then

$$(P,\mathfrak{m},\widehat{a}) \text{ is ultimate} \quad\Longleftrightarrow\quad \mathscr{E}^{\mathrm{u}}(L_P)\cap v(\widehat{a}-H)\leqslant v\mathfrak{m}. \tag{1.5}$$

Similarly, we call a slot $(P,\mathfrak{m},\widehat{a})$ of order $r\geqslant 1$ in K, where $\widehat{a}\in\widehat{K}\setminus K$, *ultimate* if for all $a\prec\mathfrak{m}$ in K we have

$$\operatorname{order}(L_{P_{+a}})=r \ \text{ and } \ \mathscr{E}^{\mathrm{u}}(L_{P_{+a}})\cap v(\widehat{a}-K) \ < \ v(\widehat{a}-a).$$

Every refinement of an ultimate slot in H remains ultimate, and likewise with K in place of H [8, Lemma 4.4.10 and remarks after Lemma 4.4.17]. By [8, Propositions 4.4.14, 4.4.18, and Remarks 4.4.15, 4.4.19] we have:

Proposition 1.13. *Let $(P,\mathfrak{m},\widehat{a})$ with $\widehat{a}\in\widehat{H}\setminus H$ be a slot in H of positive order. If $(P,\mathfrak{m},\widehat{a})$ is normal or $\deg P=1$, then $(P,\mathfrak{m},\widehat{a})$ has an ultimate refinement. Likewise, for any slot $(P,\mathfrak{m},\widehat{a})$ with $\widehat{a}\in\widehat{K}\setminus K$ in K of positive order.*

Repulsive splitting. In this subsection f ranges over K and $\mathfrak{m}$ over $H^{\times}$.

Definition 1.14. We say that f is *attractive* if $\operatorname{Re} f\succcurlyeq 1$ and $\operatorname{Re} f<0$, and *repulsive* if $\operatorname{Re} f\succcurlyeq 1$ and $\operatorname{Re} f>0$. Given $\gamma>0$ we also say that f is *γ-repulsive* if $\operatorname{Re} f>0$ or $\operatorname{Re} f\succ\mathfrak{m}^{\dagger}$ for all $\mathfrak{m}$ with $\gamma=v\mathfrak{m}$.

The following is [8, Corollary 4.5.5]:

Lemma 1.15. *Suppose f is γ-repulsive where $\gamma=v\mathfrak{m}>0$, and $\operatorname{Re} f\succcurlyeq 1$. Then f is repulsive iff $f-\mathfrak{m}^{\dagger}$ is repulsive, and f is attractive iff $f-\mathfrak{m}^{\dagger}$ is attractive.*

Given $\widehat{a}\in\widehat{H}\setminus H$, we also say that f is *$\widehat{a}$-repulsive* if it is γ-repulsive for each $\gamma\in v(\widehat{a}-H)\cap\Gamma^{>}$, that is, $\operatorname{Re} f>0$ or $\operatorname{Re} f\succ\mathfrak{m}^{\dagger}$ for all $a\in H$ and $\mathfrak{m}$ with $\mathfrak{m}\asymp\widehat{a}-a\prec 1$. Let also $A\in K[\partial]^{\neq}$ have order $r\geqslant 1$. A splitting $(g_1,\dots,g_r)$ of A over K is *$\widehat{a}$-repulsive* if $g_1,\dots,g_r$ are $\widehat{a}$-repulsive. If there is an $\widehat{a}$-repulsive splitting of A over K, then A is said to *split $\widehat{a}$-repulsively* over K. See [8, Section 4.5] for more about these notions, and Sections 5 and 9 below for their analytic significance.

Split-normal and repulsive-normal slots. Let $(P,\mathfrak{m},\widehat{a})$ be a steep slot in H of order $r\geqslant 1$ with $\widehat{a}\in\widehat{H}\setminus H$. Set $L:=L_{P_{\times\mathfrak{m}}}$, $\mathfrak{v}:=\mathfrak{v}(L)$, and $w:=\operatorname{wt}(P)$. Note that for $Q\in K\{Y\}$ we have $Q_{\geqslant 1}=Q-Q(0)$. We say that $(P,\mathfrak{m},\widehat{a})$ is

(1) *split-normal* if $(P_{\times\mathfrak{m}})_{\geqslant 1}=Q+R$ where $Q,R\in H\{Y\}$, Q is homogeneous of degree 1 and order r, L_Q splits over K, and $R\prec_{\Delta(\mathfrak{v})}\mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$;
(2) *strongly split-normal* if $P_{\times\mathfrak{m}}=Q+R$, $Q,R\in H\{Y\}$, Q homogeneous of degree 1 and order r, L_Q splits strongly over K, and $R\prec_{\Delta(\mathfrak{v})}\mathfrak{v}^{w+1}(P_{\times\mathfrak{m}})_1$;
(3) *repulsive-normal* if (1) holds with "splits $\widehat{a}/\mathfrak{m}$-repulsively over K" in place of "splits over K"; and
(4) *strongly repulsive-normal* if (2) holds with "has a strong $\widehat{a}/\mathfrak{m}$-repulsive splitting over K" in place of "splits strongly over K".

The following diagram summarizes some dependencies among these properties:

$$\begin{array}{ccccc} \text{strongly repulsive-normal} & \Longrightarrow & \text{strongly split-normal} & \Longrightarrow & \text{strictly normal} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{repulsive-normal} & \Longrightarrow & \text{split-normal} & \Longrightarrow & \text{normal} \end{array}$$

Here is the Normalization Theorem from [8] that is crucial in Section 11:

Theorem 1.16. *Suppose H is $\upomega$-free, Liouville closed, with archimedean ordered constant field C, and 1-linearly newtonian algebraic closure $K = H[\mathrm{i}]$. If H is not newtonian, then for some Z-minimal special hole $(Q, 1, \widehat{b})$ in H with* $\operatorname{order} Q \geqslant 1$ *and some active $\phi > 0$ in H with $\phi \preccurlyeq 1$, the hole $(Q^{\phi}, 1, \widehat{b})$ in H^{ϕ} is deep, strongly repulsive-normal, and ultimate.*

2. Hardy Fields

In the rest of this paper we freely use the notations about functions and germs introduced in [9, Sections 2 and 3]. We recall some of it in this section, before stating basic extension theorems about Hardy fields needed later. We finish this section with a general fact about bounding the derivatives of solutions to linear differential equations, based on [38, 59].

Functions and germs. Let a range over $\mathbb{R}$ and r over $\mathbb{N}\cup\{\omega,\infty\}$. Then $\mathcal{C}^r_a$ denotes the $\mathbb{R}$-algebra of functions $[a,+\infty) \to \mathbb{R}$ which extend to a $\mathcal{C}^r$-function $U \to \mathbb{R}$ for some open $U \supseteq [a,+\infty)$. Here, as usual, $\mathcal{C}^{\omega}$ means "analytic". Hence

$$\mathcal{C}_a \ := \ \mathcal{C}^0_a \ \supseteq \mathcal{C}^1_a \ \supseteq \ \mathcal{C}^2_a \ \supseteq \ \cdots \ \supseteq \ \mathcal{C}^{\infty}_a \ \supseteq \ \mathcal{C}^{\omega}_a.$$

Each subring $\mathcal{C}^r_a$ of $\mathcal{C}$ has its complexification $\mathcal{C}^r_a[\mathrm{i}] = \mathcal{C}^r_a + \mathcal{C}^r_a\mathrm{i}$, a subalgebra of the $\mathbb{C}$-algebra $\mathcal{C}_a[\mathrm{i}]$ of continuous functions $[a,+\infty) \to \mathbb{C}$. For $f \in \mathcal{C}_a[\mathrm{i}]$ we have

$$\overline{f} \ := \ \operatorname{Re} f - \mathrm{i} \operatorname{Im} f \ \in \ \mathcal{C}_a[\mathrm{i}], \qquad |f| \ := \ \sqrt{(\operatorname{Re} f)^2 + (\operatorname{Im} f)^2} \ \in \ \mathcal{C}_a.$$

Let $\mathcal{C}^r$ be the $\mathbb{R}$-algebra of germs at $+\infty$ of functions in $\bigcup_a \mathcal{C}^r_a$. Thus $\mathcal{C} := \mathcal{C}^0$ consists of the germs at $+\infty$ of continuous $\mathbb{R}$-valued functions on intervals $[a,+\infty)$, $a \in \mathbb{R}$, and

$$\mathcal{C} \ = \ \mathcal{C}^0 \ \supseteq \mathcal{C}^1 \ \supseteq \ \mathcal{C}^2 \ \supseteq \ \cdots \ \supseteq \ \mathcal{C}^{\infty} \ \supseteq \ \mathcal{C}^{\omega}.$$

For each r we also have the $\mathbb{C}$-subalgebra $\mathcal{C}^r[\mathrm{i}] = \mathcal{C}^r + \mathcal{C}^r\mathrm{i}$ of $\mathcal{C}[\mathrm{i}]$. For $n \geqslant 1$ we have a derivation $g \mapsto g' \colon \mathcal{C}^n[\mathrm{i}] \to \mathcal{C}^{n-1}[\mathrm{i}]$ such that $(\text{germ of } f)' = (\text{germ of } f')$ for $f \in \bigcup_a \mathcal{C}^n_a[\mathrm{i}]$, and this derivation restricts to a derivation $\mathcal{C}^n \to \mathcal{C}^{n-1}$. Therefore $\mathcal{C}^{<\infty}[\mathrm{i}] := \bigcap_n \mathcal{C}^n[\mathrm{i}]$ is naturally a differential ring with ring of constants $\mathbb{C}$, and $\mathcal{C}^{<\infty} := \bigcap_n \mathcal{C}^n$ is a differential subring of $\mathcal{C}^{<\infty}[\mathrm{i}]$ with ring of constants $\mathbb{R}$. Note that $\mathcal{C}^{<\infty}[\mathrm{i}]$ has $\mathcal{C}^{\infty}[\mathrm{i}]$ as a differential subring, $\mathcal{C}^{<\infty}$ has $\mathcal{C}^{\infty}$ as a differential subring, and the differential ring $\mathcal{C}^{\infty}$ has in turn the differential subring $\mathcal{C}^{\omega}$.

Asymptotic relations. We often use the same notation for a $\mathbb{C}$-valued function on a subset of $\mathbb{R}$ containing an interval $(a,+\infty)$, $a \in \mathbb{R}$, as for its germ if the resulting ambiguity is harmless. With this convention, given a property P of complex numbers and $g \in \mathcal{C}[\mathrm{i}]$ we say that $P\big(g(t)\big)$ *holds eventually* if $P\big(g(t)\big)$ holds for all sufficiently large real t. We equip $\mathcal{C}$ with the partial ordering given by $f \leqslant g :\Leftrightarrow f(t) \leqslant g(t)$, eventually, and equip $\mathcal{C}[i]$ with the asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ defined as follows: for $f, g \in \mathcal{C}[i]$,

$$\begin{aligned} f \preccurlyeq g \quad &:\Longleftrightarrow \quad \text{there exists } c \in \mathbb{R}^{>} \text{ such that } |f| \leqslant c|g|, \\ f \prec g \quad &:\Longleftrightarrow \quad g \in \mathcal{C}[i]^{\times} \text{ and } \lim_{t\to\infty} f(t)/g(t) = 0 \\ &\Longleftrightarrow \quad g \in \mathcal{C}[i]^{\times} \text{ and } |f| \leqslant c|g| \text{ for all } c \in \mathbb{R}^{>}, \\ f \sim g \quad &:\Longleftrightarrow \quad g \in \mathcal{C}[i]^{\times} \text{ and } \lim_{t\to\infty} f(t)/g(t) = 1 \\ &\Longleftrightarrow \quad f - g \prec g. \end{aligned}$$

We also use these notations for functions in $\mathcal{C}_a$ ($a \in \mathbb{R}$); for example, for $f \in \mathcal{C}_a$ and $g \in \mathcal{C}_b$ ($a, b \in \mathbb{R}$), $f \preccurlyeq g$ means: (germ of f) $\preccurlyeq$ (germ of g).

Let now H be a *Hausdorff field*, that is, a subring of $\mathcal{C}$ that happens to be a field. Then the restriction of the partial ordering of $\mathcal{C}$ to H is total and makes H into an ordered field. The ordered field H has a convex subring $\mathcal{O} := \{f \in H : f \preccurlyeq 1\}$, which is a valuation ring of H, and we consider H accordingly as a valued ordered field. Moreover, $H[i]$ is a subfield of $\mathcal{C}[i]$, and $\mathcal{O} + \mathcal{O}i = \{f \in H[i] : f \preccurlyeq 1\}$ is the unique valuation ring of $H[i]$ whose intersection with H is $\mathcal{O}$. In this way we consider $H[i]$ as a valued field extension of H. The asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ on $\mathcal{C}[i]$ restricted to $H[i]$ are exactly the asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ on $H[i]$ that $H[i]$ has as a valued field (cf. [ADH, (3.1.1)]; likewise with H in place of $H[i]$.

Hardy fields. In this subsection H is a *Hardy field*: a differential subfield of $\mathcal{C}^{<\infty}$. A germ $y \in \mathcal{C}$ is said to be *hardian* if it lies in a Hardy field, and *H-hardian* (or *hardian over H*) if it lies in a Hardy field extension of H. (Thus y is hardian iff y is $\mathbb{Q}$-hardian.) Every Hardy field is a Hausdorff field, and we consider H as an ordered valued differential field with the ordering and valuation on H as above. Hardy fields are pre-H-fields, and H-fields if they contain $\mathbb{R}$. We also equip the differential subfield $H[i]$ of $\mathcal{C}^{<\infty}[i]$ with the unique valuation ring lying over that of H. Then $H[i]$ is a pre-d-valued field of H-type with small derivation, and if $H \supseteq \mathbb{R}$, then $H[i]$ is d-valued with constant field $\mathbb{C}$.

Recall that H is said to be *maximal* if it has no proper Hardy field extension, and that every Hardy field has a maximal Hardy field extension. Due to our focus on d-algebraic Hardy field extensions in this paper, a weaker condition is often more natural for our purposes: we say that H is d-*maximal* if it has no proper d-algebraic Hardy field extension. Zorn yields a d-maximal d-algebraic Hardy field extension of H, hence the intersection $\mathrm{D}(H)$ of all d-maximal Hardy fields containing H is a d-algebraic Hardy field extension of H, called the d-*perfect hull* of H. If $\mathrm{D}(H) = H$, then we say that H is d-*perfect.* So

$$\text{maximal} \quad \Longrightarrow \quad \text{d-maximal} \quad \Longrightarrow \quad \text{d-perfect.}$$

The following fact summarizes some well-known extension theorems from [21, 27, 68, 71, 85] (see also [9, Proposition 4.2]):

Proposition 2.1. *If H is* d*-perfect, then H is real closed with $H \supseteq \mathbb{R}$, and for each $f \in H$ we have* $\exp f \in H$ *and $g' = f$ for some $g \in H$.*

Hence every d-perfect Hardy field is a Liouville closed H-field with constant field $\mathbb{R}$. For the following result, see [9, Corollary 6.12]:

Proposition 2.2. *If H is* d*-perfect, then $K = H[i]$ satisfies* $\mathrm{I}(K) \subseteq K^{\dagger}$.

In [10] we give an example of a d-perfect H which is not $\upomega$-free. However, if H is d-maximal, then H is $\upomega$-free; this is a consequence of the following result from [9]:

Theorem 2.3. *Every Hardy field has a* d*-algebraic* $\upomega$*-free Hardy field extension.*

Recall from the introduction that H is called a $\mathcal{C}^\infty$*-Hardy field,* if $H \subseteq \mathcal{C}^\infty$, and that a $\mathcal{C}^\infty$-Hardy field is called $\mathcal{C}^\infty$*-maximal* if it has no proper $\mathcal{C}^\infty$-Hardy field extension; likewise we defined $\mathcal{C}^\omega$-Hardy fields and $\mathcal{C}^\omega$-maximal Hardy fields. Theorem 2.3 also holds with "Hardy field" replaced by "$\mathcal{C}^\infty$-Hardy field", as well as by "$\mathcal{C}^\omega$-Hardy field". This follows from its proof in [9], and also from general results about smoothness of solutions to algebraic differential equations over H in Section 7 below. By these results every $\mathcal{C}^\infty$-maximal Hardy field is d-maximal, so if $H \subseteq \mathcal{C}^\infty$, then $\mathrm{D}(H) \subseteq \mathcal{C}^\infty$; likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$. (See Corollary 7.8.)

Bounding solutions of linear differential equations. Let $r \in \mathbb{N}^{\geqslant 1}$, and with $\boldsymbol{i}$ ranging over $\mathbb{N}^r$, let

$$P \;=\; P(Y, Y', \dots, Y^{(r-1)}) \;=\; \sum_{\|\boldsymbol{i}\| < r} P_{\boldsymbol{i}} Y^{\boldsymbol{i}} \;\in\; \mathcal{C}[\mathrm{i}]\big[Y, Y', \dots, Y^{(r-1)}\big]$$

with $P_{\boldsymbol{i}} \in \mathcal{C}[\mathrm{i}]$ for all $\boldsymbol{i}$ with $\|\boldsymbol{i}\| < r$, and $P_{\boldsymbol{i}} \neq 0$ for only finitely many such $\boldsymbol{i}$. Then P gives rise to an evaluation map

$$y \mapsto P\big(y, y', \dots, y^{(r-1)}\big) \;:\; \mathcal{C}^{r-1}[\mathrm{i}] \to \mathcal{C}[\mathrm{i}].$$

Let $y \in \mathcal{C}^r[\mathrm{i}]$ satisfy the differential equation

$$y^{(r)} \;=\; P\big(y, y', \dots, y^{(r-1)}\big). \tag{2.1}$$

In addition, $\mathfrak{m}$ with $0 < \mathfrak{m} \preccurlyeq 1$ is a hardian germ, and $\eta \in \mathcal{C}$ is eventually increasing with $\eta(t) > 0$ eventually, and $n \geqslant r$.

Proposition 2.4. *Suppose* $P_{\boldsymbol{i}} \preccurlyeq \eta$ *for all* $\boldsymbol{i}$*,* $P(0) \preccurlyeq \eta\,\mathfrak{m}^n$*, and* $y \preccurlyeq \mathfrak{m}^n$*. Then*

$$y^{(j)} \;\preccurlyeq\; \eta^j \mathfrak{m}^{n-j(1+\varepsilon)} \qquad \text{for } j = 0, \dots, r \text{ and all } \varepsilon \in \mathbb{R}^{>},$$

with $\prec$ *in place of* $\preccurlyeq$ *if* $y \prec \mathfrak{m}^n$ *and* $P(0) \prec \eta\,\mathfrak{m}^n$*.*

Proposition 2.4 for $\mathfrak{m} \asymp 1$ is covered by the following result:

Theorem 2.5 (Landau [59])**.** *Suppose* $y \preccurlyeq 1$ *and* $P_{\boldsymbol{i}} \preccurlyeq \eta$ *for all* $\boldsymbol{i}$*. Then* $y^{(j)} \preccurlyeq \eta^j$ *for* $j = 0, \dots, r$*. Moreover, if* $y \prec 1$*, then* $y^{(j)} \prec \eta^j$ *for* $j = 0, \dots, r-1$*, and if in addition* $P(0) \prec \eta$*, then also* $y^{(r)} \prec \eta^r$*.*

To deduce from Theorem 2.5 the general case of Proposition 2.4 we use:

Lemma 2.6. *Suppose that* $\mathfrak{m} \prec 1$ *and* $z \in \mathcal{C}^r[\mathrm{i}]$*. If* $z^{(j)} \preccurlyeq \eta^j$ *for* $j = 0, \dots, r$*, then* $(z\mathfrak{m}^n)^{(j)} \preccurlyeq \eta^j \mathfrak{m}^{n-j}$ *for* $j = 0, \dots, r$*, and likewise with* $\prec$ *instead of* $\preccurlyeq$*.*

Proof. Corollary 1.1.12 in [8] yields $(\mathfrak{m}^n)^{(m)} \preccurlyeq \mathfrak{m}^{n-m}$ for $m \leqslant n$. Thus if $z^{(j)} \preccurlyeq \eta^j$ for $j = 0, \dots, r$, then

$$z^{(k)} (\mathfrak{m}^n)^{(j-k)} \preccurlyeq \eta^k \mathfrak{m}^{n-(j-k)} \preccurlyeq \eta^j \mathfrak{m}^{n-j} \qquad (0 \leqslant k \leqslant j \leqslant r),$$

so $(z\mathfrak{m}^n)^{(j)} \preccurlyeq \eta^j \mathfrak{m}^{n-j}$ for $j = 0, \dots, r$, by the Product Rule. The argument with $\prec$ instead of $\preccurlyeq$ is similar. □

Proof of Proposition 2.4. We assume $\mathfrak{m} \prec 1$ (as we may). Proposition 2.1 yields a Liouville closed Hardy field $H \supseteq \mathbb{R}$ with $\mathfrak{m} \in H$. For $i = 0, \dots, r$ set

$$Y_i \ := \ \sum_{j=0}^{i} \binom{i}{j} Y^{(i-j)}(\mathfrak{m}^n)^{(j)} \in H\big[Y, Y', \dots, Y^{(i)}\big] \ \subseteq \ \mathcal{C}[\mathrm{i}]\big[Y, Y', \dots, Y^{(r)}\big].$$

Then for $z := y\,\mathfrak{m}^{-n} \preccurlyeq 1$ in $\mathcal{C}^r[\mathrm{i}]$ the Product Rule gives

$$Y_i(z, z', \dots, z^{(i)}) \ = \ (z\,\mathfrak{m}^n)^{(i)} \ = \ y^{(i)} \qquad (i = 0, \dots, r),$$

so with

$$Q \ := \ Y^{(r)} - \mathfrak{m}^{-n}\big(Y_r - P(Y_0, \dots, Y_{r-1})\big) \in \mathcal{C}[\mathrm{i}]\big[Y, Y', \dots, Y^{(r-1)}\big]$$

we have by substitution of $z, \dots, z^{(r)}$ for $Y, Y', \dots, Y^{(r)}$,

$$\begin{aligned} z^{(r)} \ &= \ Q\big(z, z', \dots, z^{(r-1)}\big) + \mathfrak{m}^{-n}\big(y^{(r)} - P(y, y', \dots, y^{(r-1)})\big) \\ &= \ Q\big(z, z', \dots, z^{(r-1)}\big). \end{aligned}$$

In $H\{Y\}$ we have $(Y^{\boldsymbol{j}})_{\times \mathfrak{m}^n} = Y_0^{j_0} \cdots Y_r^{j_r}$ for $\boldsymbol{j} = (j_0, \dots, j_r) \in \mathbb{N}^{1+r}$. Let $\varepsilon \in \mathbb{R}^>$; then $\mathfrak{m}^{-\varepsilon} \in H$. We equip $H\{Y\}$ with the gaussian extension of the valuation of H; see [ADH, 4.5]. Then $\mathfrak{m}^{-n}(Y^{\boldsymbol{j}})_{\times \mathfrak{m}^n} \preccurlyeq \mathfrak{m}^{-\varepsilon}$ for $\boldsymbol{j} \in \mathbb{N}^{1+r} \setminus \{0\}$, by [ADH, 6.1.4]. Take $Q_{\boldsymbol{i}} \in \mathcal{C}[\mathrm{i}]$ for $\|\boldsymbol{i}\| < r$ such that

$$Q \ = \ \sum_{\|\boldsymbol{i}\| < r} Q_{\boldsymbol{i}} Y^{\boldsymbol{i}}, \qquad (Q_{\boldsymbol{i}} \neq 0 \text{ for only finitely many } \boldsymbol{i}).$$

Together with $P_{\boldsymbol{i}} \preccurlyeq \eta$ for all $\boldsymbol{i}$ and $P(0) \preccurlyeq \eta\,\mathfrak{m}^n$, the remarks above yield $Q_{\boldsymbol{i}} \preccurlyeq \eta\,\mathfrak{m}^{-\varepsilon}$ for all $\boldsymbol{i}$. By Theorem 2.5 applied to P, y, η replaced by Q, z, $\eta\,\mathfrak{m}^{-\varepsilon}$, we now obtain $z^{(j)} \preccurlyeq (\eta\,\mathfrak{m}^{-\varepsilon})^j$, with $\prec$ in place of $\preccurlyeq$ if $y \prec \mathfrak{m}^n$ and $P(0) \prec \eta\,\mathfrak{m}^n$. Using Lemma 2.6 with $\eta\,\mathfrak{m}^{-\varepsilon}$ in place of η finishes the proof of Proposition 2.4. □

The following immediate consequence of Proposition 2.4 is used in Section 4. (The case $\mathfrak{m} = 1$ is due to Esclangon [38].)

Corollary 2.7. *Suppose $f_1, \dots, f_r \in \mathcal{C}[\mathrm{i}]$ and $y \in \mathcal{C}^r[\mathrm{i}]$ satisfy*

$$y^{(r)} + f_1 y^{(r-1)} + \cdots + f_r y \ = \ 0, \qquad f_1, \dots, f_r \preccurlyeq \eta, \quad y \ \preccurlyeq \ \mathfrak{m}^n.$$

Then $y^{(j)} \preccurlyeq \eta^j \mathfrak{m}^{n - j(1+\varepsilon)}$ for $j = 0, \dots, r$ and all $\varepsilon \in \mathbb{R}^>$, with $\prec$ in place of $\preccurlyeq$ if $y \prec \mathfrak{m}^n$.

3. Hardy Fields and Uniform Distribution

Section 4 gives an analytic description of the universal exponential extension U of the algebraic closure K of a Liouville closed Hardy field extending $\mathbb{R}$. The elements of U are exponential sums with coefficients and exponents in K. To extract asymptotic information about the summands in such a sum we refine here results of Boshernitzan [25] about uniform distribution mod 1 for functions whose germs are in a Hardy field. Our reference for uniform distribution mod 1 is [58, Ch. 1, §9]. We also need some facts about trigonometric polynomials, almost periodic functions, and their mean values. These are treated in [16, 31] mainly in the one-variable case; adaptations to the multivariable case, required here, generally are straightforward.

Throughout this section we assume $n \geqslant 1$. We equip $\mathbb{R}^n$ with its usual Lebesgue measure μ_n, and *measurable* means measurable with respect to μ_n. *For vectors $r = (r_1, \dots, r_n)$ and $s = (s_1, \dots, s_n)$ in $\mathbb{R}^n$ we let $r \cdot s := r_1 s_1 + \cdots + r_n s_n \in \mathbb{R}$ be* *the usual dot product of r and s. We also set $rs := (r_1 s_1, \dots, r_n s_n) \in \mathbb{R}^n$, not to be confused with $r \cdot s \in \mathbb{R}$. Moreover, we let $v, w\colon \mathbb{R}^n \to \mathbb{C}$ be complex-valued functions on $\mathbb{R}^n$, and let $s = (s_1, \dots, s_n)$ range over $\mathbb{R}^n$, and T over $\mathbb{R}^>$.* We set $|s| = |s|_\infty := \max\{|s_1|, \dots, |s_n|\}$ and $\|w\| := \sup_s |w(s)| \in [0, +\infty]$. We shall also have occasion to consider various functions $\mathbb{R}^n \to \mathbb{C}$ obtained from w: $\overline{w}$, $|w|$, as well as w_{+r} and $w_{\times r}$ (for $r \in \mathbb{R}^n$), defined by

$$\overline{w}(s) := \overline{w(s)}, \quad |w|(s) := |w(s)|, \quad w_{+r}(s) := w(r+s), \quad w_{\times r}(s) := w(rs).$$

Almost periodic functions. Let α range over $\mathbb{R}^n$. Call w a *trigonometric polynomial* if there are $w_\alpha \in \mathbb{C}$, with $w_\alpha \neq 0$ for only finitely many α, such that

$$w(s) = \sum_\alpha w_\alpha \, \mathrm{e}^{(\alpha \cdot s)\mathrm{i}} \qquad \text{for all } s. \tag{3.1}$$

The coefficients w_α in (3.1) are uniquely determined by w. (See Corollary 3.6 below.) If w is a trigonometric polynomial, then $\overline{w}$ is a trigonometric polynomial, and for $r \in \mathbb{R}^n$, so are w_{+r} and $w_{\times r}$. The trigonometric polynomials form a subalgebra of the $\mathbb{C}$-algebra of uniformly continuous bounded functions $\mathbb{R}^n \to \mathbb{C}$. The latter is a Banach algebra with respect to $\|\cdot\|$, and the elements of the closure of its subalgebra of trigonometric polynomials with respect to this norm are the *almost periodic* functions (in the sense of Bohr), which therefore form a Banach subalgebra. In particular, if v, w are almost periodic, so are $v + w$ and vw. Moreover, if w is almost periodic, then so are $\overline{w}$ and w_{+r}, $w_{\times r}$ for $r \in \mathbb{R}^n$.

We say that v is 1-*periodic* if $v_{+k} = v$ for all $k \in \mathbb{Z}^n$. If v is continuous and 1-periodic, then v is almost periodic: indeed, by "Stone-Weierstrass" there is for every $\varepsilon \in \mathbb{R}^>$ a 1-periodic trigonometric polynomial w with $\|v - w\| < \varepsilon$. (See [33, (7.4.2)] for the case $n = 1$.)

Mean value. The function w is said to have a *mean value* if w is bounded and measurable, and the limit

$$\lim_{T \to \infty} \frac{1}{T^n} \int_{[0,T]^n} w(s)\, ds \tag{3.2}$$

exists (in $\mathbb{C}$); in that case we call the quantity (3.2) the *mean value* of w and denote it by $M(w)$. One verifies easily that if v and w have a mean value, then so do the functions $v + w$, cw ($c \in \mathbb{C}$), and $\overline{w}$, with

$$M(v + w) = M(v) + M(w), \quad M(cw) = cM(w), \quad \text{and} \quad M(\overline{w}) = \overline{M(w)}.$$

If w has a mean value and $w(\mathbb{R}^n) \subseteq \mathbb{R}$, then $M(w) \in \mathbb{R}$.

Lemma 3.1. *Let $d \in \mathbb{R}^n$. Then w has a mean value iff w_{+d} has a mean value, in which case $M(w) = M(w_{+d})$.*

Proof. It suffices to treat the case $d = (d_1, 0, \dots, 0)$, $d_1 \in \mathbb{R}^>$. For $T > d_1$ we have

$$\left| \int_{[0,T]^n} w_{+d}(s)\, ds - \int_{[0,T]^n} w(s)\, ds \right| =$$

$$\left| \int_{[T, d_1 + T] \times [0,T]^{n-1}} w(s)\, ds - \int_{[0, d_1] \times [0,T]^{n-1}} w(s)\, ds \right| \leqslant 2 d_1 \|w\| T^{n-1},$$

and this yields the claim. $\square$

Corollary 3.2. *Suppose w has a mean value and $T_0 \in \mathbb{R}^>$. If $w(s) = 0$ for all $s \in (\mathbb{R}^{\geqslant})^n$ with $|s| \geqslant T_0$, then $M(w) = 0$. If $w(\mathbb{R}^n) \subseteq \mathbb{R}$ and $w(s) \geqslant 0$ for all $s \in (\mathbb{R}^{\geqslant})^n$ with $|s| \geqslant T_0$, then $M(w) \geqslant 0$.*

Lemma 3.3. *Suppose w has a mean value and $w(\mathbb{R}^n) \subseteq \mathbb{R}$. Then*

$$\liminf_{|s|\to\infty} w(s) \;\leqslant\; M(w) \;\leqslant\; \limsup_{|s|\to\infty} w(s) \qquad \text{with } s \text{ ranging over } (\mathbb{R}^{\geqslant})^n.$$

Proof. Assume $L := \limsup_{|s|\to\infty} w(s) < M(w)$. Let $\varepsilon := \frac{1}{2}\big(M(w) - L\big)$, and take $T_0 \in \mathbb{R}^>$ such that $w(s) \leqslant M(w) - \varepsilon$ for all s with $|s| \geqslant T_0$. The previous corollary then yields $M(w) \leqslant M(w) - \varepsilon$, a contradiction. This shows the second inequality; the first inequality is proved in a similar way. □

The following is routine:

Lemma 3.4. *Let (v_m) be a sequence of bounded measurable functions $\mathbb{R}^n \to \mathbb{C}$ with a mean value, such that $\lim_{m\to\infty} \|v_m - w\| = 0$, and let w be bounded and measurable. Then w has a mean value, and $\lim_{m\to\infty} M(v_m) = M(w)$.*

If $\alpha \in \mathbb{R}^n$ and $w(s) = \mathrm{e}^{\mathrm{i}(\alpha\cdot s)}$ for all s, then w has a mean value, with $M(w) = 1$ if $\alpha = 0$ and $M(w) = 0$ otherwise. Together with Lemma 3.4, this yields the well-known fact that if w is almost periodic, then w has a mean value. (See [31, Theorem 1.12] for the case $n = 1$.) Moreover, using Lemma 3.4 it is also easy to show that if w is almost periodic and $r \in (\mathbb{R}^\times)^n$, then the almost periodic function $w_{\times r}$ has the same mean value as w. If w is continuous and 1-periodic, then w has a mean value, namely $M(w) = \int_{[0,1]^n} w(s)\,ds$. For a proof of the following result in the case $n = 1$, see [31, Theorem 1.19]:

Proposition 3.5 (Bohr)**.** *If w is almost periodic, $w(\mathbb{R}^n) \subseteq \mathbb{R}^{\geqslant}$, and $M(w) = 0$, then $w = 0$.*

By Proposition 3.5, the map $(v, w) \mapsto \langle v, w\rangle := M(v\overline{w})$ is a positive definite hermitian form on the $\mathbb{C}$-linear space of almost periodic functions $\mathbb{R}^n \to \mathbb{C}$. For a trigonometric polynomial w as in (3.1) we have $w_\alpha = \langle w, \mathrm{e}^{(\alpha\cdot s)\mathrm{i}}\rangle$, and thus:

Corollary 3.6. *For w as in* (3.1)*, if $w = 0$, then $w_\alpha = 0$ for all α.*

Uniform distribution. *In this subsection $f_1, \dots, f_n \colon \mathbb{R} \to \mathbb{R}$ are measurable, and $f := (f_1, \dots, f_n) \colon \mathbb{R} \to \mathbb{R}^n$.*

Theorem 3.7 (Weyl)**.** *The following conditions on f are equivalent:*

(i) $\displaystyle\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{e}^{2\pi\mathrm{i}(k\cdot f(t))}\,dt \;=\; 0$ *for all $k \in (\mathbb{Z}^n)^{\neq}$;*

(ii) $\displaystyle\lim_{T\to\infty} \frac{1}{T}\int_0^T (w\circ f)(t)\,dt \;=\; \int_{[0,1]^n} w(s)\,ds$ *for every continuous* 1*-periodic w;*

(iii) *for every continuous* 1*-periodic w, the function $w \circ f \colon \mathbb{R} \to \mathbb{C}$ has mean value $M(w \circ f) = M(w)$.*

We say that f is *uniformly distributed* mod 1 (abbreviated: u.d. mod 1) if f satisfies condition (i) in Theorem 3.7. This is not the usual definition but is equivalent to it by [58, Theorem 9.9]. The latter, in combination with [58, Exercise 9.26] also yields the implication (i) ⇒ (ii) in Theorem 3.7. For (ii) ⇒ (i), apply (ii) to the 1-periodic trigonometric polynomial w given by $w(s) = \mathrm{e}^{2\pi\mathrm{i}(k\cdot s)}$. The equivalence (ii) $\Leftrightarrow$ (iii) is clear by earlier remarks.

In the next lemma we consider a strengthening of u.d. mod 1. For $\alpha \in \mathbb{R}^n$ set $\alpha f := (\alpha_1 f_1, \dots, \alpha_n f_n) \colon \mathbb{R} \to \mathbb{R}^n$.

Lemma 3.8. *The following conditions on f are equivalent:*

(i) *αf is u.d.* mod 1 *for all $\alpha \in (\mathbb{R}^\times)^n$;*

(ii) $\displaystyle\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{e}^{2\pi\mathrm{i}(\beta\cdot f(t))}\, dt \ = \ 0$ *for all $\beta \in (\mathbb{R}^n)^{\neq}$;*

(iii) *for every almost periodic w, $w \circ f$ has mean value $M(w \circ f) = M(w)$.*

Proof. Assume (i); let $\beta \in (\mathbb{R}^n)^{\neq}$. For $i = 1, \dots, n$ set $\alpha_i := 1$, $k_i := 0$ if $\beta_i = 0$ and $\alpha_i := \beta_i$, $k_i := 1$ if $\beta_i \neq 0$. Then $k = (k_1, \dots, k_n) \in (\mathbb{Z}^n)^{\neq}$, $\alpha = (\alpha_1, \dots, \alpha_n)$ is in $(\mathbb{R}^\times)^n$, and $\beta\cdot f(t) = k\cdot(\alpha f)(t)$ for all $t \in \mathbb{R}$. Now (ii) follows from the above definition of "u.d. mod 1" applied to αf in place of f. For (ii) $\Rightarrow$ (iii), assume (ii). Then each trigonometric polynomial v gives a function $v \circ f$ with mean value $M(v \circ f) = M(v)$. Let w be almost periodic, and take a sequence (v_m) of trigonometric polynomials $\mathbb{R}^n \to \mathbb{C}$ such that $\|v_m - w\| \to 0$ as $m \to \infty$. So $M(v_m) \to M(w)$ as $m \to \infty$, by Lemma 3.4. Also $\|(v_m \circ f) - (w \circ f)\| \to 0$ as $m \to \infty$, hence by Lemma 3.4 again, $w \circ f$ has a mean value and $M(v_m) = M(v_m \circ f) \to M(w \circ f)$ as $m \to \infty$. Therefore $M(w \circ f) = M(w)$. Finally, assume (iii), and let $\alpha \in (\mathbb{R}^\times)^n$; to show that αf is u.d. mod 1 we verify that condition (iii) in Theorem 3.7 holds for αf in place of f. Thus suppose w is continuous and 1-periodic. By (iii) applied to the almost periodic function $w_{\times\alpha}$ in place of w, the function $w_{\times\alpha} \circ f = w \circ (\alpha f)$ has a mean value and $M(w_{\times\alpha} \circ f) = M(w_{\times\alpha})$; now use that $M(w_{\times\alpha}) = M(w)$ by a remark after Lemma 3.4. □

We say that f is *uniformly distributed* (abbreviated: u.d.) if it satisfies one of the equivalent conditions in Lemma 3.8.

Corollary 3.9. *Suppose w is almost periodic, $w(\mathbb{R}^n) \subseteq \mathbb{R}^{\geqslant}$, and f is u.d. Then*

$$\limsup_{t\to+\infty} w\big(f(t)\big) = 0 \iff w = 0.$$

Proof. Lemmas 3.3 and 3.8 give $0 \leqslant M(w) = M(w \circ f) \leqslant \limsup_{t\to+\infty} w\big(f(t)\big)$. Now use Proposition 3.5. □

Application to Hardy fields. *In this subsection $f_1, \dots, f_n \colon \mathbb{R} \to \mathbb{R}$ are continuous, their germs, denoted also by $f_1, \dots, f_n$, lie in a common Hardy field, and as above $f := (f_1, \dots, f_n) \colon \mathbb{R} \to \mathbb{R}^n$.*

Theorem 3.10 (Boshernitzan [25, Theorem 1.12])**.**

$$f \text{ is u.d. mod } 1 \iff k_1 f_1 + \cdots + k_n f_n \succ \log x \text{ for all } (k_1, \dots, k_n) \in (\mathbb{Z}^n)^{\neq}.$$

For example, given $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, the map $t \mapsto (\lambda_1 t, \dots, \lambda_n t) \colon \mathbb{R} \to \mathbb{R}^n$ is u.d. mod 1 iff $\lambda_1, \dots, \lambda_n$ are $\mathbb{Q}$-linearly independent (Weyl [89]).

Corollary 3.11. *We have the following equivalence:*

$$f \text{ is u.d.} \iff \beta_1 f_1 + \cdots + \beta_n f_n \succ \log x \text{ for all } (\beta_1, \dots, \beta_n) \in (\mathbb{R}^n)^{\neq}.$$

In particular, if $\log x \prec f_1 \prec \cdots \prec f_n$, then f is u.d.

Proof. Let $\beta = (\beta_1, \dots, \beta_n) \in (\mathbb{R}^n)^{\neq}$ and define α_i, k_i as in the proof of (i) $\Rightarrow$ (ii) in Lemma 3.8. Then $\beta_1 f_1 + \cdots + \beta_n f_n = k_1(\alpha_1 f_1) + \cdots + k_n(\alpha_n f_n)$, and the germs $\alpha_1 f_1, \dots, \alpha_n f_n$ lie in a common Hardy field by Proposition 2.1. Using this observation and Theorem 3.10 yields the forward direction. The backward direction is also an easy consequence of Theorem 3.10. $\square$

We now get to the result that we actually need in Section 4.

Proposition 3.12. *Suppose w is almost periodic, $w(\mathbb{R}^n) \subseteq \mathbb{R}^{\geqslant}$, $1 \prec f_1 \prec \cdots \prec f_n$, and $\limsup_{t\to+\infty} w\big(f(t)\big) = 0$. Then $w = 0$.*

Proof. We first arrange $f_1 > \mathbb{R}$, replacing $f_1, \dots, f_n$ and w by $-f_1, \dots, -f_n$ and the function $s \mapsto w(-s)\colon \mathbb{R}^n \to \mathbb{R}^{\geqslant}$, if $f_1 < \mathbb{R}$. Pick a such that the restriction of f_1 to $\mathbb{R}^{\geqslant a}$ is strictly increasing, set $b := f_1(a)$, and let $f_1^{\text{inv}}\colon \mathbb{R}^{\geqslant b} \to \mathbb{R}$ be the compositional inverse of this restriction. Set $g_j(t) := (f_j \circ f_1^{\text{inv}})(t)$ for $t \geqslant b$ and $j = 1, \dots, n$ and consider the map

$$g \;=\; (g_1, \dots, g_n) \;=\; f \circ f_1^{\text{inv}} \;:\; \mathbb{R}^{\geqslant b} \to \mathbb{R}^n.$$

The germs of $g_1, \dots, g_n$, denoted by the same symbols, lie in a common Hardy field (see for example [9, Section 4]) and satisfy $x = g_1 \prec g_2 \prec \cdots \prec g_n$. Now f_1^{inv} is strictly increasing and moreover $f_1^{\text{inv}}(t) \to +\infty$ as $t \to +\infty$, so

$$\limsup_{t\to+\infty} w\big(f(t)\big) \;=\; \limsup_{t\to+\infty} w\big(f\big(f_1^{\text{inv}}(t)\big)\big) \;=\; \limsup_{t\to+\infty} w\big(g(t)\big) \;=\; 0.$$

Thus replacing $f_1, \dots, f_n$ by continuous functions $\mathbb{R} \to \mathbb{R}$ with the same germs as $g_1, \dots, g_n$, we arrange $x = f_1 \prec f_2 \prec \cdots \prec f_n$. Then f is u.d. by Corollary 3.11. Now use Corollary 3.9. $\square$

4. Universal Exponential Extensions of Hardy Fields

In this section $H \supseteq \mathbb{R}$ is a Liouville closed Hardy field and $K := H[\mathrm{i}]$. So K is an algebraically closed d*-valued field with constant field* $\mathbb{C}$. In order to give an analytic description of the universal differential extension $\mathrm{U} = \mathrm{U}_K$ of K we consider the differential ring extension $\mathcal{C}^{<\infty}[\mathrm{i}]$ of K with ring of constants $\mathbb{C}$. For any $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$ we have a $g \in \mathcal{C}^{<\infty}[\mathrm{i}]$ with $g' = f$, and then $u = \mathrm{e}^g \in \mathcal{C}^{<\infty}[\mathrm{i}]^\times$ satisfies $u^\dagger = f$.

Lemma 4.1. *Suppose $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$ is purely imaginary, that is, $f \in \mathrm{i}\mathcal{C}^{<\infty}$. Then there is a $u \in \mathcal{C}^{<\infty}[\mathrm{i}]^\times$ such that $u^\dagger = f$ and $|u| = 1$.*

Proof. Taking $g \in \mathrm{i}\mathcal{C}^{<\infty}$ with $g' = f$, the resulting $u = \mathrm{e}^g$ works. $\square$

We define the subgroup $\mathrm{e}^{H\mathrm{i}}$ of $\mathcal{C}^{<\infty}[\mathrm{i}]^\times$ by

$$\mathrm{e}^{H\mathrm{i}} \;:=\; \big\{\mathrm{e}^{h\mathrm{i}} :\; h \in H\big\} \;=\; \big\{u \in \mathcal{C}^{<\infty}[\mathrm{i}]^\times :\; |u| = 1,\; u^\dagger \in H\mathrm{i}\big\}.$$

Then $(\mathrm{e}^{H\mathrm{i}})^\dagger = H\mathrm{i}$ by Lemma 4.1, so $(H^\times \cdot \mathrm{e}^{H\mathrm{i}})^\dagger = K$ and thus $K[\mathrm{e}^{H\mathrm{i}}]$ is an exponential extension of K with the same ring of constants $\mathbb{C}$ as K. Fix a complement $\Lambda \subseteq H\mathrm{i}$ of the subspace $K^\dagger$ of the $\mathbb{Q}$-linear space K, and let λ range over Λ. The differential K-algebras $\mathrm{U} := K\big[\mathrm{e}(\Lambda)\big]$ and $K[\mathrm{e}^{H\mathrm{i}}]$ are isomorphic by Corollary 1.3, but we need something better:

Lemma 4.2. *There is an isomorphism $\mathrm{U} \to K[\mathrm{e}^{H\mathrm{i}}]$ of differential K-algebras that maps $\mathrm{e}(\Lambda)$ into $\mathrm{e}^{H\mathrm{i}}$.*

Proof. We have a short exact sequence of commutative groups

$$1 \to S \xrightarrow{\subseteq} \mathrm{e}^{H\mathrm{i}} \xrightarrow{\ell} H\mathrm{i} \to 0,$$

where $S = \{z \in \mathbb{C}^\times : |z| = 1\}$ and $\ell(u) := u^\dagger$ for $u \in \mathrm{e}^{H\mathrm{i}}$. Since the subgroup S of $\mathbb{C}^\times$ is divisible, this sequence splits: we have a group embedding $e\colon H\mathrm{i} \to \mathrm{e}^{H\mathrm{i}}$ such that $e(b)^\dagger = b$ for all $b \in H\mathrm{i}$. Then the group embedding

$$\mathrm{e}(\lambda) \mapsto e(\lambda) \ : \ \mathrm{e}(\Lambda) \to \mathrm{e}^{H\mathrm{i}}$$

extends uniquely to a K-algebra morphism $\mathrm{U} \to K[\mathrm{e}^{H\mathrm{i}}]$. Since $\mathrm{e}(\lambda)^\dagger = \lambda = e(\lambda)^\dagger$ for all λ, this is a differential K-algebra morphism, and even an isomorphism by Lemma 1.2 applied to $R = K[\mathrm{e}^{H\mathrm{i}}]$. $\square$

Complex conjugation $f + g\mathrm{i} \mapsto \overline{f + g\mathrm{i}} = f - g\mathrm{i}$ ($f, g \in \mathcal{C}^{<\infty}$) is an automorphism of the differential ring $\mathcal{C}^{<\infty}[\mathrm{i}]$ over H and maps $K[\mathrm{e}^{H\mathrm{i}}]$ onto itself, sending each $u \in \mathrm{e}^{H\mathrm{i}}$ to u^{-1}. Thus any isomorphism $\iota\colon \mathrm{U} \to K[\mathrm{e}^{H\mathrm{i}}]$ of differential K-algebras with $\iota(\mathrm{e}(\Lambda)) \subseteq \mathrm{e}^{H\mathrm{i}}$ as in Lemma 4.2 also satisfies $\iota(\overline{f}) = \overline{\iota(f)}$ for $f \in \mathrm{U}$, where $f \mapsto \overline{f}$ is the ring automorphism of U which extends complex conjugation on $K = H[\mathrm{i}]$ and satisfies $\overline{\mathrm{e}(\lambda)} = \mathrm{e}(-\lambda)$ for all λ; see [8, Section 2.2, subsection "The real case"].) Hence by [8, Lemma 2.2.14, Corollary 2.2.17], given such an isomorphism ι, any differential K-algebra isomorphism as in Lemma 4.2 equals $\iota \circ \sigma_\chi$ for a unique character $\chi\colon \Lambda \to \mathbb{C}^\times$ with $|\chi(\lambda)| = 1$ for all λ, where σ_χ is the differential K-algebra automorphism of U with $\sigma_\chi(\mathrm{e}(\lambda)) = \chi(\lambda)\,\mathrm{e}(\lambda)$ for all λ. Fix such an isomorphism ι and identify U with its image $K[\mathrm{e}^{H\mathrm{i}}]$ via ι.

Lemma 4.3. *Let $A \in K[\partial]^{\neq}$. Then the $\mathbb{C}$-linear space $\ker_{\mathrm{U}} A$ has a basis*

$$f_1\,\mathrm{e}^{\phi_1\mathrm{i}}, \dots, f_d\,\mathrm{e}^{\phi_d\mathrm{i}} \qquad \textit{where } f_j \in K^\times,\ \phi_j \in H\ (j = 1, \dots, d).$$

If $\mathrm{I}(K) \subseteq K^\dagger$, then we can choose the f_j, ϕ_j such that in addition, for each j, we have $\phi_j = 0$ or $\phi_j \succ 1$.

Proof. By the remarks before (1.2), $\ker_{\mathrm{U}} A$ has a basis contained in $\mathrm{U}^\times = K^\times\,\mathrm{e}(\Lambda)$. Using $\mathrm{e}(\Lambda) \subseteq \mathrm{e}^{H\mathrm{i}}$, this yields the first part. For the second part, use that if $\mathrm{I}(K) \subseteq K^\dagger$ and $\phi \in H$, $\phi \preccurlyeq 1$, then $\mathrm{e}^{\phi\mathrm{i}} \in K^\times$ by [9, Proposition 6.11]. $\square$

Remark. Let $A \in K[\partial]^{\neq}$, and suppose $\mathrm{I}(K) \subseteq K^\dagger$. Then one can choose the f_j, ϕ_j as in Lemma 4.3 such that also for all $j \neq k$, either $\phi_j - \phi_k \succ 1$, or $\phi_j = \phi_k$ and $vf_j \neq vf_k$ in $v(K^\times)$. For such f_j, ϕ_j we have $\mathscr{E}^{\mathrm{u}}(A) \supseteq \{vf_1, \dots, vf_d\}$. These statements are proved in [10] and not used later in this paper, but explain how $\mathscr{E}^{\mathrm{u}}(A)$ helps to locate the $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ with $A(y) = 0$.

We have the asymptotic relations $\preccurlyeq_{\mathrm{g}}$ and $\prec_{\mathrm{g}}$ on U coming from the gaussian extension v_{g} of the valuation on K. (See Section 1.) But we also have the asymptotic relations induced on $\mathrm{U} = K[\mathrm{e}^{H\mathrm{i}}]$ by the relations $\preccurlyeq$ and $\prec$ defined on $\mathcal{C}[\mathrm{i}]$. It is clear that for $f \in \mathrm{U}$:

$$f \preccurlyeq_{\mathrm{g}} 1 \implies f \preccurlyeq 1 \iff \text{for some } n \text{ we have } |f(t)| \leqslant n \text{ eventually},$$

$$f \prec_{\mathrm{g}} 1 \implies f \prec 1 \iff \lim_{t \to +\infty} f(t) = 0.$$

As a tool for later use we derive a converse of the implication $f \prec_{\mathrm{g}} 1 \Rightarrow f \prec 1$: Lemma 4.7 below, where we assume in addition that $\mathrm{I}(K) \subseteq K^\dagger$ and Λ is an $\mathbb{R}$-linear subspace of K. This requires the material from Section 3 and some considerations about exponential sums treated in the next subsection.

Exponential sums over Hardy fields. In this subsection $n \geqslant 1$. In the next lemma, $f = (f_1, \dots, f_m) \in H^m$ where $m \geqslant 1$ and $1 \prec f_1 \prec \cdots \prec f_m$. (In that lemma it doesn't matter which functions we use to represent the germs $f_1, \dots, f_m$.) For $r = (r_1, \dots, r_m) \in \mathbb{R}^m$ we set $r \cdot f := r_1 f_1 + \cdots + r_m f_m \in H$.

Lemma 4.4. *Let $r^1, \dots, r^n \in \mathbb{R}^m$ be distinct and $c_1, \dots, c_n \in \mathbb{C}^\times$. Then*

$$\limsup_{t\to\infty} \left| c_1 \operatorname{e}^{(r^1\cdot f)(t)i} + \cdots + c_n \operatorname{e}^{(r^n\cdot f)(t)i} \right| \ > \ 0.$$

Proof. Consider the trigonometric polynomial $w\colon \mathbb{R}^m \to \mathbb{R}^{\geqslant}$ given by

$$w(s) \ := \ \left| c_1 \operatorname{e}^{(r^1\cdot s)i} + \cdots + c_n \operatorname{e}^{(r^n\cdot s)i} \right|^2.$$

By Corollary 3.6 we have $w(s) > 0$ for some $s \in \mathbb{R}^m$. Taking continuous representatives $\mathbb{R} \to \mathbb{R}$ of $f_1, \dots, f_m$, to be denoted also by $f_1, \dots, f_m$, the lemma now follows from Proposition 3.12. $\square$

Next, let $h_1, \dots, h_n \in H$ be distinct such that $(\mathbb{R}h_1 + \cdots + \mathbb{R}h_n) \cap \operatorname{I}(H) = \{0\}$. Since H is Liouville closed we have $\phi_1, \dots, \phi_n \in H$ such that $\phi_1' = h_1, \dots, \phi_n' = h_n$.

Lemma 4.5. *Let $c_1, \dots, c_n \in \mathbb{C}^\times$. Then for $\phi_1, \dots, \phi_n$ as above,*

$$\limsup_{t\to\infty} \left| c_1 \operatorname{e}^{\phi_1(t)i} + \cdots + c_n \operatorname{e}^{\phi_n(t)i} \right| \ > \ 0.$$

Proof. The case $n = 1$ is trivial, so let $n \geqslant 2$. Then $\phi_1, \dots, \phi_n$ are not all in $\mathbb{R}$. Set $V := \mathbb{R} + \mathbb{R}\phi_1 + \cdots + \mathbb{R}\phi_n \subseteq H$, so $\partial V = \mathbb{R}h_1 + \cdots + \mathbb{R}h_n$. We claim that $V \cap \mathcal{o}_H = \{0\}$. To see this, let $\phi \in V \cap \mathcal{o}_H$; then $\phi' \in \partial(V) \cap \operatorname{I}(H) = \{0\}$ and hence $\phi \in \mathbb{R} \cap \mathcal{o}_H = \{0\}$, proving the claim. Now H is a Hahn space over $\mathbb{R}$ by [ADH, p. 109], so by [ADH, 2.3.13] we have $f_1, \dots, f_m \in V$ $(1 \leqslant m \leqslant n)$ such that $V = \mathbb{R} + \mathbb{R}f_1 + \cdots + \mathbb{R}f_m$ and $1 \prec f_1 \prec \cdots \prec f_m$. For $j = 1, \dots, n$, $k = 1, \dots, m$, take $t_j, r_{jk} \in \mathbb{R}$ such that $\phi_j = t_j + \sum_{k=1}^m r_{jk} f_k$ and set $r^j := (r_{j1}, \dots, r_{jm}) \in \mathbb{R}^m$. Since $\phi_{j_1} - \phi_{j_2} \notin \mathbb{R}$ for $j_1 \neq j_2$, we have $r^{j_1} \neq r^{j_2}$ for $j_1 \neq j_2$. It remains to apply Lemma 4.4 to $c_1 \operatorname{e}^{t_1 i}, \dots, c_n \operatorname{e}^{t_n i}$ in place of $c_1, \dots, c_n$. $\square$

Corollary 4.6. *Let $f_1, \dots, f_n \in K$ and set $f := f_1 \operatorname{e}^{\phi_1 i} + \cdots + f_n \operatorname{e}^{\phi_n i} \in \mathcal{C}^{<\infty}[i]$, and suppose $f \prec 1$. Then $f_1, \dots, f_n \prec 1$.*

Proof. We may assume $0 \neq f_1 \preccurlyeq \cdots \preccurlyeq f_n$. Towards a contradiction, suppose that $f_n \succcurlyeq 1$, and take $m \leqslant n$ minimal such that $f_m \asymp f_n$. Then with $g_j := f_j/f_n \in K^\times$ and $g := g_1 \operatorname{e}^{\phi_1 i} + \cdots + g_n \operatorname{e}^{\phi_n i}$ we have $g \prec 1$ and $g_1, \dots, g_n \preccurlyeq 1$, with $g_j \prec 1$ iff $j < m$. Replacing $f_1, \dots, f_n$ by $g_m, \dots, g_n$ and $\phi_1, \dots, \phi_n$ by $\phi_m, \dots, \phi_n$ we arrange $f_1 \asymp \cdots \asymp f_n \asymp 1$. So

$$f_1 = c_1 + \varepsilon_1, \dots, f_n = c_n + \varepsilon_n \quad \text{ with } c_1, \dots, c_n \in \mathbb{C}^\times \text{ and } \varepsilon_1, \dots, \varepsilon_n \in \mathcal{o}.$$

Then $\varepsilon_1 \operatorname{e}^{\phi_1 i} + \cdots + \varepsilon_n \operatorname{e}^{\phi_n i} \prec 1$, hence

$$c_1 \operatorname{e}^{\phi_1 i} + \cdots + c_n \operatorname{e}^{\phi_n i} \ = \ f - \left(\varepsilon_1 \operatorname{e}^{\phi_1 i} + \cdots + \operatorname{e}^{\phi_n i}\right) \ \prec \ 1.$$

Now Lemma 4.5 yields the desired contradiction. $\square$

In the rest of this section, $\operatorname{I}(K) \subseteq K^\dagger$. As noted in Section 1 we can then take $\Lambda = \Lambda_H i$ where Λ_H is an $\mathbb{R}$-linear complement of $\operatorname{I}(H)$ in H. *We assume Λ has this form,* and accordingly identify U with $K[\operatorname{e}^{Hi}]$ as explained before Lemma 4.3.

Lemma 4.7. *Let $f \in \operatorname{U}$ be such that $f \prec 1$. Then $f \prec_{\operatorname{g}} 1$.*

Proof. We have $f = f_1 \operatorname{e}(h_1 i) + \cdots + f_n \operatorname{e}(h_n i)$ with $f_1, \dots, f_n \in K$ and distinct $h_1, \dots, h_n \in \Lambda_H$, so $(\mathbb{R}h_1 + \cdots + \mathbb{R}h_n) \cap \operatorname{I}(H) = \{0\}$. For $h \in \Lambda_H$ we have $\operatorname{e}(hi) = \operatorname{e}^{\phi i}$ with $\phi \in H$ and $\phi' = h$. Hence $f = f_1 \operatorname{e}^{\phi_1 i} + \cdots + f_n \operatorname{e}^{\phi_n i}$ where $\phi_j \in H$ and $\phi_j' = h_j$ for $j = 1, \dots, n$. Now Corollary 4.6 yields $f \prec_{\mathrm{g}} 1$. □

Corollary 4.8. *Let $f \in \mathrm{U}$ and $\mathfrak{m} \in H^\times$. Then $f \prec \mathfrak{m}$ iff $f \prec_{\mathrm{g}} \mathfrak{m}$.*

An application to slots in H. Until further notice $(P, 1, \widehat{h})$ is a slot in H of order $r \geqslant 1$. We also let $A \in K[\partial]$ have order r, and we let $\mathfrak{m}$ range over the elements of $H^\times$ such that $v\mathfrak{m} \in v(\widehat{h} - H)$. We begin with an important consequence of the bounds on solutions of linear differential equations in Section 2:

Lemma 4.9. *Suppose that $(P, 1, \widehat{h})$ is Z-minimal, deep, and special, and that $\mathfrak{v}(L_P) \asymp \mathfrak{v} := \mathfrak{v}(A)$. Let $y \in \mathcal{C}^r[\mathrm{i}]$ satisfy $A(y) = 0$ and $y \prec \mathfrak{m}$ for all $\mathfrak{m}$. Then $y', \dots, y^{(r)} \prec \mathfrak{m}$ for all $\mathfrak{m}$.*

Proof. Proposition 1.9 gives an $\mathfrak{m} \preccurlyeq \mathfrak{v}$, so it is enough to show $y', \dots, y^{(r)} \prec \mathfrak{m}$ for all $\mathfrak{m} \preccurlyeq \mathfrak{v}$. Accordingly we assume $0 < \mathfrak{m} \preccurlyeq \mathfrak{v}$ below. As $\widehat{h}$ is special over H, we have $2(r+1)v\mathfrak{m} \in v(\widehat{h} - H)$, so $y \prec \mathfrak{m}^{2(r+1)}$. Then Corollary 2.7 with $n = 2(r+1)$, $\eta = |\mathfrak{v}|^{-1}$, $\varepsilon = 1/r$ gives for $j = 0, \dots, r$:

$$y^{(j)} \prec \mathfrak{v}^{-j} \mathfrak{m}^{n - j(1+\varepsilon)} \preccurlyeq \mathfrak{m}^{n - j(2+\varepsilon)} \preccurlyeq \mathfrak{m}^{n - r(2+\varepsilon)} = \mathfrak{m}. \qquad \square$$

Note that if $\dim_{\mathbb{C}} \ker_{\mathrm{U}} A = r$, then $\mathrm{U} = K[\operatorname{e}^{H\mathrm{i}}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$ contains every $y \in \mathcal{C}^r[\mathrm{i}]$ with $A(y) = 0$. Corollary 4.8 is typically used in combination with the ultimate condition. Here is a first easy application:

Lemma 4.10. *Suppose* $\deg P = 1$, *$(P, 1, \widehat{h})$ is ultimate,* $\dim_{\mathbb{C}} \ker_{\mathrm{U}} L_P = r$, *and y in $\mathcal{C}^r[\mathrm{i}]$ satisfies $L_P(y) = 0$ and $y \prec 1$. Then $y \prec \mathfrak{m}$ for all $\mathfrak{m}$.*

Proof. We have $y \in \mathrm{U}$, so $y \prec_{\mathrm{g}} 1$ by Lemma 4.7. If $y = 0$ we are done, so assume $y \neq 0$. Then (1.4) gives $0 < v_{\mathrm{g}} y \in v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} L_P) = \mathscr{E}^{\mathrm{u}}(L_P)$, hence $v_{\mathrm{g}} y > v(\widehat{h} - H)$ by (1.5), so $y \prec_{\mathrm{g}} \mathfrak{m}$ for all $\mathfrak{m}$. Now Corollary 4.8 yields the conclusion. □

Corollary 4.11. *Suppose* $\deg P = 1$, *$(P, 1, \widehat{h})$ is Z-minimal, deep, special, and ultimate, and* $\dim_{\mathbb{C}} \ker_{\mathrm{U}} L_P = r$. *Let $f, g \in \mathcal{C}^r[\mathrm{i}]$ be such that $P(f) = P(g) = 0$ and $f, g \prec 1$. Then $(f - g)^{(j)} \prec \mathfrak{m}$ for $j = 0, \dots, r$ and all $\mathfrak{m}$.*

Proof. Use Lemmas 4.9 and 4.10 for $A = L_P$ and $y = f - g$. □

In the rest of this subsection we assume that $(P, 1, \widehat{h})$ is normal and ultimate, $\dim_{\mathbb{C}} \ker_{\mathrm{U}} A = r$, *and $L_P = A + B$ where*

$$B \prec_{\Delta(\mathfrak{v})} \mathfrak{v}^{r+1} A, \qquad \mathfrak{v} := \mathfrak{v}(A) \prec^{\flat} 1.$$

Then Lemma 1.5 gives $\mathfrak{v}(L_P) \sim \mathfrak{v}$, and $v_{\mathrm{g}}(\ker^{\neq}_{\mathrm{U}} A) = \mathscr{E}^{\mathrm{u}}(A) = \mathscr{E}^{\mathrm{u}}(L_P)$ by (1.4). This yields a variant of Lemma 4.10, with a similar proof:

Proposition 4.12. *If $y \in \mathcal{C}^r[\mathrm{i}]$ and $A(y) = 0$, $y \prec 1$, then $y \prec \mathfrak{m}$ for all $\mathfrak{m}$.*

The following result will be used in establishing a crucial non-linear version of Corollary 4.11, namely Proposition 9.14.

Corollary 4.13. *If $(P, 1, \widehat{h})$ is Z-minimal, deep, and special, and $y \in \mathcal{C}^r[\mathrm{i}]$ is such that $A(y) = 0$ and $y \prec 1$, then $y, y', \dots, y^{(r)} \prec \mathfrak{m}$ for all $\mathfrak{m}$.*

Proof. Use first Proposition 4.12 and then Lemma 4.9. □

So far we didn't have to name an immediate asymptotic extension of H where $\widehat{h}$ is located, but for the "complex" version of the above we need to be more specific: Let $\widehat{H}$ be an immediate asymptotic extension of H and $\widehat{K} = \widehat{H}[\mathrm{i}] \supseteq \widehat{H}$ a corresponding immediate d-valued extension of K. The results in this subsection then go through if instead of $(P, 1, \widehat{h})$ being a slot in H of order $r \geqslant 1$ we assume that $(P, 1, \widehat{h})$ is a slot in K of order $r \geqslant 1$ with $\widehat{h} \in \widehat{K} \setminus K$, with $\mathfrak{m}$ now ranging over the elements of $K^{\times}$ such that $v\mathfrak{m} \in v(\widehat{h} - K)$.

5. Inverting Linear Differential Operators over Hardy Fields

Given a Hardy field H and $A \in H[\partial]$ we shall construe A as a $\mathbb{C}$-linear operator on various spaces of functions. We wish to construct right-inverses to such operators. A key assumption here is that A splits over $H[\mathrm{i}]$. This reduces the construction of such inverses mainly to the case of order 1, and this case is handled in the first two subsections using suitable twisted integration operators. In the third subsection we put things together and also show how to "preserve reality". In the last subsection we introduce damping factors. Throughout we pay attention to the continuity of various operators with respect to various norms, for use in Section 6.

We let a range over $\mathbb{R}$ and r over $\mathbb{N} \cup \{\infty, \omega\}$. If $r \in \mathbb{N}$, then $r-1$ and $r+1$ have the usual meaning, while for $r \in \{\infty, \omega\}$ we set $r-1 = r+1 := r$. (This convention is just to avoid case distinctions.) We have the usual absolute value on $\mathbb{C}$ given by $|a+bi| = \sqrt{a^2+b^2} \in \mathbb{R}^{\geqslant}$ for $a, b \in \mathbb{R}$, so for $f \in \mathcal{C}_a[\mathrm{i}]$ we have $|f| \in \mathcal{C}_a$.

Integration and some useful norms. For $f \in \mathcal{C}_a[\mathrm{i}]$ we define $\partial_a^{-1} f \in \mathcal{C}_a^1[\mathrm{i}]$ by

$$\partial_a^{-1} f(t) \ := \ \int_a^t f(s)\,ds \ := \ \int_a^t \operatorname{Re} f(s)\,ds + \mathrm{i}\int_a^t \operatorname{Im} f(s)\,ds,$$

so $\partial_a^{-1} f$ is the unique $g \in \mathcal{C}_a^1[\mathrm{i}]$ such that $g' = f$ and $g(a) = 0$. The integration operator $\partial_a^{-1} \colon \mathcal{C}_a[\mathrm{i}] \to \mathcal{C}_a^1[\mathrm{i}]$ is $\mathbb{C}$-linear and maps $\mathcal{C}_a^r[\mathrm{i}]$ into $\mathcal{C}_a^{r+1}[\mathrm{i}]$. For $f \in \mathcal{C}_a[\mathrm{i}]$ we have

$$\big|\partial_a^{-1} f(t)\big| \leqslant \big(\partial_a^{-1}|f|\big)(t) \qquad \text{for all } t \geqslant a.$$

Let $f \in \mathcal{C}_a[\mathrm{i}]$. Call f **integrable at** ∞ if $\lim_{t\to\infty} \int_a^t f(s)\,ds$ exists in $\mathbb{C}$. In that case we denote this limit by $\int_a^\infty f(s)\,ds$ and put

$$\int_\infty^a f(s)\,ds \ := \ -\int_a^\infty f(s)\,ds,$$

and define $\partial_\infty^{-1} f \in \mathcal{C}_a^1[\mathrm{i}]$ by

$$\partial_\infty^{-1} f(t) \ := \ \int_\infty^t f(s)\,ds \ = \ \int_\infty^a f(s)\,ds + \int_a^t f(s)\,ds \ = \ \int_\infty^a f(s)\,ds + \partial_a^{-1} f(t),$$

so $\partial_\infty^{-1} f$ is the unique $g \in \mathcal{C}_a^1[\mathrm{i}]$ such that $g' = f$ and $\lim_{t\to\infty} g(t) = 0$. Note that

$$\mathcal{C}_a[\mathrm{i}]^{\mathrm{int}} \ := \ \big\{f \in \mathcal{C}_a[\mathrm{i}] : \ f \text{ is integrable at } \infty\big\} \tag{5.1}$$

is a $\mathbb{C}$-linear subspace of $\mathcal{C}_a[\mathrm{i}]$ and that ∂_∞^{-1} defines a $\mathbb{C}$-linear operator from this subspace into $\mathcal{C}_a^1[\mathrm{i}]$ which maps $\mathcal{C}_a^r[\mathrm{i}] \cap \mathcal{C}_a[\mathrm{i}]^{\mathrm{int}}$ into $\mathcal{C}_a^{r+1}[\mathrm{i}]$. If $f \in \mathcal{C}_a[\mathrm{i}]$ and $g \in \mathcal{C}_a^{\mathrm{int}} := \mathcal{C}_a[\mathrm{i}]^{\mathrm{int}} \cap \mathcal{C}_a$ with $|f| \leqslant g$ as germs in $\mathcal{C}$, then $f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{int}}$; in particular, if $f \in \mathcal{C}_a[\mathrm{i}]$ and $|f| \in \mathcal{C}_a^{\mathrm{int}}$, then $f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{int}}$.

For $f\in\mathcal{C}_a[\mathrm{i}]$ we set

$$\|f\|_a \ :=\ \sup_{t\geqslant a}|f(t)| \ \in\ [0,\infty],$$

so (with b for "bounded"):

$$\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \ :=\ \big\{f\in\mathcal{C}_a[\mathrm{i}]:\ \|f\|_a<\infty\big\}$$

is a $\mathbb{C}$-linear subspace of $\mathcal{C}_a[\mathrm{i}]$, and $f\mapsto\|f\|_a$ is a norm on $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ making it a Banach space over $\mathbb{C}$. It is also convenient to define for $t\geqslant a$ the seminorm

$$\|f\|_{[a,t]} \ :=\ \max_{a\leqslant s\leqslant t}|f(s)|$$

on $\mathcal{C}_a[\mathrm{i}]$. More generally, let $r\in\mathbb{N}$. Then for $f\in\mathcal{C}^r_a[\mathrm{i}]$ we set

$$\|f\|_{a;r} \ :=\ \max\big\{\|f\|_a,\dots,\|f^{(r)}\|_a\big\} \ \in\ [0,\infty],$$

so

$$\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}} := \big\{f\in\mathcal{C}^r_a[\mathrm{i}]:\ \|f\|_{a;r}<\infty\big\}$$

is a $\mathbb{C}$-linear subspace of $\mathcal{C}^r_a[\mathrm{i}]$, and $f\mapsto\|f\|_{a;r}$ makes $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$ a normed vector space over $\mathbb{C}$. Note that for $f,g\in\mathcal{C}^r_a[\mathrm{i}]$ we have $\|fg\|_{a;r}\ \leqslant\ 2^r\|f\|_{a;r}\|g\|_{a;r}$, so $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$ is a subalgebra of the $\mathbb{C}$-algebra $\mathcal{C}^r_a[\mathrm{i}]$. If $f\in\mathcal{C}^{r+1}_a[\mathrm{i}]$, then $f'\in\mathcal{C}^r_a[\mathrm{i}]$ with $\|f'\|_{a;r}\leqslant\|f\|_{a;r+1}$.

With $\boldsymbol{i}=(i_0,\dots,i_r)$ ranging over $\mathbb{N}^{1+r}$, let $P=\sum_{\boldsymbol{i}}P_{\boldsymbol{i}}Y^{\boldsymbol{i}}$ (all $P_{\boldsymbol{i}}\in\mathcal{C}_a[\mathrm{i}]$) be a polynomial in $\mathcal{C}_a[\mathrm{i}]\big[Y,Y',\dots,Y^{(r)}\big]$. For $f\in\mathcal{C}^r_a[\mathrm{i}]$ we set

$$P(f)\ :=\ \sum_{\boldsymbol{i}}P_{\boldsymbol{i}}f^{\boldsymbol{i}}\in\mathcal{C}_a[\mathrm{i}]\qquad\text{where } f^{\boldsymbol{i}}:=f^{i_0}(f')^{i_1}\cdots(f^{(r)})^{i_r}\in\mathcal{C}_a[\mathrm{i}].$$

We also let

$$\|P\|_a\ :=\ \max_{\boldsymbol{i}}\|P_{\boldsymbol{i}}\|_a\in[0,\infty].$$

Then $\|P\|_a<\infty$ iff $P\in\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\big[Y,\dots,Y^{(r)}\big]$, and $\|\cdot\|_a$ is a norm on the $\mathbb{C}$-linear space $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\big[Y,\dots,Y^{(r)}\big]$. In the following assume $\|P\|_a<\infty$. Then for $j=0,\dots,r$ such that $\partial P/\partial Y^{(j)}\neq 0$ we have

$$\|\partial P/\partial Y^{(j)}\|_a\ \leqslant\ (\deg_{Y^{(j)}}P)\cdot\|P\|_a.$$

Moreover:

Lemma 5.1. *If P is homogeneous of degree $d\in\mathbb{N}$ and $f\in\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$, then*

$$\|P(f)\|_a\ \leqslant\ \binom{d+r}{r}\cdot\|P\|_a\cdot\|f\|_{a;r}^d.$$

Corollary 5.2. *Let $d\leqslant e$ in $\mathbb{N}$ be such that $P_{\boldsymbol{i}}=0$ whenever $|\boldsymbol{i}|<d$ or $|\boldsymbol{i}|>e$. Then for $f\in\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$ we have*

$$\|P(f)\|_a\ \leqslant\ D\cdot\|P\|_a\cdot\big(\|f\|_{a;r}^d+\cdots+\|f\|_{a;r}^e\big)$$

where $D=D(d,e,r):=\binom{e+r+1}{r+1}-\binom{d+r}{r+1}\in\mathbb{N}^{\geqslant1}$.

Let $B\colon V\to\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$ be a $\mathbb{C}$-linear map from a normed vector space V over $\mathbb{C}$ into $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$. Then we set

$$\|B\|_{a;r}\ :=\ \sup\big\{\|B(f)\|_{a;r}:\ f\in V,\ \|f\|\leqslant1\big\}\ \in\ [0,\infty],$$

the **operator norm of** B. Hence with the convention $\infty \cdot b := b \cdot \infty := \infty$ for $b \in [0, \infty]$ we have

$$\|B(f)\|_{a;r} \ \leqslant\ \|B\|_{a;r} \cdot \|f\| \qquad \text{for } f \in V.$$

Note that B is continuous iff $\|B\|_{a;r} < \infty$. If the map $D\colon \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a^s[\mathrm{i}]^{\mathrm{b}}$ ($s \in \mathbb{N}$) is also $\mathbb{C}$-linear, then

$$\|D \circ B\|_{a;s} \ \leqslant\ \|D\|_{a;s} \cdot \|B\|_{a;r}.$$

For $r = 0$ we drop the subscript: $\|B\|_a := \|B\|_{a;0}$.

Lemma 5.3. *Let $r \in \mathbb{N}^{\geqslant 1}$ and $\phi \in \mathcal{C}_a^{r-1}[\mathrm{i}]^{\mathrm{b}}$. Then the $\mathbb{C}$-linear operator*

$$\partial - \phi \ :\ \mathcal{C}_a^r[\mathrm{i}] \to \mathcal{C}_a^{r-1}[\mathrm{i}], \quad f \mapsto f' - \phi f$$

maps $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a^{r-1}[\mathrm{i}]^{\mathrm{b}}$, and its restriction $\partial - \phi\colon \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a^{r-1}[\mathrm{i}]^{\mathrm{b}}$ is continuous with operator norm $\|\partial - \phi\|_{a;r-1} \leqslant 1 + 2^{r-1}\|\phi\|_{a;r-1}$.

Let $r \in \mathbb{N}$, $a_0 \in \mathbb{R}$, and let a range over $[a_0, \infty)$. The $\mathbb{C}$-linear map

$$f \mapsto f|_{[a,+\infty)} \ :\ \mathcal{C}_{a_0}^r[\mathrm{i}] \to \mathcal{C}_a^r[\mathrm{i}]$$

satisfies $\|f|_{[a,+\infty)}\|_{a;r} \ \leqslant\ \|f\|_{a_0;r}$ for $f \in \mathcal{C}_{a_0}^r[\mathrm{i}]$, so it maps $\mathcal{C}_{a_0}^r[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$. For $f \in \mathcal{C}_{a_0}^0[\mathrm{i}]$ also denoting its germ at $+\infty$ and its restriction $f|_{[a,+\infty)}$, we have:

$$\begin{aligned} f \preccurlyeq 1 \quad &\Longleftrightarrow \quad \|f\|_a < \infty \text{ for some } a \quad \Longleftrightarrow \quad \|f\|_a < \infty \text{ for all } a, \\ f \prec 1 \quad &\Longleftrightarrow \quad \|f\|_a \to 0 \text{ as } a \to \infty. \end{aligned}$$

Twisted integration. For $f \in \mathcal{C}_a[\mathrm{i}]$ we have the $\mathbb{C}$-linear operator

$$g \mapsto fg \ :\ \mathcal{C}_a[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}],$$

which we also denote by f. We now fix an element $\phi \in \mathcal{C}_a[\mathrm{i}]$, and set $\Phi := \partial_a^{-1}\phi$, so $\Phi \in \mathcal{C}_a^1[\mathrm{i}]$, $\Phi(t) = \int_a^t \phi(s)\,ds$ for $t \geqslant a$, and $\Phi' = \phi$. Thus $\mathrm{e}^{\Phi}, \mathrm{e}^{-\Phi} \in \mathcal{C}_a^1[\mathrm{i}]$ with $(\mathrm{e}^{\Phi})^{\dagger} = \phi$. Consider the $\mathbb{C}$-linear operator

$$B := \mathrm{e}^{\Phi} \circ \partial_a^{-1} \circ \mathrm{e}^{-\Phi} \ :\ \mathcal{C}_a[\mathrm{i}] \to \mathcal{C}_a^1[\mathrm{i}],$$

so

$$Bf(t) \ =\ \mathrm{e}^{\Phi(t)} \int_a^t \mathrm{e}^{-\Phi(s)}\, f(s)\, ds \qquad \text{for } f \in \mathcal{C}_a[\mathrm{i}].$$

It is easy to check that B is a right inverse to $\partial - \phi\colon \mathcal{C}_a^1[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}]$ in the sense that $(\partial - \phi) \circ B$ is the identity on $\mathcal{C}_a[\mathrm{i}]$. Note that for $f \in \mathcal{C}_a[\mathrm{i}]$ we have $Bf(a) = 0$, and thus $(Bf)'(a) = f(a)$, using $(Bf)' = f + \phi B(f)$. Set $R := \operatorname{Re}\Phi$ and $S := \operatorname{Im}\Phi$, so $R, S \in \mathcal{C}_a^1$, $R' = \operatorname{Re}\phi$, $S' = \operatorname{Im}\phi$, and $R(a) = S(a) = 0$. Note also that if $\phi \in \mathcal{C}_a^r[\mathrm{i}]$, then $\mathrm{e}^{\Phi} \in \mathcal{C}_a^{r+1}[\mathrm{i}]$, so B maps $\mathcal{C}_a^r[\mathrm{i}]$ into $\mathcal{C}_a^{r+1}[\mathrm{i}]$.

Suppose $\varepsilon > 0$ and $\operatorname{Re}\phi(t) \leqslant -\varepsilon$ for all $t \geqslant a$. Then $-R$ has derivative $-R'(t) \geqslant \varepsilon$ for all $t \geqslant a$, so $-R$ is strictly increasing with image $[-R(a), \infty) = [0, \infty)$ and compositional inverse $(-R)^{\mathrm{inv}} \in \mathcal{C}_0^1$. Making the change of variables $-R(s) = u$

for $s \geqslant a$, we obtain for $t \geqslant a$ and $f \in \mathcal{C}_a[\mathrm{i}]$, and with $s := (-R)^{\mathrm{inv}}(u)$,

$$\begin{aligned} \int_a^t \mathrm{e}^{-\Phi(s)} f(s)\,ds &= \int_0^{-R(t)} \mathrm{e}^{-\Phi(s)} f(s)\frac{1}{-R'(s)}\,du, \quad \text{and thus} \\ |Bf(t)| &\leqslant \mathrm{e}^{R(t)} \cdot \left(\int_0^{-R(t)} \mathrm{e}^{u}\ du \cdot \|f\|_{[a,t]}\right) \cdot \left\|\frac{1}{\operatorname{Re}\phi}\right\|_{[a,t]} \\ &= \big[1 - \mathrm{e}^{R(t)}\,\big] \cdot \|f\|_{[a,t]} \cdot \left\|\frac{1}{\operatorname{Re}\phi}\right\|_{[a,t]} \\ &\leqslant \|f\|_{[a,t]} \cdot \left\|\frac{1}{\operatorname{Re}\phi}\right\|_{[a,t]} \ \leqslant\ \|f\|_a \cdot \left\|\frac{1}{\operatorname{Re}\phi}\right\|_a. \end{aligned}$$

Thus B maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^1[\mathrm{i}]$ and $B\colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ is continuous with operator norm $\|B\|_a \leqslant \big\|\frac{1}{\operatorname{Re}\phi}\big\|_a$.

Next, suppose $\varepsilon > 0$ and $\operatorname{Re}\phi(t) \geqslant \varepsilon$ for all $t \geqslant a$. Then $R'(t) \geqslant \varepsilon$ for all $t \geqslant a$, so $R(t) \geqslant \varepsilon \cdot (t-a)$ for such t. Hence if $f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, then $\mathrm{e}^{-\Phi} f$ is integrable at ∞. Recall from (5.1) that $\mathcal{C}_a[\mathrm{i}]^{\mathrm{int}}$ is the $\mathbb{C}$-linear subspace of $\mathcal{C}_a[\mathrm{i}]$ consisting of the $g \in \mathcal{C}_a[\mathrm{i}]$ that are integrable at ∞. We have the $\mathbb{C}$-linear maps

$$f \mapsto \mathrm{e}^{-\Phi} f\colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{int}}, \qquad \partial_\infty^{-1}\colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{int}} \to \mathcal{C}_a^1[\mathrm{i}], \qquad f \mapsto \mathrm{e}^{\Phi} f\colon \mathcal{C}_a^1[\mathrm{i}] \to \mathcal{C}_a^1[\mathrm{i}].$$

Composition yields the $\mathbb{C}$-linear operator $B\colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a^1[\mathrm{i}]$,

$$Bf(t) \ := \ \mathrm{e}^{\Phi(t)} \int_\infty^t \mathrm{e}^{-\Phi(s)} f(s)\,ds \qquad (f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}).$$

It is a right inverse to $\partial - \phi$ in the sense that $(\partial - \phi) \circ B$ is the identity on $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. Note that R is strictly increasing with image $[0,\infty)$ and compositional inverse $R^{\mathrm{inv}} \in \mathcal{C}_0^1$. Making the change of variables $R(s) = u$ for $s \geqslant a$, we obtain for $t \geqslant a$ and $f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ with $s := R^{\mathrm{inv}}(u)$,

$$\begin{aligned} \int_\infty^t \mathrm{e}^{-\Phi(s)} f(s)\,ds &= -\int_{R(t)}^\infty \mathrm{e}^{-\Phi(s)} f(s)\frac{1}{R'(s)}\,du, \quad \text{and thus} \\ |Bf(t)| &\leqslant \mathrm{e}^{R(t)} \cdot \left(\int_{R(t)}^\infty \mathrm{e}^{-u}\ du\right) \cdot \|f\|_t \cdot \left\|\frac{1}{\operatorname{Re}\phi}\right\|_t \\ &\leqslant \|f\|_t \cdot \left\|\frac{1}{\operatorname{Re}\phi}\right\|_t \ \leqslant\ \|f\|_a \cdot \left\|\frac{1}{\operatorname{Re}\phi}\right\|_a. \end{aligned}$$

Hence B maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^1[\mathrm{i}]$, and as a $\mathbb{C}$-linear operator $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, B is continuous with operator norm $\|B\|_a \leqslant \big\|\frac{1}{\operatorname{Re}\phi}\big\|_a$. If $\phi \in \mathcal{C}_a^r[\mathrm{i}]$, then B maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^r[\mathrm{i}]$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^{r+1}[\mathrm{i}]$.

The case that for some $\varepsilon > 0$ we have $\operatorname{Re}\phi(t) \leqslant -\varepsilon$ for all $t \geqslant a$ is called the *attractive case*, and the case that for some $\varepsilon > 0$ we have $\operatorname{Re}\phi(t) \geqslant \varepsilon$ for all $t \geqslant a$ is called the *repulsive case*. In both cases the above yields a continuous operator $B\colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ with operator norm $\leqslant \big\|\frac{1}{\operatorname{Re}\phi}\big\|_a$ which is right-inverse to the operator $\partial - \phi\colon \mathcal{C}_a^1[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}]$. We denote this operator B by B_ϕ if we need to indicate its dependence on ϕ. Note also its dependence on a. In both the attractive and the repulsive case, B maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^1[\mathrm{i}]$, and if $\phi \in \mathcal{C}_a^r[\mathrm{i}]$, then B maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^r[\mathrm{i}]$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^{r+1}[\mathrm{i}]$.

Given a Hardy field H and $f \in H[\mathrm{i}]$ with $\operatorname{Re} f \succcurlyeq 1$ we can choose a and a representative of f in $\mathcal{C}_a[\mathrm{i}]$, to be denoted also by f, such that $\operatorname{Re} f(t) \neq 0$ for all $t \geqslant a$, and then $f \in \mathcal{C}_a[\mathrm{i}]$ falls either under the attractive case or under the repulsive case. The original germ $f \in H[\mathrm{i}]$ as well as the function $f \in \mathcal{C}_a[\mathrm{i}]$ is accordingly said to be attractive, respectively repulsive. (This agrees with the terminology of 1.14.)

Twists and right-inverses of linear operators over Hardy fields. Let H be a Hardy field, $K := H[\mathrm{i}]$, and let $A \in K[\partial]$ be a monic operator of order $r \geqslant 1$,

$$A \;=\; \partial^r + f_1\partial^{r-1} + \cdots + f_r, \qquad f_1, \dots, f_r \in K.$$

Take a real number a_0 and functions in $\mathcal{C}_{a_0}[\mathrm{i}]$ that represent the germs $f_1, \dots, f_r$ and to be denoted also by $f_1, \dots, f_r$. Whenever we increase below the value of a_0, it is understood that we also update the functions $f_1, \dots, f_r$ accordingly, by restriction; the same holds for any function on $[a_0, \infty)$ that gets named. Throughout, a ranges over $[a_0, \infty)$, and $f_1, \dots, f_r$ denote also the restrictions of these functions to $[a, \infty)$, and likewise for any function on $[a_0, \infty)$ that we name. Thus for any a we have the $\mathbb{C}$-linear operator

$$A_a \;:\; \mathcal{C}^r_a[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}], \quad y \mapsto y^{(r)} + f_1 y^{(r-1)} + \cdots + f_r y.$$

Next, let $\mathfrak{m} \in H^\times$ be given. It gives rise to the twist $A_{\ltimes \mathfrak{m}} \in K[\partial]$,

$$A_{\ltimes \mathfrak{m}} \;:=\; \mathfrak{m}^{-1} A \mathfrak{m} \;=\; \partial^r + g_1\partial^{r-1} + \cdots + g_r, \qquad g_1, \dots, g_r \in K.$$

Now [ADH, (5.1.1), (5.1.2), (5.1.3)] gives universal expressions for $g_1, \dots, g_r$ in terms of $f_1, \dots, f_r, \mathfrak{m}, \mathfrak{m}^{-1}$; for example, $g_1 = f_1 + r\mathfrak{m}^\dagger$. Suppose the germ $\mathfrak{m}$ is represented by a function in $\mathcal{C}^r_{a_0}[\mathrm{i}]^\times$, also denoted by $\mathfrak{m}$. Let $\mathfrak{m}^{-1}$ likewise do double duty as the multiplicative inverse of $\mathfrak{m}$ in $\mathcal{C}^r_{a_0}[\mathrm{i}]$. The expressions above can be used to show that the germs $g_1, \dots, g_r$ are represented by functions in $\mathcal{C}_{a_0}[\mathrm{i}]$, to be denoted also by $g_1, \dots, g_r$, such that for all a and all $y \in \mathcal{C}^r_a[\mathrm{i}]$ we have

$$\mathfrak{m}^{-1} A_a(\mathfrak{m} y) \;=\; (A_{\ltimes \mathfrak{m}})_a(y), \text{ where } (A_{\ltimes \mathfrak{m}})_a(y) \;:=\; y^{(r)} + g_1 y^{(r-1)} + \cdots + g_r y.$$

The operator $A_a \colon \mathcal{C}^r_a[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}]$ is surjective: see [33, (10.6.3)] or [87, §19, I, II]. We aim to construct a right-inverse of A_a on the subspace $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ of $\mathcal{C}_a[\mathrm{i}]$. For this, we assume given a splitting of A over K,

$$A \;=\; (\partial - \phi_1) \cdots (\partial - \phi_r), \qquad \phi_1, \dots, \phi_r \in K.$$

Take functions in $\mathcal{C}_{a_0}[\mathrm{i}]$, to be denoted also by $\phi_1, \dots, \phi_r$, that represent the germs $\phi_1, \dots, \phi_r$. We increase a_0 to arrange $\phi_1, \dots, \phi_r \in \mathcal{C}^{r-1}_{a_0}[\mathrm{i}]$. Note that for $j = 1, \dots, r$ the $\mathbb{C}$-linear map $\partial - \phi_j \colon \mathcal{C}^1_a[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}]$ restricts to a $\mathbb{C}$-linear map $A_j \colon \mathcal{C}^j_a[\mathrm{i}] \to \mathcal{C}^{j-1}_a[\mathrm{i}]$, so that we obtain a map $A_1 \circ \cdots \circ A_r \colon \mathcal{C}^r_a[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}]$. It is routine to verify that for all sufficiently large a we have

$$A_a \;=\; A_1 \circ \cdots \circ A_r \;:\; \mathcal{C}^r_a[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}].$$

We increase a_0 so that $A_a = A_1 \circ \cdots \circ A_r$ for all a. Note that $A_1, \dots, A_r$ depend on a, but we prefer not to indicate this dependence notationally.

Now $\mathfrak{m} \in H^\times$ gives over K the splitting

$$A_{\ltimes \mathfrak{m}} \;=\; (\partial - \phi_1 + \mathfrak{m}^\dagger) \cdots (\partial - \phi_r + \mathfrak{m}^\dagger).$$

Suppose as before that the germ $\mathfrak{m}$ is represented by a function $\mathfrak{m} \in \mathcal{C}^r_{a_0}[\mathrm{i}]^\times$. With the usual notational conventions we have $\phi_j - \mathfrak{m}^\dagger \in \mathcal{C}^{r-1}_{a_0}[\mathrm{i}]$, giving the $\mathbb{C}$-linear map $\widetilde{A}_j := \partial - (\phi_j - \mathfrak{m}^\dagger)\colon \mathcal{C}_a^j[\mathrm{i}] \to \mathcal{C}_a^{j-1}[\mathrm{i}]$ for $j = 1, \dots, r$, which for all sufficiently large a gives, just as for A_a, a factorization

$$(A_{\ltimes \mathfrak{m}})_a \;=\; \widetilde{A}_1 \circ \cdots \circ \widetilde{A}_r.$$

To construct a right-inverse of A_a we now assume $\operatorname{Re}\phi_1, \dots, \operatorname{Re}\phi_r \succcurlyeq 1$. Then we increase a_0 once more so that for all $t \geqslant a_0$,

$$\operatorname{Re}\phi_1(t), \dots, \operatorname{Re}\phi_r(t) \neq 0.$$

Recall that for $j = 1, \dots, r$ we have the continuous $\mathbb{C}$-linear operator

$$B_j \;:=\; B_{\phi_j} \;:\; \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$$

from the previous subsection. The subsection on twisted integration now yields:

Lemma 5.4. *The continuous $\mathbb{C}$-linear operator*

$$A_a^{-1} \;:=\; B_r \circ \cdots \circ B_1 \;:\; \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$$

is a right-inverse of A_a: it maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^r[\mathrm{i}]$, and $A_a \circ A_a^{-1}$ is the identity on $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. For its operator norm we have $\|A_a^{-1}\|_a \;\leqslant\; \prod_{j=1}^r \big\|\frac{1}{\operatorname{Re}\phi_j}\big\|_a$.

Suppose A is real in the sense that $A \in H[\partial]$. Then by increasing a_0 we arrange that $f_1, \dots, f_r \in \mathcal{C}_{a_0}$. Next, set

$$\mathcal{C}_a^{\mathrm{b}} \;:=\; \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a \;=\; \big\{f \in \mathcal{C}_a : \|f\|_a < \infty\big\},$$

an $\mathbb{R}$-linear subspace of $\mathcal{C}_a$. Then the real part

$$\operatorname{Re} A_a^{-1} \;:\; \mathcal{C}_a^{\mathrm{b}} \to \mathcal{C}_a^{\mathrm{b}}, \qquad (\operatorname{Re} A_a^{-1})(f) \;:=\; \operatorname{Re}\big(A_a^{-1}(f)\big)$$

is $\mathbb{R}$-linear and maps $\mathcal{C}_a^{\mathrm{b}}$ into $\mathcal{C}_a^r$. Moreover, it is right-inverse to A_a on $\mathcal{C}_a^{\mathrm{b}}$ in the sense that $A_a \circ \operatorname{Re} A_a^{-1}$ is the identity on $\mathcal{C}_a^{\mathrm{b}}$, and for $f \in \mathcal{C}_a^{\mathrm{b}}$,

$$\|(\operatorname{Re} A_a^{-1})(f)\|_a \;\leqslant\; \|A_a^{-1}(f)\|_a.$$

Damping factors. Here H, K, A, $f_1, \dots, f_r$, $\phi_1, \dots, \phi_r$, a_0 are as in Lemma 5.4. In particular, $r \in \mathbb{N}^{\geqslant 1}$, $\operatorname{Re}\phi_1, \dots, \operatorname{Re}\phi_r \succcurlyeq 1$, and a ranges over $[a_0, \infty)$. For later use we choose damping factors u to make the operator uA_a^{-1} more manageable than A_a^{-1}. For $j = 0, \dots, r$ we set

$$A_j^\circ \;:=\; A_1 \circ \cdots \circ A_j \;:\; \mathcal{C}_a^j[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}], \tag{5.2}$$

with A_0° the identity on $\mathcal{C}_a[\mathrm{i}]$ and $A_r^\circ = A_a$, and

$$B_j^\circ \;:=\; B_j \circ \cdots \circ B_1 \;:\; \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}, \tag{5.3}$$

where B_0° is the identity on $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ and $B_r^\circ = A_a^{-1}$. Then B_j° maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^j[\mathrm{i}]$ and $A_j^\circ \circ B_j^\circ$ is the identity on $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ by Lemma 5.4.

Lemma 5.5. *Let $u \in \mathcal{C}_a^r[\mathrm{i}]^\times$. Then for $i = 0, \dots, r$ and $f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$,*

$$\big[u \cdot A_a^{-1}(f)\big]^{(i)} \;=\; \sum_{j=r-i}^{r} u_{i,j} \cdot u \cdot B_j^\circ(f) \quad \text{in } \mathcal{C}_a^{r-i}[\mathrm{i}] \tag{5.4}$$

with coefficient functions $u_{i,j} \in \mathcal{C}_a^{r-i}[\mathrm{i}]$ given by $u_{i,r-i} = 1$, and for $0 \leqslant i < r$,

$$u_{i+1,j} \;=\; \begin{cases} u_{i,r}' + u_{i,r}(u^\dagger + \phi_r) & \text{if } j = r, \\ u_{i,j}' + u_{i,j}(u^\dagger + \phi_j) + u_{i,j+1} & \text{if } r - i \leqslant j < r. \end{cases}$$

Proof. Recall that for $j=1,\dots,r$ and $f\in\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ we have $B_j(f)'=f+\phi_jB_j(f)$. It is obvious that (5.4) holds for $i=0$. Assuming (5.4) for a certain $i<r$ we get

$$\big[uA_a^{-1}(f)\big]^{(i+1)} \ =\ \sum_{j=r-i}^{r} u_{i,j}'\cdot uB_j^{\circ}(f)+\sum_{j=r-i}^{r} u_{i,j}\cdot\big[uB_j^{\circ}(f)\big]',$$

and for $j=r-i,\dots,r$,

$$\big[uB_j^{\circ}(f)\big]' \ =\ u'B_j^{\circ}(f)+u\cdot\big[B_j^{\circ}(f)\big]' \ =\ u^{\dagger}\cdot uB_j^{\circ}(f)+uB_{j-1}^{\circ}(f)+\phi_j uB_j^{\circ}(f),$$

which gives the desired result. □

Let $\mathfrak{v}\in\mathcal{C}_{a_0}^{r}$ be such that $\mathfrak{v}(t)>0$ for all $t\geqslant a_0$, $\mathfrak{v}\in H$, $\mathfrak{v}\prec 1$. Then we have the convex subgroup

$$\Delta\ :=\ \big\{\gamma\in v(H^{\times}):\ \gamma=o(v\mathfrak{v})\big\}$$

of $v(H^{\times})$. *We assume that $\phi_1,\dots,\phi_r\preccurlyeq_{\Delta}\mathfrak{v}^{-1}$ in the asymptotic field K, where ϕ_j and $\mathfrak{v}$ also denote their germs.* For real $\nu>0$ we have $\mathfrak{v}^{\nu}\in(\mathcal{C}_{a_0}^{r})^{\times}$, so

$$u\ :=\ \mathfrak{v}^{\nu}|_{[a,\infty)}\in(\mathcal{C}_a^{r})^{\times},\qquad \|u\|_a<\infty.$$

In the next proposition u has this meaning, a meaning which accordingly varies with a. Recall that A_a^{-1} maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\cap\mathcal{C}_a^{r}[\mathrm{i}]$ with $\|A_a^{-1}\|_a<\infty$.

Proposition 5.6. *Assume H is real closed and $\nu\in\mathbb{Q}$, $\nu>r$. Then:*

- (i) *the $\mathbb{C}$-linear operator $uA_a^{-1}\colon\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\to\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a^{r}[\mathrm{i}]^{\mathrm{b}}$;*
- (ii) *$uA_a^{-1}\colon\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\to\mathcal{C}_a^{r}[\mathrm{i}]^{\mathrm{b}}$ is continuous;*
- (iii) *there is a real constant $c\geqslant 0$ such that $\|uA_a^{-1}\|_{a;r}\leqslant c$ for all a;*
- (iv) *for all $f\in\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ we have $uA_a^{-1}(f)\preccurlyeq\mathfrak{v}^{\nu}\prec 1$;*
- (v) *$\|uA_a^{-1}\|_{a;r}\to 0$ as $a\to\infty$.*

Proof. Note that $\mathfrak{v}^{\dagger}\preccurlyeq_{\Delta}1$ by [ADH, 9.2.10(iv)]. Denoting the germ of u also by u we have $u\in H$ and $u^{\dagger}=\nu\mathfrak{v}^{\dagger}\preccurlyeq_{\Delta}1$, in particular, $u^{\dagger}\preccurlyeq\mathfrak{v}^{-1/2}$. Note that the $u_{i,j}$ from Lemma 5.5—that is, their germs—lie in K. Induction on i gives $u_{i,j}\preccurlyeq_{\Delta}\mathfrak{v}^{-i}$ for $r-i\leqslant j\leqslant r$. Hence $uu_{i,j}\prec_{\Delta}\mathfrak{v}^{\nu-i}\prec_{\Delta}1$ for $r-i\leqslant j\leqslant r$. Thus for $i=0,\dots,r$ we have a real constant

$$c_{i,a}\ :=\ \sum_{j=r-i}^{r}\|u\,u_{i,j}\|_a\cdot\|B_j\|_a\cdots\|B_1\|_a\in[0,\infty)$$

with $\big\|\big[uA_a^{-1}(f)\big]^{(i)}\big\|_a\leqslant c_{i,a}\|f\|_a$ for all $f\in\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. Therefore uA_a^{-1} maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a^{r}[\mathrm{i}]^{\mathrm{b}}$, and the operator $uA_a^{-1}\colon\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\to\mathcal{C}_a^{r}[\mathrm{i}]^{\mathrm{b}}$ is continuous with

$$\|uA_a^{-1}\|_{a;r}\ \leqslant\ c_a:=\max\{c_{0,a},\dots,c_{r,a}\}.$$

As to (iii), this is because for all i, j, $\|u\,u_{ij}\|_a$ is decreasing as a function of a, and $\|B_j\|_a\leqslant\big\|\frac{1}{\operatorname{Re}\phi_j}\big\|_a$ for all j. For $f\in\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ we have $A_a^{-1}(f)\in\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, so (iv) holds. As to (v), $u\,u_{i,j}\prec 1$ gives $\|uu_{ij}\|_a\to 0$ as $a\to\infty$, for all i, j. In view of $\|B_j\|_a\leqslant\big\|\frac{1}{\operatorname{Re}\phi_j}\big\|_a$ for all j, this gives $c_{i,a}\to 0$ as $a\to\infty$ for $i=0,\dots,r$, so $c_a\to 0$ as $a\to\infty$. □

6. Solving Split-Normal Equations over Hardy Fields

We construct here solutions of suitable algebraic differential equations over Hardy fields. These solutions lie in rings $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$ ($r \in \mathbb{N}^{\geqslant 1}$) and are obtained as fixed points of certain contractive maps, as is common in solving differential equations. Here we use that $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$ is a Banach space with respect to the norm $\|\cdot\|_{a;r}$. It will take some effort to define the right contractions using the operators from Section 5.

In this section H, K, A, $f_1,\dots,f_r$, $\phi_1,\dots,\phi_r$, a_0 are as in Lemma 5.4. In particular, H is a Hardy field, $K = H[\mathrm{i}]$, and

$$A = (\partial - \phi_1)\cdots(\partial-\phi_r) \qquad \text{where } r \in \mathbb{N}^{\geqslant 1},\ \phi_1,\dots,\phi_r \in K,\ \operatorname{Re}\phi_1,\dots,\operatorname{Re}\phi_r \succcurlyeq 1.$$

Here a_0 is chosen so that we have representatives for $\phi_1,\dots,\phi_r$ in $\mathcal{C}^{r-1}_{a_0}[\mathrm{i}]$, denoted also by $\phi_1,\dots,\phi_r$. We let a range over $[a_0,\infty)$. In addition we assume that H is real closed, and that we are given a germ $\mathfrak{v} \in H^{>}$ such that $\mathfrak{v} \prec 1$ and $\phi_1,\dots,\phi_r \preccurlyeq_\Delta \mathfrak{v}^{-1}$ for the convex subgroup

$$\Delta \ :=\ \big\{\gamma \in v(H^\times) :\ \gamma = o(v\mathfrak{v})\big\}$$

of $v(H^\times)$. We increase a_0 so that $\mathfrak{v}$ is represented by a function in $\mathcal{C}^r_{a_0}$, also denoted by $\mathfrak{v}$, with $\mathfrak{v}(t) > 0$ for all $t \geqslant a_0$.

Constructing fixed points over H. Consider a differential equation

$$(*) \qquad\qquad A(y) \ =\ R(y), \qquad y \prec 1,$$

where $R \in K\{Y\}$ has order $\leqslant r$, degree $\leqslant d \in \mathbb{N}^{\geqslant 1}$ and weight $\leqslant w \in \mathbb{N}^{\geqslant r}$, with $R \prec_\Delta \mathfrak{v}^w$. Now $R = \sum_{\boldsymbol{j}} R_{\boldsymbol{j}} Y^{\boldsymbol{j}}$ with $\boldsymbol{j}$ ranging here and below over the tuples $(j_0,\dots,j_r) \in \mathbb{N}^{1+r}$ with $|\boldsymbol{j}| \leqslant d$ and $\|\boldsymbol{j}\| \leqslant w$; likewise for $\boldsymbol{i}$. For each $\boldsymbol{j}$ we take a function in $\mathcal{C}_{a_0}[\mathrm{i}]$ that represents the germ $R_{\boldsymbol{j}} \in K$ and let $R_{\boldsymbol{j}}$ denote this function as well as its restriction to any $[a,\infty)$. Thus R is represented on $[a,\infty)$ by a polynomial $\sum_{\boldsymbol{j}} R_{\boldsymbol{j}} Y^{\boldsymbol{j}} \in \mathcal{C}_a[\mathrm{i}]\big[Y,\dots,Y^{(r)}\big]$, to be denoted also by R for simplicity. This yields for each a an evaluation map

$$f \mapsto R(f) := \sum_{\boldsymbol{j}} R_{\boldsymbol{j}} f^{\boldsymbol{j}} \ :\ \mathcal{C}^r_a[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}].$$

As in [ADH, 4.2] we also have for every $\boldsymbol{i}$ the formal partial derivative

$$R^{(\boldsymbol{i})} \ :=\ \frac{\partial^{|\boldsymbol{i}|} R}{\partial^{i_0} Y \cdots \partial^{i_r} Y^{(r)}} \ \in\ \mathcal{C}_a[\mathrm{i}]\big[Y,\dots,Y^{(r)}\big]$$

with $R^{(\boldsymbol{i})} = \sum_{\boldsymbol{j}} R^{(\boldsymbol{i})}_{\boldsymbol{j}} Y^{\boldsymbol{j}}$, all $R^{(\boldsymbol{i})}_{\boldsymbol{j}} \in \mathcal{C}_a[\mathrm{i}]$ having their germs in K.

A *solution of* $(*)$ *on* $[a,\infty)$ is a function $f \in \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$ such that $A_a(f) = R(f)$ and $f \prec 1$. One might try to obtain a solution of $(*)$ as a fixed point of the operator $f \mapsto A_a^{-1}\big(R(f)\big)$, but this operator might fail to be contractive on a useful space of functions. Therefore we twist A and arrange things so that we can use Proposition 5.6. In the rest of this section we fix $\nu \in \mathbb{Q}$ with $\nu > w$ (so $\nu > r$) such that $R \prec_\Delta \mathfrak{v}^\nu$ and $\nu\mathfrak{v}^\dagger \not\sim \operatorname{Re}\phi_j$ in H for $j = 1,\dots,r$. (Note that such ν exists.) Then the twist $\widetilde{A} := A_{\ltimes \mathfrak{v}^\nu} = \mathfrak{v}^{-\nu} A \mathfrak{v}^\nu \in K[\partial]$ splits over K as follows:

$$\widetilde{A} \ =\ (\partial - \phi_1 + \nu\mathfrak{v}^\dagger)\cdots(\partial - \phi_r + \nu\mathfrak{v}^\dagger), \qquad \text{with}$$
$$\phi_j - \nu\mathfrak{v}^\dagger \ \preccurlyeq_\Delta\ \mathfrak{v}^{-1}, \quad \operatorname{Re}\phi_j - \nu\mathfrak{v}^\dagger \ \succcurlyeq\ 1 \qquad (j = 1,\dots,r).$$

We also increase a_0 so that $\operatorname{Re}\phi_j(t) - \nu\mathfrak{v}^\dagger(t) \neq 0$ for all $t \geqslant a_0$ and such that for all a and $u := \mathfrak{v}^\nu|_{[a,\infty)} \in (\mathcal{C}_a^r)^\times$ the operator $\widetilde{A}_a : \mathcal{C}_a^r[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}]$ satisfies

$$\widetilde{A}_a(y) \;=\; u^{-1}A_a(uy) \qquad (y \in \mathcal{C}_a^r[\mathrm{i}]).$$

(See the explanations before Lemma 5.4 for definitions of A_a and $\widetilde{A}_a$.) We now increase a_0 once more, fixing it for the rest of the section except in the subsection "Preserving reality", so as to obtain as in Lemma 5.4, with $\widetilde{A}$ in the role of A, a right-inverse $\widetilde{A}_a^{-1} : \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ for such $\widetilde{A}_a$.

Lemma 6.1. *We have a continuous operator* (*not necessarily* $\mathbb{C}$*-linear*)

$$\Xi_a \;:\; \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}, \quad f \mapsto u\widetilde{A}_a^{-1}\big(u^{-1}R(f)\big).$$

It has the property that $\Xi_a(f) \preccurlyeq \mathfrak{v}^\nu \prec 1$ *and* $A_a\big(\Xi_a(f)\big) = R(f)$ *for all* $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$.

Proof. We have $\|u^{-1}R_{\boldsymbol{i}}\|_a < \infty$ for all $\boldsymbol{i}$, so $u^{-1}R(f) = \sum_{\boldsymbol{i}} u^{-1}R_{\boldsymbol{i}} f^{\boldsymbol{i}} \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ for all $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$, and thus $u\widetilde{A}_a^{-1}\big(u^{-1}R(f)\big) \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ for such f, by Proposition 5.6(i). Continuity of Ξ_a follows from Proposition 5.6(ii) and continuity of $f \mapsto u^{-1}R(f) : \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. For $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ we have $\Xi_a(f) \preccurlyeq \mathfrak{v}^\nu \prec 1$ by Proposition 5.6(iv), and

$$u^{-1}A_a\big(\Xi_a(f)\big) \;=\; u^{-1}A_a\big[u\widetilde{A}_a^{-1}\big(u^{-1}R(f)\big)\big] \;=\; \widetilde{A}_a\big[\widetilde{A}_a^{-1}\big(u^{-1}R(f)\big)\big] \;=\; u^{-1}R(f),$$

so $A_a\big(\Xi_a(f)\big) = R(f)$. ∎

By Lemma 6.1, each $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ with $\Xi_a(f) = f$ is a solution of $(*)$ on $[a,\infty)$.

Lemma 6.2. *There is a constant* $C_a \in \mathbb{R}^{\geqslant}$ *such that for all* $f, g \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$,

$$\|\Xi_a(f+g) - \Xi_a(f)\|_{a;r} \;\leqslant\; C_a \cdot \max\big\{1, \|f\|_{a;r}^d\big\} \cdot \big(\|g\|_{a;r} + \cdots + \|g\|_{a;r}^d\big).$$

We can take these C_a *such that* $C_a \to 0$ *as* $a \to \infty$*, and we do so below.*

Proof. Let $f, g \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$. We have the Taylor expansion

$$R(f+g) \;=\; \sum_{\boldsymbol{i}} \frac{1}{\boldsymbol{i}!} R^{(\boldsymbol{i})}(f) g^{\boldsymbol{i}} \;=\; \sum_{\boldsymbol{i}} \frac{1}{\boldsymbol{i}!}\Big[\sum_{\boldsymbol{j}} R_{\boldsymbol{j}}^{(\boldsymbol{i})} f^{\boldsymbol{j}}\Big] g^{\boldsymbol{i}}.$$

Now for all $\boldsymbol{i}$, $\boldsymbol{j}$ we have $R_{\boldsymbol{j}}^{(\boldsymbol{i})} \prec_\Delta \mathfrak{v}^\nu$ in K, so $u^{-1}R_{\boldsymbol{j}}^{(\boldsymbol{i})} \prec 1$. Hence

$$D_a \;:=\; \sum_{\boldsymbol{i},\boldsymbol{j}} \big\|u^{-1}R_{\boldsymbol{j}}^{(\boldsymbol{i})}\big\|_a \;\in\; [0,\infty)$$

has the property that $D_a \to 0$ as $a \to \infty$, and

$$\big\|u^{-1}\big(R(f+g) - R(f)\big)\big\|_a \;\leqslant\; D_a \cdot \max\big\{1, \|f\|_{a;r}^d\big\} \cdot \big(\|g\|_{a;r} + \cdots + \|g\|_{a;r}^d\big).$$

So $h := u^{-1}\big(R(f+g) - R(f)\big) \in \mathcal{C}_a^0[\mathrm{i}]^{\mathrm{b}}$ gives $\Xi_a(f+g) - \Xi_a(f) = u\widetilde{A}_a^{-1}(h)$, and

$$\|\Xi_a(f+g) - \Xi_a(f)\|_{a;r} = \|u\widetilde{A}_a^{-1}(h)\|_{a;r} \leqslant \|u\widetilde{A}_a^{-1}\|_{a;r} \cdot \|h\|_a.$$

Thus the lemma holds for $C_a := \|u\widetilde{A}_a^{-1}\|_{a;r} \cdot D_a$. ∎

In the proof of the next theorem we use the well-known fact that the normed vector space $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ over $\mathbb{C}$ is actually a Banach space. Thus if $S \subseteq \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ is nonempty and closed and $\Phi : S \to S$ is contractive (that is, there is a $\lambda \in [0,1)$ such that $\|\Phi(f) - \Phi(g)\|_{a;r} \leqslant \lambda\|f-g\|_{a;r}$ for all $f, g \in S$), then Φ has a unique fixed point f_0, and $\Phi^n(f) \to f_0$ as $n \to \infty$, for every $f \in S$. (See, for example, [87, Ch. II, §5, IX].)

Theorem 6.3. *For all sufficiently large a the operator Ξ_a maps the closed ball*

$$\big\{f \in \mathcal{C}_a^r[\mathrm{i}] : \|f\|_{a;r} \leqslant 1/2\big\}$$

of the Banach space $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ into itself and has a unique fixed point on this ball.

Proof. We have $\Xi_a(0) = u\widetilde{A}_a^{-1}(u^{-1}R_0)$, so $\|\Xi_a(0)\|_{a;r} \leqslant \|u\widetilde{A}_a^{-1}\|_{a;r}\|u^{-1}R_0\|_a$. Now $\|u^{-1}R_0\|_a \to 0$ as $a \to \infty$, so by Proposition 5.6(iii) we can take a so large that $\|u\widetilde{A}_a^{-1}\|_{a;r}\|u^{-1}R_0\|_a \leqslant \frac{1}{4}$. For f, g in the closed ball above we have by Lemma 6.2,

$$\|\Xi_a(f) - \Xi_a(g)\|_{a;r} = \|\Xi_a(f + (g-f)) - \Xi_a(f)\|_{a;r} \leqslant C_a \cdot d\|f-g\|_{a;r}.$$

Take a so large that also $C_a d \leqslant \frac{1}{2}$. Then $\|\Xi_a(f) - \Xi_a(g)\|_{a;r} \leqslant \frac{1}{2}\|f-g\|_{a;r}$. Applying this to $g = 0$ we see that Ξ_a maps the closed ball above to itself. Thus Ξ_a has a unique fixed point on this ball. □

If $\deg R \leqslant 0$ (so $R = R_0$), then $\Xi_a(f) = u\widetilde{A}_a^{-1}(u^{-1}R_0)$ is independent of $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$, so for sufficiently large a, the fixed point $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ of Ξ_a with $\|f\|_{a;r} \leqslant 1/2$ is $f = \Xi_a(0) = u\widetilde{A}_a^{-1}(u^{-1}R_0)$.

Next we investigate the difference between solutions of $(*)$ on $[a_0, \infty)$:

Lemma 6.4. *Suppose $f, g \in \mathcal{C}_{a_0}^r[\mathrm{i}]^{\mathrm{b}}$ and $A_{a_0}(f) = R(f)$, $A_{a_0}(g) = R(g)$. Then there are positive reals E, ε such that for all a there exists an $h_a \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ with the property that for $\theta_a := (f-g)|_{[a,\infty)}$,*

$$A_a(h_a) = 0, \quad \theta_a - h_a \prec \mathfrak{v}^w, \quad \|\theta_a - h_a\|_{a;r} \leqslant E \cdot \|\mathfrak{v}^\varepsilon\|_a \cdot \big(\|\theta_a\|_{a;r} + \cdots + \|\theta_a\|_{a;r}^d\big),$$

and thus $h_a \prec 1$ in case $f - g \prec 1$.

Proof. Set $\eta_a := A_a(\theta_a) = R(f) - R(g)$, where f and g stand for their restrictions to $[a, \infty)$. From $R \prec \mathfrak{v}^\nu$ we get $u^{-1}R(f) \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ and $u^{-1}R(g) \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, so $u^{-1}\eta_a \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. By Proposition 5.6(i),(iv) we have

$$\xi_a := u\widetilde{A}_a^{-1}(u^{-1}\eta_a) \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}, \qquad \xi_a \prec \mathfrak{v}^w.$$

Now $\widetilde{A}_a(u^{-1}\xi_a) = u^{-1}\eta_a$, that is, $A_a(\xi_a) = \eta_a$. Note that then $h_a := \theta_a - \xi_a$ satisfies $A_a(h_a) = 0$. Now $\xi_a = \theta_a - h_a$ and $\xi_a = \Xi_a(g + \theta_a) - \Xi_a(g)$, hence by Lemma 6.2 and its proof,

$$\|\theta_a - h_a\|_{a;r} = \|\xi_a\|_{a;r} \leqslant C_a \cdot \max\big\{1, \|g\|_{a;r}^d\big\} \cdot \big(\|\theta\|_{a;r} + \cdots + \|\theta\|_{a;r}^d\big), \text{ with}$$

$$C_a := \big\|u\widetilde{A}_a^{-1}\big\|_{a;r} \cdot \sum_{\boldsymbol{i},\boldsymbol{j}} \big\|u^{-1}R_{\boldsymbol{j}}^{(\boldsymbol{i})}\big\|_a.$$

Take a real $\varepsilon > 0$ such that $R \prec \mathfrak{v}^{\nu+\varepsilon}$. This gives a real $e > 0$ such that $\sum_{\boldsymbol{i},\boldsymbol{j}} \big\|u^{-1}R_{\boldsymbol{j}}^{(\boldsymbol{i})}\big\|_a \leqslant e\|\mathfrak{v}^\varepsilon\|_a$ for all a. Now use Proposition 5.6(iii). □

The situation we have in mind in the lemma above is that f and g are close at infinity, in the sense that $\|f-g\|_{a;r} \to 0$ as $a \to \infty$. Then the lemma yields solutions of $A(y) = 0$ that are *very* close to $f - g$ at infinity. However, being very close at infinity as stated in Lemma 6.4, namely $\theta_a - h_a \prec \mathfrak{v}^w$ and the rest, is too weak for later use. We take up this issue again in Section 9 below. (In Corollary 6.13 later in the present section we already show: if $f \neq g$ as germs, then $h_a \neq 0$ for sufficiently large a.)

Preserving reality. We now assume in addition that A and R are real, that is, $A \in H[\partial]$ and $R \in H\{Y\}$. It is not clear that the fixed points constructed in the proof of Theorem 6.3 are then also real. Therefore we slightly modify this construction using real parts. We first apply the discussion following Lemma 5.4 to $\widetilde{A}$ as well as to A, increasing a_0 so that for all a the $\mathbb{R}$-linear real part $\operatorname{Re}\widetilde{A}_a^{-1}\colon \mathcal{C}_a^{\mathrm{b}} \to \mathcal{C}_a^{\mathrm{b}}$ maps $\mathcal{C}_a^{\mathrm{b}}$ into $\mathcal{C}_a^r$ and is right-inverse to $\widetilde{A}_a$ on $(\mathcal{C}_a^0)^{\mathrm{b}}$, with

$$\big\|(\operatorname{Re}\widetilde{A}_a^{-1})(f)\big\|_a \leqslant \big\|\widetilde{A}_a^{-1}(f)\big\|_a \qquad \text{for all } f \in \mathcal{C}_a^{\mathrm{b}}.$$

Next we set

$$(\mathcal{C}_a^r)^{\mathrm{b}} \ := \ \big\{f \in \mathcal{C}_a^r : \ \|f\|_{a;r} < \infty\big\} \ = \ \mathcal{C}_a^r[i]^{\mathrm{b}} \cap \mathcal{C}_a^r,$$

which is a real Banach space with respect to $\|\cdot\|_{a;r}$. Finally, this increasing of a_0 is done so that the original $R_{\boldsymbol{j}} \in \mathcal{C}_{a_0}[i]$ restrict to updated functions $R_{\boldsymbol{j}} \in \mathcal{C}_{a_0}$. For all a, take u, Ξ_a as in Lemma 6.1. This lemma has the following real analogue as a consequence:

Lemma 6.5. *The operator*

$$\operatorname{Re}\,\Xi_a \ : \ (\mathcal{C}_a^r)^{\mathrm{b}} \to (\mathcal{C}_a^r)^{\mathrm{b}}, \quad f \mapsto \operatorname{Re}\big(\Xi_a(f)\big)$$

satisfies $(\operatorname{Re}\,\Xi_a)(f) \preccurlyeq \mathfrak{v}^\nu$ *for* $f \in (\mathcal{C}_a^r)^{\mathrm{b}}$*, and any fixed point of* $\operatorname{Re}\,\Xi_a$ *is a solution of* $(*)$ *on* $[a,\infty)$*.*

Below the constants C_a are as in Lemma 6.2.

Lemma 6.6. *For* $f, g \in (\mathcal{C}_a^r)^{\mathrm{b}}$*,*

$$\big\|(\operatorname{Re}\,\Xi_a)(f+g) - (\operatorname{Re}\Xi_a)(f)\big\|_{a;r} \ \leqslant \ C_a \cdot \max\big\{1, \|f\|_{a;r}^d\big\} \cdot \big(\|g\|_{a;r} + \dots + \|g\|_{a;r}^d\big).$$

The next corollary is derived from Lemma 6.6 in the same way as Theorem 6.3 from Lemma 6.2:

Corollary 6.7. *For all sufficiently large a the operator* $\operatorname{Re}\,\Xi_a$ *maps the closed ball*

$$\big\{f \in \mathcal{C}_a^r : \ \|f\|_{a;r} \leqslant 1/2\big\}$$

of the Banach space $(\mathcal{C}_a^r)^{\mathrm{b}}$ *into itself and has a unique fixed point on this ball.*

We also have a real analogue of Lemma 6.4:

Corollary 6.8. *Suppose* $f, g \in (\mathcal{C}_{a_0}^r)^{\mathrm{b}}$ *and* $A_{a_0}(f) = R(f)$, $A_{a_0}(g) = R(g)$*. Then there are positive reals* E, ε *such that for all a there exists an* $h_a \in (\mathcal{C}_a^r)^{\mathrm{b}}$ *with the property that for* $\theta_a := (f-g)|_{[a,\infty)}$*,*

$$A_a(h_a) = 0, \quad \theta_a - h_a \prec \mathfrak{v}^w, \quad \|\theta_a - h_a\|_{a;r} \ \leqslant \ E \cdot \|\mathfrak{v}^\varepsilon\|_a \cdot \big(\|\theta_a\|_{a;r} + \dots + \|\theta_a\|_{a;r}^d\big).$$

Proof. Take h_a to be the real part of an h_a as in Lemma 6.4. □

Some useful bounds. To prepare for Section 9 we derive in this subsection some bounds from Lemmas 6.2 and 6.4. Throughout we assume $d, r \in \mathbb{N}^{\geqslant 1}$. We begin with an easy inequality:

Lemma 6.9. *Let* $(V, \|\cdot\|)$ *be a normed* $\mathbb{C}$*-linear space, and* $f, g \in V$*. Then*

$$\|f+g\|^d \ \leqslant \ 2^d \cdot \max\big\{1, \|f\|^d\big\} \cdot \max\big\{1, \|g\|^d\big\}.$$

Proof. Use that $\|f+g\| \leqslant \|f\| + \|g\| \leqslant 2\max\big\{1, \|f\|\big\} \cdot \max\big\{1, \|g\|\big\}$. □

Now let u, Ξ_a be as in Lemma 6.1. By that lemma, the operator

$$\Phi_a \ :\ \mathcal{C}^r_a[i]^{\mathrm{b}} \times \mathcal{C}^r_a[i]^{\mathrm{b}} \to \mathcal{C}^r_a[i]^{\mathrm{b}}, \quad (f,y) \mapsto \Xi_a(f+y) - \Xi_a(f)$$

is continuous. Furthermore $\Phi_a(f,0)=0$ for $f \in \mathcal{C}^r_a[i]^{\mathrm{b}}$ and

$$(6.1) \qquad \Phi_a(f,g+y) - \Phi_a(f,g) \ =\ \Phi_a(f+g,y) \qquad \text{for } f,g,y \in \mathcal{C}^r_a[i]^{\mathrm{b}}.$$

Lemma 6.10. *There are $C_a, C_a^+ \in \mathbb{R}^{\geqslant}$ such that for all $f,g,y \in \mathcal{C}^r_a[i]^{\mathrm{b}}$,*

$$(6.2) \qquad \|\Phi_a(f,y)\|_{a;r} \ \leqslant\ C_a \cdot \max\bigl\{1, \|f\|^d_{a;r}\bigr\} \cdot \bigl(\|y\|_{a;r} + \dots + \|y\|^d_{a;r}\bigr),$$

$$(6.3) \quad \|\Phi_a(f,g+y) - \Phi_a(f,g)\|_{a;r} \ \leqslant \\ C_a^+ \cdot \max\bigl\{1, \|f\|^d_{a;r}\bigr\} \cdot \max\bigl\{1, \|g\|^d_{a;r}\bigr\} \cdot \bigl(\|y\|_{a;r} + \dots + \|y\|^d_{a;r}\bigr).$$

We can take these C_a, C_a^+ such that $C_a, C_a^+ \to 0$ as $a \to \infty$, and do so below.

Proof. The C_a as in Lemma 6.2 satisfy the requirements on the C_a here. Now let $f,g,y \in \mathcal{C}^r_a[i]^{\mathrm{b}}$. Then by (6.1) and (6.2) we have

$$\|\Phi_a(f,g+y) - \Phi_a(f,g)\|_{a;r} \ \leqslant\ C_a \cdot \max\bigl\{1, \|f+g\|^d_{a;r}\bigr\} \cdot \bigl(\|y\|_{a;r} + \dots + \|y\|^d_{a;r}\bigr).$$

Thus by Lemma 6.9, $C_a^+ := 2^d \cdot C_a$ has the required property. $\square$

Next, let f, g be as in the hypothesis of Lemma 6.4 and take E, ε, and h_a (for each a) as in its conclusion. Thus for all a and $\theta_a := (f-g)|_{[a,\infty)}$,

$$\|\theta_a - h_a\|_{a;r} \ \leqslant\ E \cdot \|\mathfrak{v}^\varepsilon\|_a \cdot \bigl(\|\theta_a\|_{a;r} + \dots + \|\theta_a\|^d_{a;r}\bigr),$$

and if $f - g \prec 1$, then $h_a \prec 1$. So

$$\|\theta_a - h_a\|_{a;r} \ \leqslant\ E \cdot \|\mathfrak{v}^\varepsilon\|_a \cdot \bigl(\rho + \dots + \rho^d\bigr), \quad \rho := \|f-g\|_{a_0;r}.$$

We let

$$B_a \ :=\ \bigl\{y \in \mathcal{C}^r_a[i]^{\mathrm{b}} :\ \|y - h_a\|_{a;r} \leqslant 1/2\bigr\}$$

be the closed ball of radius $1/2$ around h_a in $\mathcal{C}^r_a[i]^{\mathrm{b}}$. Using $\mathfrak{v}^\varepsilon \prec 1$ we take $a_1 \geqslant a_0$ so that $\theta_a \in B_a$ for all $a \geqslant a_1$. Then for $a \geqslant a_1$ we have

$$\|h_a\|_{a;r} \ \leqslant\ \|h_a - \theta_a\|_{a;r} + \|\theta_a\|_{a;r} \ \leqslant\ \tfrac{1}{2} + \rho,$$

and hence for $y \in B_a$,

$$(6.4) \qquad \|y\|_{a;r} \ \leqslant\ \|y - h_a\|_{a;r} + \|h_a\|_{a;r} \ \leqslant\ \tfrac{1}{2} + \bigl(\tfrac{1}{2} + \rho\bigr) \ =\ 1 + \rho.$$

Consider now the continuous operators

$$\Phi_a, \Psi_a \ :\ \mathcal{C}^r_a[i]^{\mathrm{b}} \to \mathcal{C}^r_a[i]^{\mathrm{b}}, \qquad \Phi_a(y) := \Xi_a(g+y) - \Xi_a(g), \quad \Psi_a(y) := \Phi_a(y) + h_a.$$

In the notation introduced above, $\Phi_a(y) = \Phi_a(g,y)$ for $y \in \mathcal{C}^r_a[i]^{\mathrm{b}}$. With ξ_a as in the proof of Lemma 6.4 we also have $\Phi_a(\theta_a) = \xi_a$ and $\Psi_a(\theta_a) = \xi_a + h_a = \theta_a$. Below we reconstruct the fixed point θ_a of Ψ_a from h_a, for sufficiently large a.

Lemma 6.11. *There exists $a_2 \geqslant a_1$ such that for all $a \geqslant a_2$ we have $\Psi_a(B_a) \subseteq B_a$, and $\|\Psi_a(y) - \Psi_a(z)\|_{a;r} \leqslant \frac{1}{2}\|y - z\|_{a;r}$ for all $y, z \in B_a$.*

Proof. Take C_a as in Lemma 6.10, and let $y \in B_a$. Then by (6.2),

$$\|\Phi_a(y)\|_{a;r} \ \leqslant\ C_a \cdot \max\bigl\{1, \|g\|^d_{a;r}\bigr\} \cdot \bigl(\|y\|_{a;r} + \dots + \|y\|^d_{a;r}\bigr), \quad \theta_a \in B_a, \text{ so}$$

$$\|\Psi_a(y) - h_a\|_{a;r} \ \leqslant\ C_a M, \quad M := \max\bigl\{1, \|g\|^d_{a_0;r}\bigr\} \cdot \bigl((1+\rho) + \dots + (1+\rho)^d\bigr).$$

Recall that $C_a \to 0$ as $a \to \infty$. Suppose $a \geqslant a_1$ is so large that $C_a M \leqslant 1/2$. Then $\Psi_a(B_a) \subseteq B_a$. With C_a^+ as in Lemma 6.10, (6.3) gives for $y, z \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$,

$$\|\Phi_a(y) - \Phi_a(z)\|_{a;r} \leqslant$$
$$C_a^+ \cdot \max\{1, \|g\|_{a;r}^d\} \cdot \max\{1, \|z\|_{a;r}^d\} \cdot \big(\|y-z\|_{a;r} + \cdots + \|y-z\|_{a;r}^d\big).$$

Hence with $N := \max\{1, \|g\|_{a_0;r}^d\} \cdot (1+\rho)^d \cdot d$ we obtain for $y, z \in B_a$ that

$$\|\Psi_a(y) - \Psi_a(z)\|_{a;r} \ \leqslant\ C_a^+ N \|y-z\|_{a;r},$$

so $\|\Psi_a(y) - \Psi_a(z)\|_{a;r} \leqslant \frac{1}{2}\|y-z\|_{a;r}$ if $C_a^+ N \leqslant 1/2$. □

Below a_2 is as in Lemma 6.11.

Corollary 6.12. *If $a \geqslant a_2$, then $\lim_{n\to\infty} \Psi_a^n(h_a) = \theta_a$ in $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$.*

Proof. Let $a \geqslant a_2$. Then Ψ_a has a unique fixed point on B_a. As $\Psi_a(\theta_a) = \theta_a \in B_a$, this fixed point is θ_a. □

Corollary 6.13. *If $f \neq g$ as germs, then $h_a \neq 0$ for $a \geqslant a_2$.*

Proof. Let $a \geqslant a_2$. Then $\lim_{n\to\infty} \Psi_a^n(h_a) = \theta_a$. If $h_a = 0$, then $\Psi_a = \Phi_a$, and hence $\theta_a = 0$, since $\Phi_a(0) = 0$. □

7. Smoothness Considerations

We assume $r \in \mathbb{N}$ in this section. We prove here as much smoothness of solutions of algebraic differential equations over Hardy fields as could be hoped for. In particular, the solutions in $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ of the equation $(*)$ in Section 6 actually have their germs in $\mathcal{C}^{<\infty}[\mathrm{i}]$. To make this precise, consider a "differential" polynomial

$$P \ =\ P(Y, \dots, Y^{(r)}) \ \in\ \mathcal{C}^n[\mathrm{i}]\big[Y, \dots, Y^{(r)}\big].$$

We put *differential* in quotes since $\mathcal{C}^n[\mathrm{i}]$ is not naturally a differential ring. Nevertheless, P defines an obvious evaluation map

$$f \mapsto P\big(f, \dots, f^{(r)}\big) \ :\ \mathcal{C}^r[\mathrm{i}] \to \mathcal{C}[\mathrm{i}].$$

We also have the "separant" of P:

$$S_P \ :=\ \frac{\partial P}{\partial Y^{(r)}} \ \in\ \mathcal{C}^n[\mathrm{i}]\big[Y, \dots, Y^{(r)}\big].$$

Proposition 7.1. *Assume $n \geqslant 1$. Let $f \in \mathcal{C}^r[\mathrm{i}]$ be such that*

$$P\big(f, \dots, f^{(r)}\big) = 0 \in \mathcal{C}[\mathrm{i}] \quad \text{and} \quad S_P\big(f, \dots, f^{(r)}\big) \in \mathcal{C}[\mathrm{i}]^\times.$$

Then $f \in \mathcal{C}^{n+r}[\mathrm{i}]$. Thus if $P \in \mathcal{C}^{<\infty}[\mathrm{i}]\big[Y, \dots, Y^{(r)}\big]$, then $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$. Moreover, if $P \in \mathcal{C}^{\infty}[\mathrm{i}]\big[Y, \dots, Y^{(r)}\big]$, then $f \in \mathcal{C}^{\infty}[\mathrm{i}]$, and likewise with $\mathcal{C}^{\omega}[\mathrm{i}]$ in place of $\mathcal{C}^{\infty}[\mathrm{i}]$.

We deduce this from the lemma below, which has a complex-analytic hypothesis. Let $U \subseteq \mathbb{R} \times \mathbb{C}^{1+r}$ be open. Let t range over $\mathbb{R}$ and $z_0, \dots, z_r$ over $\mathbb{C}$, and set $x_j := \operatorname{Re} z_j$, $y_j := \operatorname{Im} z_j$ for $j = 0, \dots, r$, and

$$U(t, z_0, \dots, z_{r-1}) \ :=\ \big\{z_r : (t, z_0, \dots, z_{r-1}, z_r) \in U\big\},$$

an open subset of $\mathbb{C}$. Assume $\Phi\colon U \to \mathbb{C}$ and $n \geqslant 1$ are such that $\operatorname{Re}\Phi, \operatorname{Im}\Phi\colon U \to \mathbb{R}$ are $\mathcal{C}^n$-functions of $(t, x_0, y_0, \dots, x_r, y_r)$, and for all $t, z_0, \dots, z_{r-1}$ the function

$$z_r \mapsto \Phi(t, z_0, \dots, z_{r-1}, z_r) \ :\ U(t, z_0, \dots, z_{r-1}) \to \mathbb{C}$$

is holomorphic (the complex-analytic hypothesis alluded to).

Lemma 7.2. *Let $I \subseteq \mathbb{R}$ be a nonempty open interval and suppose $f \in \mathcal{C}^r(I)[\mathrm{i}]$ is such that for all $t \in I$,*

- $\big(t, f(t), \dots, f^{(r)}(t)\big) \in U$;
- $\Phi\big(t, f(t), \dots, f^{(r)}(t)\big) = 0$; *and*
- $(\partial\Phi/\partial z_r)\big(t, f(t), \dots, f^{(r)}(t)\big) \neq 0$.

Then $f \in \mathcal{C}^{n+r}(I)[\mathrm{i}]$.

Proof. Set $A := \operatorname{Re}\Phi$, $B := \operatorname{Im}\Phi$ and $g := \operatorname{Re} f$, $h := \operatorname{Im} f$. Then for all $t \in I$,

$$\begin{aligned} A\big(t, g(t), h(t), g'(t), h'(t) \dots, g^{(r)}(t), h^{(r)}(t)\big) &= 0 \\ B\big(t, g(t), h(t), g'(t), h'(t) \dots, g^{(r)}(t), h^{(r)}(t)\big) &= 0. \end{aligned}$$

Consider the $\mathcal{C}^n$-map $(A, B)\colon U \to \mathbb{R}^2$, with U identified in the usual way with an open subset of $\mathbb{R}^{1+2(1+r)}$. The Cauchy-Riemann equations give

$$\frac{\partial\Phi}{\partial z_r} = \frac{\partial A}{\partial x_r} + \mathrm{i}\frac{\partial B}{\partial x_r}, \qquad \frac{\partial A}{\partial x_r} = \frac{\partial B}{\partial y_r}, \qquad \frac{\partial B}{\partial x_r} = -\frac{\partial A}{\partial y_r}.$$

Thus the Jacobian matrix of the map (A, B) with respect to its last two variables x_r and y_r has determinant

$$D = \left(\frac{\partial A}{\partial x_r}\right)^2 + \left(\frac{\partial B}{\partial x_r}\right)^2 = \left|\frac{\partial\Phi}{\partial z_r}\right|^2 : U \to \mathbb{R}.$$

Let $t_0 \in I$. Then

$$D\big(t_0, g(t_0), h(t_0), \dots, g^{(r)}(t_0), h^{(r)}(t_0)\big) \neq 0,$$

so by the Implicit Mapping Theorem [33, (10.2.2), (10.2.3)] we have a connected open neighborhood V of the point

$$\big(t_0, g(t_0), h(t_0), \dots, g^{(r-1)}(t_0), h^{(r-1)}(t_0)\big) \in \mathbb{R}^{1+2r},$$

open intervals $J, K \subseteq \mathbb{R}$ containing $g^{(r)}(t_0)$, $h^{(r)}(t_0)$, respectively, and a $\mathcal{C}^n$-map

$$(G, H)\colon V \to J \times K$$

such that $V \times J \times K \subseteq U$ and the zero set of Φ on $V \times J \times K$ equals the graph of (G, H). Take an open subinterval I_0 of I with $t_0 \in I_0$ such that for all $t \in I_0$,

$$\big(t, g(t), h(t), g'(t), h'(t), \dots, g^{(r-1)}(t), h^{(r-1)}(t), g^{(r)}(t), h^{(r)}(t)\big) \in V \times J \times K.$$

Then the above gives that for all $t \in I_0$ we have

$$\begin{aligned} g^{(r)}(t) &= G\big(t, g(t), h(t), g'(t), h'(t), \dots, g^{(r-1)}(t), h^{(r-1)}(t)\big), \\ h^{(r)}(t) &= H\big(t, g(t), h(t), g'(t), h'(t), \dots, g^{(r-1)}(t), h^{(r-1)}(t)\big). \end{aligned}$$

It follows easily from these two equalities that g, h are of class $\mathcal{C}^{n+r}$ on I_0. □

Let f continue to be as in Lemma 7.2. If $\operatorname{Re}\Phi$, $\operatorname{Im}\Phi$ are $\mathcal{C}^\infty$, then by taking n arbitrarily high we conclude that $f \in \mathcal{C}^\infty(I)[\mathrm{i}]$. Moreover:

Corollary 7.3. *If $\operatorname{Re}\Phi$, $\operatorname{Im}\Phi$ are real-analytic, then $f \in \mathcal{C}^\omega(I)[\mathrm{i}]$.*

Proof. Same as that of Lemma 7.2, with the reference to [33, (10.2.3)] replaced by [33, (10.2.4)] to obtain that G, H are real-analytic, and noting that then the last displayed relations for $t \in I_0$ force g, h to be real-analytic on I_0 by [33, (10.5.3)]. □

Lemma 7.4. *Let $I \subseteq \mathbb{R}$ be a nonempty open interval, $n \geqslant 1$, and*
$$P \;=\; P\big(Y,\dots,Y^{(r)}\big) \;\in\; \mathcal{C}^n(I)[\mathrm{i}]\big[Y,\dots,Y^{(r)}\big].$$
Let $f \in \mathcal{C}^r(I)[\mathrm{i}]$ be such that
$$P\big(f,\dots,f^{(r)}\big) = 0 \in \mathcal{C}(I)[\mathrm{i}] \quad \text{and} \quad \big(\partial P/\partial Y^{(r)}\big)\big(f,\dots,f^{(r)}\big) \in \mathcal{C}(I)[\mathrm{i}]^{\times}.$$
Then $f \in \mathcal{C}^{n+r}(I)[\mathrm{i}]$. Moreover, if $P \in \mathcal{C}^{\infty}(I)[\mathrm{i}]\big[Y,\dots,Y^{(r)}\big]$, then $f \in \mathcal{C}^{\infty}(I)[\mathrm{i}]$, and likewise with $\mathcal{C}^{\omega}(I)[\mathrm{i}]$ in place of $\mathcal{C}^{\infty}(I)[\mathrm{i}]$.

Proof. Let $P = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$ where all $P_{\boldsymbol{i}} \in \mathcal{C}^n(I)[\mathrm{i}]$. Set $U := I \times \mathbb{C}^{1+r}$, and consider the map $\Phi\colon U \to \mathbb{C}$ given by
$$\Phi(t, z_0, \dots, z_r) := \sum_{\boldsymbol{i}} P_{\boldsymbol{i}}(t) z^{\boldsymbol{i}} \quad \text{where } z^{\boldsymbol{i}} := z_0^{i_0} \cdots z_r^{i_r} \text{ for } \boldsymbol{i} = (i_0,\dots,i_r) \in \mathbb{N}^{1+r}.$$
From Lemma 7.2 we obtain $f \in \mathcal{C}^{n+r}(I)[\mathrm{i}]$. In view of Corollary 7.3 and the remark preceding it, and replacing n by ∞ respectively ω, this argument also gives the second part of the lemma. □

Proposition 7.1 follows from Lemma 7.4 by taking suitable representatives of the germs involved. Let now H be a Hardy field and $P \in H[\mathrm{i}]\{Y\}$ of order r. Then $P \in \mathcal{C}^{<\infty}[\mathrm{i}]\big[Y,\dots,Y^{(r)}\big]$, and so $P(f) := P\big(f,\dots,f^{(r)}\big) \in \mathcal{C}[\mathrm{i}]$ for $f \in \mathcal{C}^r[\mathrm{i}]$ as explained in the beginning of this section.

For notational convenience we set
$$\mathcal{C}^n[\mathrm{i}]^{\preccurlyeq} := \big\{f \in \mathcal{C}^n[\mathrm{i}] : f, f', \dots, f^{(n)} \preccurlyeq 1\big\}, \qquad (\mathcal{C}^n)^{\preccurlyeq} := \mathcal{C}^n[\mathrm{i}]^{\preccurlyeq} \cap \mathcal{C}^n,$$
and likewise with $\prec$ instead of $\preccurlyeq$. Then $\mathcal{C}^n[\mathrm{i}]^{\preccurlyeq}$ is a $\mathbb{C}$-subalgebra of $\mathcal{C}^n[\mathrm{i}]$ and $(\mathcal{C}^n)^{\preccurlyeq}$ is an $\mathbb{R}$-subalgebra of $\mathcal{C}^n$. Also, $\mathcal{C}^n[\mathrm{i}]^{\prec}$ is an ideal of $\mathcal{C}^n[\mathrm{i}]^{\preccurlyeq}$, and likewise with $\mathcal{C}^n$ instead of $\mathcal{C}^n[\mathrm{i}]$. We have $\mathcal{C}^n[\mathrm{i}]^{\preccurlyeq} \supseteq \mathcal{C}^{n+1}[\mathrm{i}]^{\preccurlyeq}$ and $(\mathcal{C}^n)^{\preccurlyeq} \supseteq (\mathcal{C}^{n+1})^{\preccurlyeq}$, and likewise with $\prec$ instead of $\preccurlyeq$. For $R \in \mathcal{C}^n[\mathrm{i}]\big[Y,\dots,Y^{(r)}\big]$ we put $R_{\geqslant 1} := R - R(0)$.

Corollary 7.5. *Suppose*
$$P = Y^{(r)} + f_1 Y^{(r-1)} + \cdots + f_r Y - R \quad \text{with } f_1,\dots,f_r \text{ in } H[\mathrm{i}] \text{ and } R_{\geqslant 1} \prec 1.$$
Let $f \in \mathcal{C}^r[\mathrm{i}]^{\preccurlyeq}$ be such that $P(f) = 0$. Then $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$. Moreover, if $H \subseteq \mathcal{C}^{\infty}$, then $f \in \mathcal{C}^{\infty}[\mathrm{i}]$, and if $H \subseteq \mathcal{C}^{\omega}$, then $f \in \mathcal{C}^{\omega}[\mathrm{i}]$.

Proof. We have $S_P = \frac{\partial P}{\partial Y^{(r)}} = 1 - S$ with $S := \frac{\partial R_{\geqslant 1}}{\partial Y^{(r)}} \prec 1$ and thus
$$S_P(f,\dots,f^{(r)}) \;=\; 1 - S(f,\dots,f^{(r)}), \qquad S(f,\dots,f^{(r)}) \prec 1,$$
so $S_P(f,\dots,f^{(r)}) \in \mathcal{C}[\mathrm{i}]^{\times}$. Now appeal to Proposition 7.1. □

Thus the germ of any solution on $[a,\infty)$ of the asymptotic equation $(*)$ of Section 6 lies in $\mathcal{C}^{<\infty}[\mathrm{i}]$, and even in $\mathcal{C}^{\infty}[\mathrm{i}]$ (respectively $\mathcal{C}^{\omega}[\mathrm{i}]$) if H is in addition a $\mathcal{C}^{\infty}$-Hardy field (respectively, a $\mathcal{C}^{\omega}$-Hardy field).

For the differential subfield $K := H[\mathrm{i}]$ of the differential ring $\mathcal{C}^{<\infty}[\mathrm{i}]$ we have:

Corollary 7.6. *Suppose $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$, $P(f) = 0$, and f generates a differential subfield $K\langle f\rangle$ of $\mathcal{C}^{<\infty}[\mathrm{i}]$ over K. If H is a $\mathcal{C}^{\infty}$-Hardy field, then $K\langle f\rangle \subseteq \mathcal{C}^{\infty}[\mathrm{i}]$, and likewise with $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{\infty}$.*

Proof. Suppose H is a $\mathcal{C}^{\infty}$-Hardy field; it suffices to show $f \in \mathcal{C}^{\infty}[\mathrm{i}]$. We may assume that P is a minimal annihilator of f over K; then $S_P(f) \neq 0$ in $K\langle f\rangle$ and so $S_P(f) \in \mathcal{C}[\mathrm{i}]^{\times}$. Hence the claim follows from Proposition 7.1. □

With H replacing K in this proof we obtain the "real" version:

Corollary 7.7. *Suppose $f \in \mathcal{C}^{<\infty}$ is hardian over H and $P(f) = 0$ for some P in $H\{Y\}^{\neq}$. Then $H \subseteq \mathcal{C}^{\infty} \Rightarrow f \in \mathcal{C}^{\infty}$, and $H \subseteq \mathcal{C}^{\omega} \Rightarrow f \in \mathcal{C}^{\omega}$.*

This leads to:

Corollary 7.8. *Suppose H is a $\mathcal{C}^{\infty}$-Hardy field. Then every* d*-algebraic Hardy field extension of H is a $\mathcal{C}^{\infty}$-Hardy field; in particular,* $\mathrm{D}(H) \subseteq \mathcal{C}^{\infty}$. *Likewise with $\mathcal{C}^{\infty}$ replaced by $\mathcal{C}^{\omega}$.*

In particular, $\mathrm{D}(\mathbb{Q}) \subseteq \mathcal{C}^{\omega}$ [22, Theorems 14.3, 14.9]. Let now H be a $\mathcal{C}^{\infty}$-Hardy field. Then by Corollary 7.8, H is d-maximal iff H has no proper d-algebraic $\mathcal{C}^{\infty}$-Hardy field extension; thus every $\mathcal{C}^{\infty}$-maximal Hardy field is d-maximal, and H has a d-maximal d-algebraic $\mathcal{C}^{\infty}$-Hardy field extension. The same remarks apply with ω in place of ∞.

8. Application to Filling Holes in Hardy Fields

This section combines the analytic material above with results from [8], some of it summarized in Lemma 1.10. *Throughout H is a Hardy field with $H \not\subseteq \mathbb{R}$, and $r \in \mathbb{N}^{\geqslant 1}$.* Thus $K := H[i] \subseteq \mathcal{C}^{<\infty}[i]$ is an H-asymptotic extension of H. (Later we impose extra assumptions on H like being real closed with asymptotic integration.) Note that $v(H^{\times}) \neq \{0\}$: take $f \in H \setminus \mathbb{R}$; then $f' \neq 0$, and if $f \asymp 1$, then $f' \prec 1$.

Evaluating differential polynomials at germs. Any $Q \in K\{Y\}$ of order $\leqslant r$ can be evaluated at any germ $y \in \mathcal{C}^r[i]$ to give a germ $Q(y) \in \mathcal{C}[i]$, with $Q(y) \in \mathcal{C}$ for $Q \in H\{Y\}$ of order $\leqslant r$ and $y \in \mathcal{C}^r$. (See the beginning of Section 7.) Here is a variant that we shall need. Let $\phi \in H^{\times}$; with ∂ denoting the derivation of K, the derivation of the differential field K^{ϕ} is then $\delta := \phi^{-1}\partial$. We also let δ denote its extension $f \mapsto \phi^{-1}f' \colon \mathcal{C}^1[i] \to \mathcal{C}[i]$, which maps $\mathcal{C}^{n+1}[i]$ into $\mathcal{C}^n[i]$ and $\mathcal{C}^{n+1}$ into $\mathcal{C}^n$, for all n. Thus for $j \leqslant r$ we have the maps

$$\mathcal{C}^r[i] \xrightarrow{\ \delta\ } \mathcal{C}^{r-1}[i] \xrightarrow{\ \delta\ } \cdots \xrightarrow{\ \delta\ } \mathcal{C}^{r-j+1}[i] \xrightarrow{\ \delta\ } \mathcal{C}^{r-j}[i],$$

which by composition yield $\delta^j \colon \mathcal{C}^r[i] \to \mathcal{C}^{r-j}[i]$, mapping $\mathcal{C}^r$ into $\mathcal{C}^{r-j}$. This allows us to define for $Q \in K^{\phi}\{Y\}$ of order $\leqslant r$ and $y \in \mathcal{C}^r[i]$ the germ $Q(y) \in \mathcal{C}[i]$ by

$$Q(y) := q\big(y, \delta(y), \dots, \delta^r(y)\big) \quad \text{where } Q = q\big(Y, \dots, Y^{(r)}\big) \in K^{\phi}\big[Y, \dots, Y^{(r)}\big].$$

Note that H^{ϕ} is a differential subfield of K^{ϕ}, and if $Q \in H^{\phi}\{Y\}$ is of order $\leqslant r$ and $y \in \mathcal{C}^r$, then $Q(y) \in \mathcal{C}$.

Lemma 8.1. *Let $y \in \mathcal{C}^r[i]$ and $\mathfrak{m} \in K^{\times}$. Each of the following conditions implies $y \in \mathcal{C}^r[i]^{\preccurlyeq}$:*

(i) *$\phi \preccurlyeq 1$ and $\delta^0(y), \dots, \delta^r(y) \preccurlyeq 1$;*
(ii) *$\mathfrak{m} \preccurlyeq 1$ and $y \in \mathfrak{m}\,\mathcal{C}^r[i]^{\preccurlyeq}$.*

Moreover, if $\mathfrak{m} \preccurlyeq 1$ and $(y/\mathfrak{m})^{(0)}, \dots, (y/\mathfrak{m})^{(r)} \prec 1$, then $y^{(0)}, \dots, y^{(r)} \prec 1$.

Proof. For (i), use the smallness of the derivation of H and the transformation formulas in [ADH, 5.7] expressing the iterates of ∂ in terms of iterates of δ. For (ii) and the "moreover" part, set $y = \mathfrak{m}z$ with $z = y/\mathfrak{m}$ and use the Product Rule and the smallness of the derivation of K. □

Equations over Hardy fields and over their complexifications. Let $\phi > 0$ be active in H. We recall here from [9, Section 3] how the the asymptotic field $K^\phi = H[i]^\phi$ (with derivation $\delta = \phi^{-1}\partial$) is isomorphic to the asymptotic field $K^\circ := H^\circ[i]$ for a certain Hardy field H°: Let $\ell \in \mathcal{C}^1$ be such that $\ell' = \phi$; then $\ell > \mathbb{R}$, $\ell \in \mathcal{C}^{<\infty}$, and $\ell^{\text{inv}} \in \mathcal{C}^{<\infty}$ for the compositional inverse ℓ^{inv} of ℓ. The $\mathbb{C}$-algebra automorphism $f \mapsto f^\circ := f \circ \ell^{\text{inv}}$ of $\mathcal{C}[i]$ (with inverse $g \mapsto g \circ \ell$) maps $\mathcal{C}^n[i]$ and $\mathcal{C}^n$ onto themselves, and hence restricts to a $\mathbb{C}$-algebra automorphism of $\mathcal{C}^{<\infty}[i]$ and $\mathcal{C}^{<\infty}$ mapping $\mathcal{C}^{<\infty}$ onto itself. Moreover,

$$(\partial, \circ, \delta) \qquad (f^\circ)' = (\phi^\circ)^{-1}(f')^\circ = \delta(f)^\circ \qquad \text{for } f \in \mathcal{C}^1[i].$$

Thus we have an isomorphism $f \mapsto f^\circ : (\mathcal{C}^{<\infty}[i])^\phi \to \mathcal{C}^{<\infty}[i]$ of differential rings, and likewise with $\mathcal{C}^{<\infty}$ in place of $\mathcal{C}^{<\infty}[i]$. Then

$$H^\circ := \{h^\circ : h \in H\} \subseteq \mathcal{C}^{<\infty}$$

is a Hardy field, and $f \mapsto f^\circ$ restricts to an isomorphism $H^\phi \to H^\circ$ of pre-H-fields, and to an isomorphism $K^\phi \to K^\circ$ of asymptotic fields. We extend the latter to the isomorphism

$$Q \mapsto Q^\circ : K^\phi\{Y\} \to K^\circ\{Y\}$$

of differential rings given by $Y^\circ = Y$, which restricts to a differential ring isomorphism $H^\phi\{Y\} \to H^\circ\{Y\}$. Using the identity $(\partial, \circ, \delta)$ it is routine to check that for $Q \in K^\phi\{Y\}$ of order $\leqslant r$ and $y \in \mathcal{C}^r[i]$, we have $Q(y)^\circ = (Q^\circ)(y^\circ)$. This allows us to translate algebraic differential equations over K into algebraic differential equations over K°: Let $P \in K\{Y\}$ have order $\leqslant r$ and let $y \in \mathcal{C}^r[i]$.

Lemma 8.2. *$P(y)^\circ = P^\phi(y)^\circ = P^{\phi\circ}(y^\circ)$ where $P^{\phi\circ} := (P^\phi)^\circ \in K^\circ\{Y\}$, hence*

$$P(y) = 0 \iff P^{\phi\circ}(y^\circ) = 0.$$

Moreover, $y \prec \mathfrak{m} \iff y^\circ \prec \mathfrak{m}^\circ$, for $\mathfrak{m} \in K^\times$, so asymptotic side conditions are automatically taken care of under this "translation". Also, if $\phi \preccurlyeq 1$ and $y^\circ \in \mathcal{C}^r[i]^\preccurlyeq$, then $y \in \mathcal{C}^r[i]^\preccurlyeq$, by Lemma 8.1(i) and $(\partial, \circ, \delta)$.

In the rest of this section $H \supseteq \mathbb{R}$ is real closed with asymptotic integration. Then H is an H-field, and $K = H[i]$ is the algebraic closure of H, a d-valued field with small derivation extending H, constant field $\mathbb{C}$, and value group $\Gamma := v(K^\times) = v(H^\times)$.

Slots in Hardy fields and compositional conjugation. *In this subsection we let $\phi > 0$ be active in H; as in the previous subsection we take $\ell \in \mathcal{C}^1$ such that $\ell' = \phi$ and use the superscript $\circ$ accordingly: $f^\circ := f \circ \ell^{\text{inv}}$ for $f \in \mathcal{C}[i]$.*

Let $(P, \mathfrak{m}, \widehat{a})$ be a slot in K of order r, so $\widehat{a} \notin K$ is an element of an immediate asymptotic extension $\widehat{K}$ of K with $P \in Z(K, \widehat{a})$ and $\widehat{a} \prec \mathfrak{m}$. We associate to $(P, \mathfrak{m}, \widehat{a})$ a slot in K° as follows: choose an immediate asymptotic extension $\widehat{K}^\circ$ of K° and an isomorphism $\widehat{f} \mapsto \widehat{f}^\circ : \widehat{K}^\phi \to \widehat{K}^\circ$ of asymptotic fields extending the isomorphism $f \mapsto f^\circ : K^\phi \to K^\circ$. Then $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ is a slot in K° of the same complexity as $(P, \mathfrak{m}, \widehat{a})$. The equivalence class of $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ does not depend on the choice of $\widehat{K}^\circ$ and the isomorphism $\widehat{K}^\phi \to \widehat{K}^\circ$. If $(P, \mathfrak{m}, \widehat{a})$ is a hole in K, then $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ is a hole in K°, and likewise with "minimal hole" in place of "hole". By [8, Lemmas 3.1.20, 3.3.20, 3.3.40] we have:

Lemma 8.3. *If $(P, \mathfrak{m}, \widehat{a})$ is Z-minimal, then so is $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$, and likewise with "quasilinear" and "special" in place of "Z-minimal". If $(P, \mathfrak{m}, \widehat{a})$ is steep and $\phi \preccurlyeq 1$,* *then* $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ *is steep, and likewise with "deep", "normal", and "strictly normal" in place of "steep".*

Next, let $(P, \mathfrak{m}, \widehat{a})$ be a slot in H of order r, so $\widehat{a} \notin H$ is an element of an immediate asymptotic extension $\widehat{H}$ of H with $P \in Z(H, \widehat{a})$ and $\widehat{a} \prec \mathfrak{m}$. We associate to $(P, \mathfrak{m}, \widehat{a})$ a slot in H° as follows: choose an immediate asymptotic extension $\widehat{H}^\circ$ of H° and an isomorphism $\widehat{f} \mapsto \widehat{f}^\circ \colon \widehat{H}^\phi \to \widehat{H}^\circ$ of asymptotic fields extending the isomorphism $f \mapsto f^\circ \colon H^\phi \to H^\circ$. Then $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ is a slot in H° of the same complexity as $(P, \mathfrak{m}, \widehat{a})$. The equivalence class of $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ does not depend on the choice of $\widehat{H}^\circ$ and the isomorphism $\widehat{H}^\phi \to \widehat{H}^\circ$. If $(P, \mathfrak{m}, \widehat{a})$ is a hole in H, then $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ is a hole in H°, and likewise with "minimal hole" in place of "hole". Lemma 8.3 goes through in this setting. Also, recalling [9, Lemma 4.5], if H is Liouville closed and $(P, \mathfrak{m}, \widehat{a})$ is ultimate, then $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ is ultimate.

Moreover, by [8, Lemmas 4.3.5, 4.3.28 , Corollaries 4.5.23, 4.5.39]:

Lemma 8.4.
(i) *If* $\phi \preccurlyeq 1$ *and* $(P, \mathfrak{m}, \widehat{a})$ *is split-normal, then* $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ *is split-normal; likewise with "split-normal" replaced by "strongly split-normal".*
(ii) *If* $\phi \prec 1$ *and* $(P, \mathfrak{m}, \widehat{a})$ *is* Z*-minimal, deep, and repulsive-normal, then* $(P^{\phi\circ}, \mathfrak{m}^\circ, \widehat{a}^\circ)$ *is repulsive-normal; likewise with "repulsive-normal" replaced by "strongly repulsive-normal".*

Reformulations. We reformulate here some results of Sections 6 and 7 to facilitate their use. For $\mathfrak{v} \in K^\times$ with $\mathfrak{v} \prec 1$ we set:

$$\Delta(\mathfrak{v}) \ := \ \big\{\gamma \in \Gamma : \gamma = o(v\mathfrak{v})\big\},$$

a proper convex subgroup of Γ. In the next lemma, $P \in K\{Y\}$ has order r and $P = Q - R$, where $Q, R \in K\{Y\}$ and Q is homogeneous of degree 1 and order r. We set $w := \mathrm{wt}(P)$, so $w \geqslant r \geqslant 1$.

Lemma 8.5. *Suppose that* L_Q *splits strongly over* K*,* $\mathfrak{v}(L_Q) \prec^\flat 1$*, and*

$$R \prec_\Delta \mathfrak{v}(L_Q)^{w+1} Q, \quad \Delta := \Delta\big(\mathfrak{v}(L_Q)\big).$$

Then $P(y) = 0$ *and* $y', \dots, y^{(r)} \preccurlyeq 1$ *for some* $y \prec \mathfrak{v}(L_Q)^w$ *in* $\mathcal{C}^{<\infty}[i]$*. Moreover:*
(i) *if* $P, Q \in H\{Y\}$*, then there is such* y *in* $\mathcal{C}^{<\infty}$*;*
(ii) *if* $H \subseteq \mathcal{C}^\infty$*, then for any* $y \in \mathcal{C}^r[i]^{\preccurlyeq}$ *with* $P(y) = 0$ *we have* $y \in \mathcal{C}^\infty[i]$*; likewise with* $\mathcal{C}^\omega$ *in place of* $\mathcal{C}^\infty$*.*

Proof. Set $\mathfrak{v} := |\mathfrak{v}(L_Q)| \in H^>$, so $\mathfrak{v} \asymp \mathfrak{v}(L_Q)$. Take $f \in K^\times$ such that $A := f^{-1} L_Q$ is monic; then $\mathfrak{v}(A) = \mathfrak{v}(L_Q) \asymp \mathfrak{v}$ and $f^{-1}R \prec_\Delta f^{-1}\mathfrak{v}^{w+1}Q \asymp \mathfrak{v}^w$. We have $A = (\partial - \phi_1) \cdots (\partial - \phi_r)$ where $\phi_j \in K$ and $\operatorname{Re}\phi_j \succcurlyeq \mathfrak{v}^\dagger \succcurlyeq 1$ for $j = 1, \dots, r$ by the strong splitting assumption. Also $\phi_1, \dots, \phi_r \preccurlyeq \mathfrak{v}^{-1}$ by (1.3). The claims now follow from various results in Section 6 applied to the equation $A(y) = f^{-1}R(y)$, $y \prec 1$ in the role of $(*)$, using also Corollary 7.5. □

In the next two lemmas ϕ *is active in* H *with* $0 < \phi \preccurlyeq 1$*.*

Lemma 8.6. *Let* $(P, \mathfrak{n}, \widehat{h})$ *be a slot in* H *of order* r *such that* $(P^\phi, \mathfrak{n}, \widehat{h})$ *is strongly split-normal. Then for some* y *in* $\mathcal{C}^{<\infty}$*,*

$$P(y) \ = \ 0, \quad y \ \prec \ \mathfrak{n}, \quad y \in \mathfrak{n}\,(\mathcal{C}^r)^{\preccurlyeq}.$$

If $H \subseteq \mathcal{C}^\infty$*, then there exists such* y *in* $\mathcal{C}^\infty$*, and likewise with* $\mathcal{C}^\omega$ *in place of* $\mathcal{C}^\infty$*.*

Proof. First we consider the case $\phi = 1$. Replace $(P, \mathfrak{n}, \widehat{h})$ by $(P_{\times \mathfrak{n}}, 1, \widehat{h}/\mathfrak{n})$ to arrange $\mathfrak{n} = 1$. Then L_P has order r, $\mathfrak{v} := \mathfrak{v}(L_P) \prec^\flat 1$, and $P = Q - R$ where $Q, R \in H\{Y\}$, Q is homogeneous of degree 1 and order r, $L_Q \in H[\partial]$ splits strongly over K, and $R \prec_\Delta \mathfrak{v}^{w+1} P_1$, where $\Delta := \Delta(\mathfrak{v})$ and $w := \operatorname{wt}(P)$. Now $P_1 = Q - R_1$, so $\mathfrak{v} \sim \mathfrak{v}(L_Q)$ by Lemma 1.5(ii), and thus $\Delta = \Delta\big(\mathfrak{v}(L_Q)\big)$. Lemma 8.5 gives y in $\mathcal{C}^{<\infty}$ such that $y \prec \mathfrak{v}^w \prec 1$, $P(y) = 0$, and $y^{(j)} \preccurlyeq 1$ for $j = 1, \dots, r$. Then y has for $\mathfrak{n} = 1$ the properties displayed in the lemma.

Now suppose ϕ is arbitrary. Employing $(\ )^\circ$ as explained earlier in this section, the slot $(P^{\phi\circ}, \mathfrak{n}^\circ, \widehat{h}^\circ)$ in the Hardy field H° is strongly split-normal, hence by the case $\phi = 1$ we have $z \in \mathcal{C}^{<\infty}$ with $P^{\phi\circ}(z) = 0$, $z \prec \mathfrak{n}^\circ$, and $(z/\mathfrak{n}^\circ)^{(j)} \preccurlyeq 1$ for $j = 1, \dots, r$. Take $y \in \mathcal{C}^{<\infty}$ with $y^\circ = z$. Then $P(y) = 0$, $y \prec \mathfrak{n}$, and $y \in \mathfrak{n}\,(\mathcal{C}^r)^{\preccurlyeq}$ by Lemma 8.2 and a subsequent remark. Moreover, if $\phi, z \in \mathcal{C}^\infty$, then $y \in \mathcal{C}^\infty$, and likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$. □

In the next "complex" version, $(P, \mathfrak{m}, \widehat{a})$ is a slot in K of order r with $\mathfrak{m} \in H^\times$.

Lemma 8.7. *Suppose the slot $(P^\phi, \mathfrak{m}, \widehat{a})$ in K^ϕ is strictly normal, and its linear part splits strongly over K^ϕ. Then for some $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ we have*

$$P(y) \ = \ 0, \quad y \ \prec \ \mathfrak{m}, \quad y \in \mathfrak{m}\,\mathcal{C}^r[\mathrm{i}]^{\preccurlyeq}.$$

If $H \subseteq \mathcal{C}^\infty$, then there is such y in $\mathcal{C}^\infty[\mathrm{i}]$. If $H \subseteq \mathcal{C}^\omega$, then there is such y in $\mathcal{C}^\omega[\mathrm{i}]$.

Proof. Consider first the case $\phi = 1$. Replacing $(P, \mathfrak{m}, \widehat{a})$ by $(P_{\times\mathfrak{m}}, 1, \widehat{a}/\mathfrak{m})$ we arrange $\mathfrak{m} = 1$. Set $L := L_P \in K[\partial]$, $Q := P_1$, and $R := P - Q$. Since $(P, 1, \widehat{a})$ is strictly normal, we have $\operatorname{order}(L) = r$, $\mathfrak{v} := \mathfrak{v}(L) \prec^\flat 1$, and $R \prec_\Delta \mathfrak{v}^{w+1} Q$ where $\Delta := \Delta(\mathfrak{v})$, $w := \operatorname{wt}(P)$. As L splits strongly over K, Lemma 8.5 gives y in $\mathcal{C}^{<\infty}[\mathrm{i}]$ such that $P(y) = 0$, $y \prec \mathfrak{v}^w \prec 1$, and $y^{(j)} \preccurlyeq 1$ for $j = 1, \dots, r$. For the last part of the lemma, use the last part of Lemma 8.5. The general case reduces to this special case as in the proof of Lemma 8.6. □

Finding germs in holes. *In this subsection $\widehat{H}$ is an immediate asymptotic extension of H.* This fits into the setting of [8, Section 4.3] on split-normal slots: $K = H[\mathrm{i}]$ and $\widehat{H}$ have H as a common asymptotic subfield and $\widehat{K} := \widehat{H}[\mathrm{i}]$ as a common asymptotic extension, $\widehat{H}$ is an H-field, and $\widehat{K}$ is d-valued.

Assume also that H is $\upomega$-free. Then K is $\upomega$-free by [ADH, 11.7.23]. *Let $(P, \mathfrak{m}, \widehat{a})$ with $\mathfrak{m} \in H^\times$ and $\widehat{a} \in \widehat{K} \setminus K$ be a minimal hole in K of order $r \geqslant 1$.*

Proposition 8.8. *Suppose* $\deg P > 1$. *Then for some $y \in \mathcal{C}^{<\infty}[\mathrm{i}]$ we have*

$$P(y) \ = \ 0, \quad y \ \prec \ \mathfrak{m}, \quad y \in \mathfrak{m}\,\mathcal{C}^r[\mathrm{i}]^{\preccurlyeq}.$$

If $\mathfrak{m} \preccurlyeq 1$, then $y \in \mathcal{C}^r[\mathrm{i}]^{\preccurlyeq}$ for such y. Moreover, if $H \subseteq \mathcal{C}^\infty$, then we can take such y in $\mathcal{C}^\infty[\mathrm{i}]$, and if $H \subseteq \mathcal{C}^\omega$, then we can take such y in $\mathcal{C}^\omega[\mathrm{i}]$.

Proof. Lemma 1.10 gives a refinement $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ of $(P, \mathfrak{m}, \widehat{a})$ with $\mathfrak{n} \in H^\times$ and an active ϕ in H with $0 < \phi \preccurlyeq 1$ such that the hole $(P^\phi_{+a}, \mathfrak{n}, \widehat{a} - a)$ in K^ϕ is strictly normal and its linear part splits strongly over K^ϕ. Lemma 8.7 applied to $(P_{+a}, \mathfrak{n}, \widehat{a} - a)$ in place of $(P, \mathfrak{m}, \widehat{a})$ yields $z \in \mathcal{C}^{<\infty}[\mathrm{i}]$ with $P_{+a}(z) = 0$, $z \prec \mathfrak{n}$ and $(z/\mathfrak{n})^{(j)} \preccurlyeq 1$ for $j = 1, \dots, r$. Lemma 8.1(ii) applied to $z/\mathfrak{m}$, $\mathfrak{n}/\mathfrak{m}$ in place of y, $\mathfrak{m}$, respectively, yields $(z/\mathfrak{m})^{(j)} \preccurlyeq 1$ for $j = 0, \dots, r$. Also, $a \prec \mathfrak{m}$ (in K), hence $(a/\mathfrak{m})^{(j)} \prec 1$ for $j = 0, \dots, r$. Set $y := a + z$; then $P(y) = 0$, $y \prec \mathfrak{m}$, and $(y/\mathfrak{m})^{(j)} \preccurlyeq 1$ for $j = 1, \dots, r$. For the rest use Lemma 8.1(ii) and the last part of Lemma 8.7. □

Next we treat the linear case:

Proposition 8.9. *Suppose* $\deg P = 1$. *Then for some* $y \in \mathcal{C}^{<\infty}[i]$ *we have*

$$P(y) = 0, \quad y \prec \mathfrak{m}, \quad (y/\mathfrak{m})' \preccurlyeq 1.$$

If $\mathfrak{m} \preccurlyeq 1$, *then* $y \prec 1$ *and* $y' \preccurlyeq 1$ *for each such* y. *Moreover, if* $H \subseteq \mathcal{C}^{\infty}$, *then we can take such* y *in* $\mathcal{C}^{\infty}[i]$, *and if* $H \subseteq \mathcal{C}^{\omega}$, *then we can take such* y *in* $\mathcal{C}^{\omega}[i]$.

Proof. We have $r = 1$ by [8, Corollary 3.2.8]. If $\partial K = K$ and $\mathrm{I}(K) \subseteq K^{\dagger}$, then Lemma 1.11 applies, and we can argue as in the proof of Proposition 8.8, using this lemma instead of Lemma 1.10. We reduce the general case to this special case as follows: Set $H_1 := \mathrm{D}(H)$; then H_1 is an ω-free Hardy field by [8, Theorem 1.3.1], and $K_1 := H_1[i]$ satisfies $\partial K_1 = K_1$ and $\mathrm{I}(K_1) \subseteq K_1^{\dagger}$ by Propositions 2.1 and 2.2. Moreover, by Corollary 7.8, if $H \subseteq \mathcal{C}^{\infty}$, then $H_1 \subseteq \mathcal{C}^{\infty}$, and likewise with $\mathcal{C}^{\omega}$ in place of $\mathcal{C}^{\infty}$. The newtonization $\widehat{H}_1$ of H_1 is an immediate asymptotic extension of H_1, and $\widehat{K}_1 := \widehat{H}_1[i]$ is newtonian [ADH, 14.5.7]. Lemma 1.6 gives an embedding $K\langle\widehat{a}\rangle \to \widehat{K}_1$ over K; let $\widehat{a}_1$ be the image of $\widehat{a}$ under this embedding. If $\widehat{a}_1 \in K_1$, then we are done by taking $y := \widehat{a}_1$, so we may assume $\widehat{a}_1 \notin K_1$. Then $(P, \mathfrak{m}, \widehat{a}_1)$ is a minimal hole in K_1, and the above applies with H, K, $\widehat{a}$ replaced by H_1, K_1, $\widehat{a}_1$, respectively. □

We can improve on these results in a useful way:

Corollary 8.10. *Suppose* $\widehat{a} \sim a \in K$. *Then for some* $y \in \mathcal{C}^{<\infty}[i]$ *we have*

$$P(y) = 0, \quad y \sim a, \quad (y/a)^{(j)} \prec 1 \ \text{ for } j = 1, \dots, r.$$

If $H \subseteq \mathcal{C}^{\infty}$, *then there is such* y *in* $\mathcal{C}^{\infty}[i]$. *If* $H \subseteq \mathcal{C}^{\omega}$, *then there is such* y *in* $\mathcal{C}^{\omega}[i]$.

Proof. Take $a_1 \in K$ and $\mathfrak{n} \in H^{\times}$ such that $\mathfrak{n} \asymp \widehat{a} - a \sim a_1$, and set $b := a + a_1$. Then $(P_{+b}, \mathfrak{n}, \widehat{a} - b)$ is a refinement of $(P, \mathfrak{m}, \widehat{a})$, and Propositions 8.8 and 8.9 give $z \in \mathcal{C}^{<\infty}[i]$ with $P(b + z) = 0$, $z \prec \mathfrak{n}$ and $(z/\mathfrak{n})^{(j)} \preccurlyeq 1$ for $j = 1, \dots, r$. We have $(a_1/a)^{(j)} \prec 1$ for $j = 0, \dots, r$, since K has small derivation. Likewise, $(\mathfrak{n}/a)^{(j)} \prec 1$ for $j = 0, \dots, r$, and hence $(z/a)^{(j)} \prec 1$ for $j = 0, \dots, r$, by $z/a = (z/\mathfrak{n}) \cdot (\mathfrak{n}/a)$ and the Product Rule. So $y := b + z$ has the desired property. The rest follows from the "moreover" parts of these propositions. □

Remark 8.11. If we replace our standing assumption that H is ω-free and $(P, \mathfrak{m}, \widehat{a})$ is a minimal hole in K by the assumption that H is λ-free, $\partial K = K$, $\mathrm{I}(K) \subseteq K^{\dagger}$, and $(P, \mathfrak{m}, \widehat{a})$ is a slot in K of order and degree 1, then Proposition 8.9 and Corollary 8.10 go through by Remark 1.12.

Can we choose y in Corollary 8.10 such that additionally $\operatorname{Re} y$, $\operatorname{Im} y$ are hardian over H? At this stage we cannot claim this. In the next section we introduce weights and their corresponding norms as a more refined tool. This will allow us to obtain Corollary 9.20 as a key approximation result for later use.

9. Weights

In this section we prove Proposition 9.14 to strengthen Lemma 6.4. This uses the material on repulsive-normal slots from [8, Section 4.5], but we also need more refined norms for differentiable functions, to which we turn now.

The final result, Corollary 9.20, is the only part of this section used towards our main result, Theorem 11.19. But this use, in the proof of Theorem 11.10, is essential, and obtaining this corollary requires the entire section.

Weighted spaces of differentiable functions. *In this subsection we fix $r \in \mathbb{N}$ and a* weight *function* $\tau \in \mathcal{C}_a[\mathrm{i}]^\times$. For $f \in \mathcal{C}^r_a[\mathrm{i}]$ we set

$$\|f\|^\tau_{a;r} \ :=\ \max\left\{\|\tau^{-1}f\|_a, \|\tau^{-1}f'\|_a, \dots, \|\tau^{-1}f^{(r)}\|_a\right\} \ \in\ [0,+\infty],$$

and $\|f\|^\tau_a := \|f\|^\tau_{a;0}$ for $f \in \mathcal{C}_a[\mathrm{i}]$. Then

$$\mathcal{C}^r_a[\mathrm{i}]^\tau \ :=\ \left\{f \in \mathcal{C}^r_a[\mathrm{i}] : \|f\|^\tau_{a;r} < +\infty\right\}$$

is a $\mathbb{C}$-linear subspace of

$$\mathcal{C}_a[\mathrm{i}]^\tau \ :=\ \mathcal{C}^0_a[\mathrm{i}]^\tau \ =\ \tau\,\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \ =\ \left\{f \in \mathcal{C}_a[\mathrm{i}] : f \preccurlyeq \tau\right\}.$$

Below we consider the $\mathbb{C}$-linear space $\mathcal{C}^r_a[\mathrm{i}]^\tau$ to be equipped with the norm

$$f \mapsto \|f\|^\tau_{a;r}.$$

Recall from Section 5 the convention $b \cdot \infty = \infty \cdot b = \infty$ for $b \in [0,\infty]$. Note that

$$\|fg\|^\tau_{a;r} \ \leqslant\ 2^r \|f\|_{a;r}\, \|g\|^\tau_{a;r} \quad \text{for } f,g \in \mathcal{C}^r_a[\mathrm{i}], \tag{9.1}$$

so $\mathcal{C}^r_a[\mathrm{i}]^\tau$ is a $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$-submodule of $\mathcal{C}^r_a[\mathrm{i}]$. Note also that $\|1\|^\tau_{a;r} = \|\tau^{-1}\|_a$, hence

$$\|f\|^\tau_{a;r} \ \leqslant\ 2^r \|f\|_{a;r}\, \|\tau^{-1}\|_a \quad \text{for } f \in \mathcal{C}^r_a[\mathrm{i}]$$

and

$$\tau^{-1} \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \quad\Longleftrightarrow\quad 1 \in \mathcal{C}^r_a[\mathrm{i}]^\tau \quad\Longleftrightarrow\quad \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}} \subseteq \mathcal{C}^r_a[\mathrm{i}]^\tau.$$

We have

$$\|f\|_{a;r} \ \leqslant\ \|f\|^\tau_{a;r}\, \|\tau\|_a \quad \text{for } f \in \mathcal{C}^r_a[\mathrm{i}], \tag{9.2}$$

and thus

$$\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \quad\Longleftrightarrow\quad \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}} \subseteq \mathcal{C}^r_a[\mathrm{i}]^{\tau^{-1}} \quad\Longrightarrow\quad \mathcal{C}^r_a[\mathrm{i}]^\tau \subseteq \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}. \tag{9.3}$$

Hence if $\tau, \tau^{-1} \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, then $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}} = \mathcal{C}^r_a[\mathrm{i}]^\tau$, and the norms $\|\cdot\|^\tau_{a;r}$ and $\|\cdot\|_{a;r}$ on this $\mathbb{C}$-linear space are equivalent. (In later use, $\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, $\tau^{-1} \notin \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$.) If $\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, then $\mathcal{C}^r_a[\mathrm{i}]^\tau$ is an ideal of the commutative ring $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$. From (9.1) and (9.2) we obtain

$$\|fg\|^\tau_{a;r} \ \leqslant\ 2^r\, \|\tau\|_a\, \|f\|^\tau_{a;r}\, \|g\|^\tau_{a;r} \quad \text{for } f,g \in \mathcal{C}^r_a[\mathrm{i}].$$

For $f \in \mathcal{C}^{r+1}_a[\mathrm{i}]^\tau$ we have $\|f\|^\tau_{a;r}, \|f'\|^\tau_{a;r} \leqslant \|f\|^\tau_{a;r+1}$. From (9.2) and (9.3):

Lemma 9.1. *Suppose $\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ (so $\mathcal{C}^r_a[\mathrm{i}]^\tau \subseteq \mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$) and $f \in \mathcal{C}^r_a[\mathrm{i}]^\tau$. If (f_n) is a sequence in $\mathcal{C}^r_a[\mathrm{i}]^\tau$ and $f_n \to f$ in $\mathcal{C}^r_a[\mathrm{i}]^\tau$, then also $f_n \to f$ in $\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$.*

This is used to show:

Lemma 9.2. *Suppose $\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. Then the $\mathbb{C}$-linear space $\mathcal{C}^r_a[\mathrm{i}]^\tau$ equipped with the norm $\|\cdot\|^\tau_{a;r}$ is complete.*

Proof. We proceed by induction on r. Let (f_n) be a cauchy sequence in the normed space $\mathcal{C}_a[\mathrm{i}]^\tau$. Then the sequence $(\tau^{-1}f_n)$ in the Banach space $\mathcal{C}^0_a[\mathrm{i}]^{\mathrm{b}}$ is cauchy, hence has a limit $g \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, so with $f := \tau g \in \mathcal{C}_a[\mathrm{i}]^\tau$ we have $\tau^{-1}f_n \to \tau^{-1}f$ in $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ and hence $f_n \to f$ in $\mathcal{C}_a[\mathrm{i}]^\tau$. Thus the lemma holds for $r=0$. Suppose the lemma holds for a certain value of r, and let (f_n) be a cauchy sequence in $\mathcal{C}^{r+1}_a[\mathrm{i}]^\tau$. Then (f_n') is a cauchy sequence in $\mathcal{C}^r_a[\mathrm{i}]^\tau$ and hence has a limit $g \in \mathcal{C}^r_a[\mathrm{i}]^\tau$, by inductive hypothesis. By Lemma 9.1, $f_n' \to g$ in $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. Now (f_n) is also a cauchy sequence in $\mathcal{C}_a[\mathrm{i}]^\tau$, hence has a limit $f \in \mathcal{C}_a[\mathrm{i}]^\tau$ (by the case $r=0$), and by Lemma 9.1 again, $f_n \to f$ in $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. Thus f is differentiable and $f' = g$ by [33, (8.6.4)]. This yields $f_n \to f$ in $\mathcal{C}^{r+1}_a[\mathrm{i}]^\tau$. ∎

Lemma 9.3. *Suppose* $\tau \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$*. If* $f \in \mathcal{C}_a^r[\mathrm{i}]$ *and* $f^{(k)} \preccurlyeq \tau^{r-k+1}$ *for* $k = 0, \dots, r$*, then* $f\tau^{-1} \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$*. (Thus* $\mathcal{C}_a^r[\mathrm{i}]^{\tau^{r+1}} \subseteq \tau\,\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$*.)*

Proof. An easy induction on $n \leqslant r$ shows that there are $Q_k^n \in \mathbb{Z}[X_0, X_1, \dots, X_{n-k}]$ $(0 \leqslant k \leqslant n)$ such that for all $f \in \mathcal{C}_a^r[\mathrm{i}]$ and $n \leqslant r$:

$$(f\tau^{-1})^{(n)} \;=\; \sum_{k=0}^{n} Q_k^n(\tau, \tau', \dots, \tau^{(n-k)}) f^{(k)} \tau^{k-n-1}.$$

Now use that $Q_k^n(\tau, \tau', \dots, \tau^{(n-k)}) \preccurlyeq 1$ for $n \leqslant r$ and $k = 0, \dots, n$. $\square$

Next we generalize the inequality (9.1):

Lemma 9.4. *Let* $f_1, \dots, f_{m-1}, g \in \mathcal{C}_a^r[\mathrm{i}]$*,* $m \geqslant 1$*; then*

$$\|f_1 \cdots f_{m-1} g\|_{a;r}^{\tau} \;\leqslant\; m^r \, \|f_1\|_{a;r} \cdots \|f_{m-1}\|_{a;r} \, \|g\|_{a;r}^{\tau}.$$

Proof. Use the generalized Product Rule [ADH, p. 199] and the well-known identity $\sum \frac{n!}{i_1! \cdots i_m!} = m^n$ with the sum over all $(i_1, \dots, i_m) \in \mathbb{N}^m$ with $i_1 + \dots + i_m = n$. $\square$

With $\boldsymbol{i}$ ranging over $\mathbb{N}^{1+r}$, let $P = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$ (all $P_{\boldsymbol{i}} \in \mathcal{C}_a[\mathrm{i}]$) be a polynomial in $\mathcal{C}_a[\mathrm{i}]\big[Y, Y', \dots, Y^{(r)}\big]$; for $f \in \mathcal{C}_a^r[\mathrm{i}]$ we have $P(f) = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} f^{\boldsymbol{i}} \in \mathcal{C}_a[\mathrm{i}]$. (See also the beginning of Section 6.) We set

$$\|P\|_a \;:=\; \max_{\boldsymbol{i}} \|P_{\boldsymbol{i}}\|_a \in [0, \infty].$$

In the rest of this subsection we assume $\|P\|_a < \infty$, that is, $P \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\big[Y, \dots, Y^{(r)}\big]$. Hence if $\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, $P(0) \in \mathcal{C}_a[\mathrm{i}]^{\tau}$, and $f \in \mathcal{C}_a^r[\mathrm{i}]^{\tau}$, then $P(f) \in \mathcal{C}_a[\mathrm{i}]^{\tau}$. Here are weighted versions of Lemma 5.1 and 5.2:

Lemma 9.5. *Suppose* P *is homogeneous of degree* $d \geqslant 1$*, and let* $\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ *and* $f \in \mathcal{C}_a^r[\mathrm{i}]^{\tau}$*. Then*

$$\|P(f)\|_a^{\tau} \;\leqslant\; \binom{d+r}{r} \cdot \|P\|_a \cdot \|f\|_{a;r}^{d-1} \cdot \|f\|_{a;r}^{\tau}.$$

Proof. For $j = 0, \dots, r$ we have $\|f^{(j)}\|_a \leqslant \|f\|_{a;r}$ and $\|f^{(j)}\|_a^{\tau} \leqslant \|f\|_{a;r}^{\tau}$. Now $f^{\boldsymbol{i}}$, where $\boldsymbol{i} = (i_0, \dots, i_r) \in \mathbb{N}^{r+1}$ and $i_0 + \dots + i_r = d$, is a product of d such factors $f^{(j)}$, so Lemma 9.4 with $m := d$, $r := 0$, gives

$$\|f^{\boldsymbol{i}}\|_a^{\tau} \;\leqslant\; \|f\|_{a;r}^{d-1} \cdot \|f\|_{a;r}^{\tau}.$$

It remains to note that by (9.1) we have $\|P_{\boldsymbol{i}} f^{\boldsymbol{i}}\|_a^{\tau} \leqslant \|P_{\boldsymbol{i}}\|_a \cdot \|f^{\boldsymbol{i}}\|_a^{\tau}$. $\square$

Corollary 9.6. *Let* $1 \leqslant d \leqslant e$ *in* $\mathbb{N}$ *be such that* $P_{\boldsymbol{i}} = 0$ *if* $|\boldsymbol{i}| < d$ *or* $|\boldsymbol{i}| > e$*. Then for* $f \in \mathcal{C}_a^r[\mathrm{i}]^{\tau}$ *and* $\tau \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ *we have*

$$\|P(f)\|_a^{\tau} \;\leqslant\; D \cdot \|P\|_a \cdot \big(\|f\|_{a;r}^{d-1} + \dots + \|f\|_{a;r}^{e-1}\big) \cdot \|f\|_{a;r}^{\tau}$$

where $D \;=\; D(d, e, r) \;:=\; \binom{e+r+1}{r+1} - \binom{d+r}{r+1} \in \mathbb{N}^{\geqslant 1}$.

Doubly-twisted integration. In this subsection we adopt the setting in *Twisted integration* of Section 5. Thus $\phi \in \mathcal{C}_a[\mathrm{i}]$ and $\Phi = \partial_a^{-1}\phi$. Let $\mathfrak{r} \in \mathcal{C}_a^1$ satisfy $\mathfrak{r}(s) > 0$ for $s \geqslant a$, and set $\widetilde{\phi} := \phi - \mathfrak{r}^\dagger \in \mathcal{C}_a[\mathrm{i}]$ and $\widetilde{\Phi} := \partial_a^{-1}\widetilde{\phi}$. Thus

$$\widetilde{\Phi}(t) \;=\; \int_a^t (\phi - \mathfrak{r}^\dagger)(s)\,ds \;=\; \Phi(t) - \log \mathfrak{r}(t) + \log \mathfrak{r}(a) \qquad \text{for } t \geqslant a.$$

Consider the right inverses $B, \widetilde{B}\colon \mathcal{C}_a[\mathrm{i}] \to \mathcal{C}_a^1[\mathrm{i}]$ to, respectively, $\partial - \phi\colon \mathcal{C}_a^1[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}]$ and $\partial - \widetilde{\phi}\colon \mathcal{C}_a^1[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}]$, given by

$$B \;:=\; \mathrm{e}^{\Phi} \circ \partial_a^{-1} \circ \mathrm{e}^{-\Phi}, \quad \widetilde{B} \;:=\; \mathrm{e}^{\widetilde{\Phi}} \circ \partial_a^{-1} \circ \mathrm{e}^{-\widetilde{\Phi}}.$$

For $f \in \mathcal{C}_a[\mathrm{i}]$ and $t \geqslant a$ we have

$$\begin{aligned}
\widetilde{B}f(t) \;&=\; \mathrm{e}^{\widetilde{\Phi}(t)} \int_a^t \mathrm{e}^{-\widetilde{\Phi}(s)}\, f(s)\,ds \\
&=\; \mathfrak{r}(t)^{-1}\mathfrak{r}(a)\,\mathrm{e}^{\Phi(t)} \int_a^t \mathrm{e}^{-\Phi(s)}\,\mathfrak{r}(s)\mathfrak{r}(a)^{-1} f(s)\,ds \\
&=\; \mathfrak{r}(t)^{-1}\,\mathrm{e}^{\Phi(t)} \int_a^t \mathrm{e}^{-\Phi(s)}\,\mathfrak{r}(s) f(s)\,ds \;=\; \mathfrak{r}^{-1}(t)\big(B(\mathfrak{r}f)\big)(t)
\end{aligned}$$

and so $\widetilde{B} = \mathfrak{r}^{-1} \circ B \circ \mathfrak{r}$. Hence if $\widetilde{\phi}$ is attractive, then $B_{\ltimes \mathfrak{r}} := \mathfrak{r}^{-1} \circ B \circ \mathfrak{r}$ maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^1[\mathrm{i}]$, and the operator $B_{\ltimes \mathfrak{r}}\colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ is continuous with $\|B_{\ltimes \mathfrak{r}}\|_a \leqslant \big\|\frac{1}{\operatorname{Re}\widetilde{\phi}}\big\|_a$; if in addition $\widetilde{\phi} \in \mathcal{C}_a^r[\mathrm{i}]$, then $B_{\ltimes \mathfrak{r}}$ maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^r[\mathrm{i}]$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^{r+1}[\mathrm{i}]$. Note that if $\phi \in \mathcal{C}_a^r[\mathrm{i}]$ and $\mathfrak{r} \in \mathcal{C}_a^{r+1}$, then $\widetilde{\phi} \in \mathcal{C}_a^r[\mathrm{i}]$.

Next, suppose ϕ, $\widetilde{\phi}$ are both repulsive. Then we have the $\mathbb{C}$-linear operators $B, \widetilde{B}\colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a^1[\mathrm{i}]$ given, for $f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ and $t \geqslant a$, by

$$Bf(t) \;:=\; \mathrm{e}^{\Phi(t)} \int_\infty^t \mathrm{e}^{-\Phi(s)}\, f(s)\,ds, \qquad \widetilde{B}f(t) \;:=\; \mathrm{e}^{\widetilde{\Phi}(t)} \int_\infty^t \mathrm{e}^{-\widetilde{\Phi}(s)}\, f(s)\,ds.$$

Now assume $\mathfrak{r} \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. Then we have the $\mathbb{C}$-linear operator

$$B_{\ltimes \mathfrak{r}} \;:=\; \mathfrak{r}^{-1} \circ B \circ \mathfrak{r} \;:\; \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a^1[\mathrm{i}].$$

A computation as above shows $\widetilde{B} = B_{\ltimes \mathfrak{r}}$; thus $B_{\ltimes \mathfrak{r}}$ maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^1[\mathrm{i}]$, and the operator $B_{\ltimes \mathfrak{r}}\colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ is continuous with $\|B_{\ltimes \mathfrak{r}}\|_a \leqslant \big\|\frac{1}{\operatorname{Re}\widetilde{\phi}}\big\|_a$. If $\widetilde{\phi} \in \mathcal{C}_a^r[\mathrm{i}]$, then $B_{\ltimes \mathfrak{r}}$ maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^r[\mathrm{i}]$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \cap \mathcal{C}_a^{r+1}[\mathrm{i}]$.

More on twists and right-inverses of linear operators over Hardy fields. In this subsection we adopt the assumptions in force for Lemma 5.4, which we repeat here. Thus H is a Hardy field, $K = H[\mathrm{i}]$, $r \in \mathbb{N}^{\geqslant 1}$, and $f_1, \dots, f_r \in K$. We fix $a_0 \in \mathbb{R}$ and functions in $\mathcal{C}_{a_0}[\mathrm{i}]$ representing the germs $f_1, \dots, f_r$, denoted by the same symbols. We let a range over $[a_0, \infty)$, and we denote the restriction of each $f \in \mathcal{C}_{a_0}[\mathrm{i}]$ to $[a, \infty)$ also by f. For each a we then have the $\mathbb{C}$-linear map $A_a\colon \mathcal{C}_a^r[\mathrm{i}] \to \mathcal{C}_a[\mathrm{i}]$ given by

$$A_a(y) \;=\; y^{(r)} + f_1 y^{(r-1)} + \cdots + f_r y.$$

We are in addition given a splitting $(\phi_1, \dots, \phi_r)$ of the linear differential operator $A = \partial^r + f_1\partial^{r-1} + \cdots + f_r \in K[\partial]$ over K with $\operatorname{Re}\phi_1, \dots, \operatorname{Re}\phi_r \succcurlyeq 1$, as well as functions in $\mathcal{C}_{a_0}^{r-1}[\mathrm{i}]$ representing $\phi_1, \dots, \phi_r$, denoted by the same symbols and satisfying $\operatorname{Re}\phi_1,\dots,\operatorname{Re}\phi_r \in (\mathcal{C}_{a_0})^\times$. This gives rise to the continuous $\mathbb{C}$-linear operators

$$B_j \ := \ B_{\phi_j}\ :\ \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \qquad (j=1,\dots,r)$$

and the right-inverse

$$A_a^{-1} \ := \ B_r\circ\cdots\circ B_1\ :\ \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$$

of A_a with the properties stated in Lemma 5.4.

Now let $\mathfrak{m}\in H^\times$ with $\mathfrak{m}\prec 1$, and set $\widetilde{A} := A_{\ltimes\mathfrak{m}} \in K[\partial]$. Let $\mathfrak{r}\in(\mathcal{C}^r_{a_0})^\times$ be a representative of $\mathfrak{m}$. Then $\mathfrak{r}\in(\mathcal{C}^r_{a_0})^{\mathrm{b}}$ and $\widetilde{\phi}_j := \phi_j - \mathfrak{r}^\dagger \in \mathcal{C}^{r-1}_{a_0}[\mathrm{i}]$ for $j=1,\dots,r$. We have the $\mathbb{C}$-linear maps

$$\widetilde{A}_j \ := \ \partial - \widetilde{\phi}_j\ :\ \mathcal{C}^j_a[\mathrm{i}] \to \mathcal{C}^{j-1}_a[\mathrm{i}] \qquad (j=1,\dots,r)$$

and for sufficiently large a a factorization

$$\widetilde{A}_a \ = \ \widetilde{A}_1\circ\cdots\circ\widetilde{A}_r\ :\ \mathcal{C}^r_a[\mathrm{i}]\to\mathcal{C}_a[\mathrm{i}].$$

Below we assume this holds for all a, as can be arranged by increasing a_0. We call $f,g\in\mathcal{C}_a[\mathrm{i}]$ **alike** if f, g are both attractive or both repulsive. In the same way we define when germs $f,g\in\mathcal{C}[\mathrm{i}]$ are alike. Suppose that ϕ_j, $\widetilde{\phi}_j$ are alike for $j=1,\dots,r$. Then we have continuous $\mathbb{C}$-linear operators

$$\widetilde{B}_j \ := \ B_{\widetilde{\phi}_j}\ :\ \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \qquad (j=1,\dots,r)$$

and the right-inverse

$$\widetilde{A}_a^{-1} \ := \ \widetilde{B}_r\circ\cdots\circ\widetilde{B}_1\ :\ \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$$

of $\widetilde{A}_a$, and the arguments in the previous subsection show that $\widetilde{B}_j = (B_j)_{\ltimes\mathfrak{r}} = \mathfrak{r}^{-1}\circ B_j\circ\mathfrak{r}$ for $j=1,\dots,r$, and hence $\widetilde{A}_a^{-1} = \mathfrak{r}^{-1}\circ A_a^{-1}\circ\mathfrak{r}$. For $j=0,\dots,r$ we set, in analogy with A_j° and B_j° from (5.2) and (5.3),

$$\widetilde{A}_j^\circ \ := \ \widetilde{A}_1\circ\cdots\circ\widetilde{A}_j\ :\ \mathcal{C}^j_a[\mathrm{i}]\to\mathcal{C}_a[\mathrm{i}], \quad \widetilde{B}_j^\circ \ := \ \widetilde{B}_j\circ\cdots\circ\widetilde{B}_1\ :\ \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\to\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}.$$

Then $\widetilde{B}_j$ maps $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ into $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}\cap\mathcal{C}^j_a[\mathrm{i}]$, $\widetilde{A}_j^\circ\circ\widetilde{B}_j^\circ$ is the identity on $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$, and $\widetilde{B}_j^\circ = \mathfrak{r}^{-1}\circ B_j^\circ\circ\mathfrak{r}$ by the above.

A weighted version of Proposition 5.6. We adopt the setting of the subsection *Damping factors* of Section 5, and make the same assumptions as in the paragraph before Proposition 5.6. Thus H, K, A, $f_1,\dots,f_r$, $\phi_1,\dots,\phi_r$, a_0 are as in the previous subsection, $\mathfrak{v}\in\mathcal{C}^r_{a_0}$ satisfies $\mathfrak{v}(t)>0$ for all $t\geqslant a_0$, and its germ $\mathfrak{v}$ is in H with $\mathfrak{v}\prec 1$. As part of those assumptions we also have $\phi_1,\dots,\phi_r\preccurlyeq_\Delta \mathfrak{v}^{-1}$ in the asymptotic field K, for the convex subgroup

$$\Delta \ := \ \big\{\gamma\in v(H^\times):\ \gamma = o(v\mathfrak{v})\big\}$$

of $v(H^\times)=v(K^\times)$. Also $\nu\in\mathbb{R}^>$ and $u := \mathfrak{v}^\nu|_{[a,\infty)}\in(\mathcal{C}^r_a)^\times$.

To state a weighted version of Proposition 5.6, let $\mathfrak{m}\in H^\times$, $\mathfrak{m}\prec 1$, and let $\mathfrak{m}$ also denote a representative in $(\mathcal{C}^r_{a_0})^\times$ of the germ $\mathfrak{m}$. Set $\mathfrak{r} := \mathfrak{m}|_{[a,\infty)}$, so we have $\mathfrak{r}\in(\mathcal{C}^r_a)^\times\cap(\mathcal{C}^r_a)^{\mathrm{b}}$ and thus $\mathcal{C}^r_a[\mathrm{i}]^{\mathfrak{r}}\subseteq\mathcal{C}^r_a[\mathrm{i}]^{\mathrm{b}}$. (Note that $\mathfrak{r}$, like u, depends on a, but we do not indicate this dependence notationally.) With notations as in the previous subsection we assume that for all a we have the factorization

$$\widetilde{A}_a \ = \ \widetilde{A}_1\circ\cdots\circ\widetilde{A}_r\ :\ \mathcal{C}^r_a[\mathrm{i}]\to\mathcal{C}_a[\mathrm{i}],$$

as can be arranged by increasing a_0 if necessary.

Proposition 9.7. *Assume H is real closed, $\nu \in \mathbb{Q}$, $\nu > r$, and the elements ϕ_j, $\phi_j - \mathfrak{m}^\dagger$ of $\mathcal{C}_{a_0}[\mathrm{i}]$ are alike for $j = 1, \dots, r$. Then:*

(i) *the $\mathbb{C}$-linear operator $uA_a^{-1}\colon \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ maps $\mathcal{C}_a[\mathrm{i}]^{\mathfrak{r}}$ into $\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$;*
(ii) *its restriction to a $\mathbb{C}$-linear map $\mathcal{C}_a[\mathrm{i}]^{\mathfrak{r}} \to \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$ is continuous; and*
(iii) *denoting this restriction also by uA_a^{-1}, we have $\|uA_a^{-1}\|_{a;r}^{\mathfrak{r}} \to 0$ as $a \to \infty$.*

Proof. Let $f \in \mathcal{C}_a[\mathrm{i}]^{\mathfrak{r}}$, so $g := \mathfrak{r}^{-1} f \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. Let $i \in \{0, \dots, r\}$; then with $\widetilde{B}_j^\circ$ as in the previous subsection and $u_{i,j}$ as in Lemma 5.5, that lemma gives

$$\mathfrak{r}^{-1}\big(uA_a^{-1}(f)\big)^{(i)} = \sum_{j=r-i}^{r} u_{i,j} u \cdot \mathfrak{r}^{-1} B_j^\circ(\mathfrak{r} g) \; = \sum_{j=r-i}^{r} u_{i,j} u \widetilde{B}_j^\circ(g).$$

The proof of Proposition 5.6 shows $u_{i,j}u \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$ with $\|u_{i,j}u\|_a \to 0$ as $a \to \infty$. Set

$$\widetilde{c}_{i,a} \; := \; \sum_{j=r-i}^{r} \|u\, u_{i,j}\|_a \cdot \|\widetilde{B}_j\|_a \cdots \|\widetilde{B}_1\|_a \in [0,\infty) \qquad (i = 0, \dots, r).$$

Then $\big\|\mathfrak{r}^{-1}\big[uA_a^{-1}(f)\big]^{(i)}\big\|_a \leqslant \widetilde{c}_{i,a}\|g\|_a = \widetilde{c}_{i,a}\|f\|_a^{\mathfrak{r}}$ where $\widetilde{c}_{i,a} \to 0$ as $a \to \infty$. This yields (i)–(iii). □

Weighted variants of results in Section 6. In this subsection we adopt the hypotheses in force for Lemma 6.1. To summarize those, H, K, A, $f_1, \dots, f_r$, $\phi_1, \dots, \phi_r$, a_0, $\mathfrak{v}$, ν, u, Δ are as in the previous subsection, $d, r \in \mathbb{N}^{\geqslant 1}$, H is real closed, $R \in K\{Y\}$ has order $\leqslant d$ and weight $\leqslant w \in \mathbb{N}^{\geqslant r}$. Also $\nu \in \mathbb{Q}$, $\nu > w$, $R \prec_\Delta \mathfrak{v}^\nu$, $\nu\mathfrak{v}^\dagger \not\sim \operatorname{Re}\phi_j$ and $\operatorname{Re}\phi_j - \nu\mathfrak{v}^\dagger \in (\mathcal{C}_{a_0})^\times$ for $j = 1, \dots, r$. Finally, $\widetilde{A} := A_{\ltimes \mathfrak{v}^\nu} \in K[\partial]$ and $\widetilde{A}_a(y) = u^{-1}A_a(uy)$ for $y \in \mathcal{C}_a^r[\mathrm{i}]$. Next, let $\mathfrak{m}$, $\mathfrak{r}$ be as in the previous subsection. As in Lemma 6.10 we consider the continuous operator

$$\Phi_a\colon \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} \times \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$$

given by

$$\Phi_a(f,y) \; := \; \Xi_a(f+y) - \Xi_a(f) \; = \; u\widetilde{A}_a^{-1}\big(u^{-1}\big(R(f+y) - R(f)\big)\big).$$

Here is our weighted version of Lemma 6.10:

Lemma 9.8. *Suppose the elements $\phi_j - \nu\mathfrak{v}^\dagger$, $\phi_j - \nu\mathfrak{v}^\dagger - \mathfrak{m}^\dagger$ of $\mathcal{C}_{a_0}[\mathrm{i}]$ are alike, for $j = 1, \dots, r$, and let $f \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$. Then the operator $y \mapsto \Phi_a(f,y)$ maps $\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$ into itself. Moreover, there are $E_a, E_a^+ \in \mathbb{R}^{\geqslant}$ such that for all $g \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ and $y \in \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$,*

$$\|\Phi_a(f,y)\|_{a;r}^{\mathfrak{r}} \; \leqslant \; E_a \cdot \max\big\{1, \|f\|_{a;r}^d\big\} \cdot \big(1 + \|y\|_{a;r} + \cdots + \|y\|_{a;r}^{d-1}\big) \cdot \|y\|_{a;r}^{\mathfrak{r}},$$

$$\|\Phi_a(f,g+y) - \Phi_a(f,g)\|_{a;r}^{\mathfrak{r}} \; \leqslant$$
$$E_a^+ \cdot \max\big\{1, \|f\|_{a;r}^d\big\} \cdot \max\big\{1, \|g\|_{a;r}^d\big\} \cdot \big(1 + \|y\|_{a;r} + \cdots + \|y\|_{a;r}^{d-1}\big) \cdot \|y\|_{a;r}^{\mathfrak{r}}.$$

We can take these E_a, E_a^+ such that $E_a, E_a^+ \to 0$ as $a \to \infty$, and do so below.

Proof. Let $y \in \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$. By Taylor expansion we have

$$R(f+y) - R(f) \; = \; \sum_{|\boldsymbol{i}|>0} \frac{1}{\boldsymbol{i}!} R^{(\boldsymbol{i})}(f) y^{\boldsymbol{i}} \; = \; \sum_{|\boldsymbol{i}|>0} S_{\boldsymbol{i}}(f) y^{\boldsymbol{i}} \quad \text{where } S_{\boldsymbol{i}}(f) := \frac{1}{\boldsymbol{i}!} \sum_{\boldsymbol{j}} R_{\boldsymbol{j}}^{(\boldsymbol{i})} f^{\boldsymbol{j}},$$

and $u^{-1}S_{\boldsymbol{i}}(f) \in \mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. So $h := u^{-1}\big(R(f+y) - R(f)\big) \in \mathcal{C}_a[\mathrm{i}]^{\mathfrak{r}}$, since $\mathcal{C}_a[\mathrm{i}]^{\mathfrak{r}}$ is an ideal of $\mathcal{C}_a[\mathrm{i}]^{\mathrm{b}}$. Applying Proposition 9.7(i) with $\phi_j - \nu\mathfrak{v}^\dagger$ in the role of ϕ_j

yields $\Phi_a(f,y) = u\widetilde{A}_a^{-1}(h) \in \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$, establishing the first claim. Next, let $g \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$. Then $\Phi_a(f,g+y) - \Phi_a(f,g) = \Phi_a(f+g,y)$ by (6.1). Therefore,

$$\Phi_a(f,g+y) - \Phi_a(f,g) \ = \ u\widetilde{A}_a^{-1}(h), \quad h \ := \ u^{-1}\big(R(f+g+y) - R(f+g)\big), \text{ so}$$
$$\|\Phi_a(f,g+y) - \Phi_a(f,g)\|_{a;r}^{\mathfrak{r}} \ = \ \|u\widetilde{A}_a^{-1}(h)\|_{a;r}^{\mathfrak{r}} \ \leqslant \ \|u\widetilde{A}_a^{-1}\|_{a;r}^{\mathfrak{r}} \cdot \|h\|_a^{\mathfrak{r}}.$$

By Corollary 9.6 we have

$$\|h\|_a^{\mathfrak{r}} \ \leqslant \ D \cdot \max_{|\boldsymbol{i}|>0} \|u^{-1}S_{\boldsymbol{i}}(f+g)\|_a \cdot \big(1 + \|y\|_{a;r} + \cdots + \|y\|_{a;r}^{d-1}\big) \cdot \|y\|_{a;r}^{\mathfrak{r}}$$

where $D = D(d,r) := \left(\binom{d+r+1}{r+1} - 1\right)$. Let D_a be as in the proof of Lemma 6.2. Then $D_a \to 0$ as $a \to \infty$, and Lemma 6.9 gives for $|\boldsymbol{i}| > 0$,

$$\begin{aligned} \|u^{-1}S_{\boldsymbol{i}}(f)\|_a \ &\leqslant \ D_a \cdot \max\big\{1, \|f\|_{a;r}^d\big\} \\ \|u^{-1}S_{\boldsymbol{i}}(f+g)\|_a \ &\leqslant \ D_a \cdot \max\big\{1, \|f+g\|_{a;r}^d\big\} \\ &\leqslant \ 2^d D_a \cdot \max\big\{1, \|f\|_{a;r}^d\big\} \cdot \max\big\{1, \|g\|_{a;r}^d\big\}. \end{aligned}$$

This gives the desired result for $E_a := \|u\widetilde{A}_a^{-1}\|_{a;r}^{\mathfrak{r}} \cdot D \cdot D_a$ and $E_a^+ := 2^d E_a$, using also Proposition 9.7(iii) with $\phi_j - \nu\mathfrak{v}^\dagger$ in the role of ϕ_j. □

Lemma 9.8 allows us to refine Theorem 6.3 as follows:

Corollary 9.9. *Suppose the elements $\phi_j - \nu\mathfrak{v}^\dagger$, $\phi_j - \nu\mathfrak{v}^\dagger - \mathfrak{m}^\dagger$ of $\mathcal{C}_{a_0}[\mathrm{i}]$ are alike, for $j = 1,\dots,r$, and $R(0) \preccurlyeq \mathfrak{v}^\nu\mathfrak{m}$. Then for sufficiently large a the operator Ξ_a maps the closed ball $B_a := \big\{f \in \mathcal{C}_a^r[\mathrm{i}] : \|f\|_{a;r} \leqslant 1/2\big\}$ of the normed space $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ into itself, has a unique fixed point in B_a, and this fixed point lies in $\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$.*

Proof. Take a such that $\|\mathfrak{r}\|_a \leqslant 1$. Then by (9.2), B_a contains the closed ball

$$B_a^{\mathfrak{r}} \ := \ \big\{f \in \mathcal{C}_a^r[\mathrm{i}] : \|f\|_{a;r}^{\mathfrak{r}} \leqslant 1/2\big\}$$

of the normed space $\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$. Let $f,g \in B_a^{\mathfrak{r}}$. Then $\Xi_a(g) - \Xi_a(f) = \Phi_a(f, g-f)$ lies in $\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$ by Lemma 9.8, and with E_a as in that lemma,

$$\begin{aligned} \|\Xi_a(f) - \Xi_a(g)\|_{a;r}^{\mathfrak{r}} \ &= \ \|\Phi_a(f,g-f)\|_{a;r}^{\mathfrak{r}} \\ &\leqslant \ E_a \cdot \max\big\{1, \|f\|_{a;r}^d\big\} \cdot \big(1 + \cdots + \|g-f\|_{a;r}^{d-1}\big) \cdot \|g-f\|_{a;r}^{\mathfrak{r}} \\ &\leqslant \ E_a \cdot d \cdot \|g-f\|_{a;r}^{\mathfrak{r}}. \end{aligned}$$

Taking a so that moreover $E_a d \leqslant \frac{1}{2}$ we obtain

$$\|\Xi_a(f) - \Xi_a(g)\|_{a;r}^{\mathfrak{r}} \leqslant \tfrac{1}{2}\|f-g\|_{a;r}^{\mathfrak{r}} \quad \text{for all } f,g \in B_a^{\mathfrak{r}}. \tag{9.4}$$

Next we consider the case $g = 0$. Our hypothesis $R(0) \preccurlyeq \mathfrak{v}^\nu\mathfrak{m}$ gives $u^{-1}R(0) \in \mathcal{C}_a[\mathrm{i}]^{\mathfrak{r}}$. Proposition 9.7(i),(ii) with $\phi_j - \nu\mathfrak{v}^\dagger$ in the role of ϕ_j gives $\Xi_a(0) \in \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$ and $\|\Xi_a(0)\|_{a;r}^{\mathfrak{r}} \leqslant \|u\widetilde{A}_a^{-1}\|_{a;r}^{\mathfrak{r}}\|u^{-1}R(0)\|_a^{\mathfrak{r}}$. Using Proposition 9.7(iii) we now take a so large that $\|\Xi_a(0)\|_{a;r}^{\mathfrak{r}} \leqslant \frac{1}{4}$. Then (9.4) for $g = 0$ gives $\Xi_a(B_a^{\mathfrak{r}}) \subseteq B_a^{\mathfrak{r}}$. By Lemma 9.2 the normed space $\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$ is complete, hence Ξ_a has a unique fixed point in $B_a^{\mathfrak{r}}$. □

Now suppose in addition that $A \in H[\partial]$ and $R \in H\{Y\}$. Set

$$(\mathcal{C}_a^r)^{\mathfrak{r}} \ := \ \big\{f \in \mathcal{C}_a^r : \|f\|_{a;r}^{\mathfrak{r}} < \infty\big\} \ = \ \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}} \cap \mathcal{C}_a^r,$$

a real Banach space with respect to $\|\cdot\|_{a;r}^{\mathfrak{r}}$. Increase a_0 as at the beginning of the subsection *Preserving reality* of Section 6. Then we have the map

$$\operatorname{Re}\Phi_a \colon (\mathcal{C}_a^r)^{\mathrm{b}} \times (\mathcal{C}_a^r)^{\mathrm{b}} \to (\mathcal{C}_a^r)^{\mathrm{b}}, \qquad (f,y) \mapsto \operatorname{Re}\big(\Phi_a(f,y)\big).$$

Suppose the elements $\phi_j - \nu\mathfrak{v}^\dagger$, $\phi_j - \nu\mathfrak{v}^\dagger - \mathfrak{m}^\dagger$ are alike for $j = 1,\dots,r$, and let a and E_a, E_a^+ be as in Lemma 9.8. Then this lemma yields:

Lemma 9.10. *Let $f, g \in (\mathcal{C}_a^r)^{\mathrm{b}}$ and $y \in (\mathcal{C}_a^r)^{\mathfrak{r}}$. Then $(\operatorname{Re}\Phi_a)(f,y) \in (\mathcal{C}_a^r)^{\mathfrak{r}}$ and*

$$\|\operatorname{Re}(\Phi_a)(f,y)\|_{a;r}^{\mathfrak{r}} \ \leqslant\ E_a \cdot \max\bigl\{1, \|f\|_{a;r}^d\bigr\} \cdot \bigl(1 + \|y\|_{a;r} + \cdots + \|y\|_{a;r}^{d-1}\bigr) \cdot \|y\|_{a;r}^{\mathfrak{r}},$$

$$\|(\operatorname{Re}\Phi_a)(f, g+y) - (\operatorname{Re}\Phi_a)(f,g)\|_{a;r}^{\mathfrak{r}} \ \leqslant$$
$$E_a^+ \cdot \max\bigl\{1, \|f\|_{a;r}^d\bigr\} \cdot \max\bigl\{1, \|g\|_{a;r}^d\bigr\} \cdot \bigl(1 + \|y\|_{a;r} + \cdots + \|y\|_{a;r}^{d-1}\bigr) \cdot \|y\|_{a;r}^{\mathfrak{r}}.$$

The same way we derived Corollary 9.9 from Lemma 9.8, this leads to:

Corollary 9.11. *If $R(0) \preccurlyeq \mathfrak{v}^\nu\mathfrak{m}$, then for sufficiently large a the operator $\operatorname{Re}\Xi_a$ maps the closed ball $B_a := \bigl\{f \in \mathcal{C}_a^r : \|f\|_{a;r} \leqslant 1/2\bigr\}$ of the normed space $(\mathcal{C}_a^r)^{\mathrm{b}}$ into itself, has a unique fixed point in B_a, and this fixed point lies in $(\mathcal{C}_a^r)^{\mathfrak{r}}$.*

Revisiting Lemma 6.11. We adopt the setting of the previous subsection. As usual, a ranges over $[a_0,\infty)$. We continue the investigation of the differences $f-g$ between solutions f, g of the equation $(*)$ on $[a_0,\infty)$ that began in Lemma 6.4, and so we take f, g, E, ε, h_a, θ_a as in that lemma. Recall that in the remarks preceding Lemma 6.11 we defined continuous operators $\Phi_a, \Psi_a \colon \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} \to \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ by

$$\Phi_a(y) \ :=\ \Phi_a(g,y) \ =\ \Xi_a(g+y) - \Xi_a(g), \quad \Psi_a(y) \ :=\ \Phi_a(y) + h_a \qquad (y \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}).$$

As in those remarks, we set $\rho := \|f-g\|_{a_0;r}$ and

$$B_a \ :=\ \bigl\{y \in \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}} :\ \|y - h_a\|_{a;r} \leqslant 1/2\bigr\},$$

and take $a_1 \geqslant a_0$ so that $\theta_a \in B_a$ for all $a \geqslant a_1$. Then by (6.4) we have $\|y\|_{a;r} \leqslant 1+\rho$ for $a \geqslant a_1$ and $y \in B_a$. Next, take $a_2 \geqslant a_1$ as in Lemma 6.11; thus for $a \geqslant a_2$ and $y, z \in B_a$ we have $\Psi_a(y) \in B_a$ and $\|\Psi_a(y) - \Psi_a(z)\|_{a;r} \leqslant \frac{1}{2}\|y-z\|_{a;r}$. As in the previous subsection, $\mathfrak{m} \in H^\times$, $\mathfrak{m} \prec 1$, $\mathfrak{m}$ denotes also a representative in $(\mathcal{C}_{a_0}^r)^\times$ of the germ $\mathfrak{m}$, and $\mathfrak{r} := \mathfrak{m}|_{[a,\infty)} \in (\mathcal{C}_a^r)^\times \cap (\mathcal{C}_a^r)^{\mathrm{b}}$, so $\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}} \subseteq \mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$.

In the rest of this subsection $\phi_1 - \nu\mathfrak{v}^\dagger, \dots, \phi_r - \nu\mathfrak{v}^\dagger \in K$ are γ-repulsive for $\gamma := v\mathfrak{m} \in v(H^\times)^>$, and $h_a \in \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$ for all $a \geqslant a_2$. Then Lemma 1.15 gives $a_3 \geqslant a_2$ such that for all $a \geqslant a_3$ and $j = 1,\dots,r$, the functions $\phi_j - u^\dagger, \phi_j - (u\mathfrak{r})^\dagger \in \mathcal{C}_a[\mathrm{i}]$ are alike and hence $\Psi_a\bigl(\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}\bigr) \subseteq \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$ by Lemma 9.8. Thus $\Psi_a^n(h_a) \in \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$ for all n and $a \geqslant a_3$.

For $a \geqslant a_2$ we have $\lim_{n\to\infty} \Psi_a^n(h_a) = \theta_a$ in $\mathcal{C}_a^r[\mathrm{i}]^{\mathrm{b}}$ by Corollary 6.12; we now aim to strengthen this to "in $\mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$" (possibly for a larger a_2). Towards this:

Lemma 9.12. *There exists $a_4 \geqslant a_3$ such that $\|\Psi_a(y) - \Psi_a(z)\|_{a;r}^{\mathfrak{r}} \leqslant \frac{1}{2}\|y-z\|_{a;r}^{\mathfrak{r}}$ for all $a \geqslant a_4$ and $y, z \in B_a \cap \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$.*

Proof. For $a \geqslant a_3$ and $y, z \in \mathcal{C}_a^r[\mathrm{i}]^{\mathfrak{r}}$, and with E_a^+ as in Lemma 9.8 we have

$$\|\Psi_a(y) - \Psi_a(z)\|_{a;r}^{\mathfrak{r}} \ \leqslant$$
$$E_a^+ \cdot \max\bigl\{1, \|g\|_{a;r}^d\bigr\} \cdot \max\bigl\{1, \|z\|_{a;r}^d\bigr\} \cdot \bigl(1 + \|y-z\|_{a;r} + \cdots + \|y-z\|_{a;r}^{d-1}\bigr) \cdot \|y-z\|_{a;r}^{\mathfrak{r}}.$$

For each $a \geqslant a_1$ and $y, z \in B_a$ we then have

$$\max\bigl\{1, \|z\|_{a;r}^d\bigr\} \cdot \bigl(1 + \|y-z\|_{a;r} + \cdots + \|y-z\|_{a;r}^{d-1}\bigr) \ \leqslant\ (1+\rho)^d \cdot d,$$

so taking $a_4 \geqslant a_3$ with

$$E_a^+ \max\bigl\{1, \|g\|_{a_0;r}^d\bigr\}(1+\rho)^d d \leqslant 1/2 \quad \text{ for all } a \geqslant a_4,$$

we have $\|\Psi_a(y)-\Psi_a(z)\|_{a;r}^{\tau} \leqslant \frac{1}{2}\|y-z\|_{a;r}^{\tau}$ for all $a \geqslant a_4$ and $y,z \in B_a \cap \mathcal{C}_a^r[i]^{\tau}$. $\square$

Let a_4 be as in the previous lemma.

Corollary 9.13. *Suppose $a \geqslant a_4$. Then $\theta_a \in \mathcal{C}_a^r[i]^{\tau}$ and $\lim_{n\to\infty} \Psi_a^n(h_a) = \theta_a$ in the normed space $\mathcal{C}_a^r[i]^{\tau}$. In particular, $f-g, (f-g)', \dots, (f-g)^{(r)} \preccurlyeq \mathfrak{m}$.*

Proof. We have $\Phi_a(h_a) = \Psi_a(h_a) - h_a \in \mathcal{C}_a^r[i]^{\tau}$, so $M := \|\Phi_a(h_a)\|_{a;r}^{\tau} < \infty$. Since $\Psi_a(B_a) \subseteq B_a$, induction on n using Lemma 9.12 gives

$$\|\Psi_a^{n+1}(h_a) - \Psi_a^n(h_a)\|_{a;r}^{\tau} \leqslant M/2^n \qquad \text{for all } n.$$

Thus $\big(\Psi_a^n(h_a)\big)$ is a cauchy sequence in the normed space $\mathcal{C}_a^r[i]^{\tau}$, and so converges in $\mathcal{C}_a^r[i]^{\tau}$ by Lemma 9.2. In the normed space $\mathcal{C}_a^r[i]^{\mathrm{b}}$ we have $\lim_{n\to\infty} \Psi_a^n(h_a) = \theta_a$, by Corollary 6.12. Thus $\lim_{n\to\infty} \Psi_a^n(h_a) = \theta_a$ in $\mathcal{C}_a^r[i]^{\tau}$ by Lemma 9.1. $\square$

An application to slots in H. Here we adopt the setting of the subsection *An application to slots in H* in Section 4. Thus $H \supseteq \mathbb{R}$ is a Liouville closed Hardy field, $K := H[i]$, $\mathrm{I}(K) \subseteq K^{\dagger}$, and $(P, 1, \widehat{h})$ is a slot in H of order $r \geqslant 1$; we set $w := \mathrm{wt}(P)$, $d := \deg P$. *Assume also that K is 1-linearly surjective if $r \geqslant 3$.*

Proposition 9.14. *Suppose $(P, 1, \widehat{h})$ is special, ultimate, Z-minimal, deep, and strongly repulsive-normal. Let $f, g \in \mathcal{C}^r[i]$ and $\mathfrak{m} \in H^{\times}$ be such that*

$$P(f) \;=\; P(g) \;=\; 0, \qquad f,g \;\prec\; 1, \qquad v\mathfrak{m} \in v(\widehat{h} - H).$$

Then $(f-g)^{(j)} \preccurlyeq \mathfrak{m}$ for $j = 0, \dots, r$.

Proof. We arrange $\mathfrak{m} \prec 1$. Let $\mathfrak{v} := |\mathfrak{v}(L_P)| \in H^{>}$, so $\mathfrak{v} \prec^{\flat} 1$, and set $\Delta := \Delta(\mathfrak{v})$. Take $Q, R \in H\{Y\}$ where Q is homogeneous of degree 1 and order r, $A := L_Q \in H[\partial]$ has a strong $\widehat{h}$-repulsive splitting over K, $P = Q - R$, and $R \prec_{\Delta} \mathfrak{v}^{w+1} P_1$, so $\mathfrak{v}(A) \sim \mathfrak{v}(L_P)$ by Lemma 1.5. Multiplying P, Q, R by some $b \in H^{\times}$ we arrange that A is monic, so $A = \partial^r + f_1\partial^{r-1} + \dots + f_r$ with $f_1, \dots, f_r \in H$ and $R \prec_{\Delta} \mathfrak{v}^w$. Let $(\phi_1, \dots, \phi_r) \in K^r$ be a strong $\widehat{h}$-repulsive splitting of A over K, so $\phi_1, \dots, \phi_r$ are $\widehat{h}$-repulsive and

$$A \;=\; (\partial - \phi_1)\cdots(\partial - \phi_r), \qquad \operatorname{Re}\phi_1, \dots, \operatorname{Re}\phi_r \;\succcurlyeq\; \mathfrak{v}^{\dagger} \;\succcurlyeq\; 1.$$

By (1.3) we have $\phi_1, \dots, \phi_r \preccurlyeq \mathfrak{v}^{-1}$. Thus we can take $a_0 \in \mathbb{R}$ and functions on $[a_0, \infty)$ representing the germs $\phi_1, \dots, \phi_r$, $f_1, \dots, f_r$, f, g, as well as the $R_{\boldsymbol{j}}$ with $\boldsymbol{j} \in \mathbb{N}^{1+r}$, $|\boldsymbol{j}| \leqslant d$, $\|\boldsymbol{j}\| \leqslant w$ (using the same symbols for the germs mentioned as for their chosen representatives) so as to be in the situation described in the beginning of Section 6, with f and g solutions on $[a_0, \infty)$ of the differential equation $(*)$ there. As there, we take $\nu \in \mathbb{Q}$ with $\nu > w$ so that $R \prec_{\Delta} \mathfrak{v}^{\nu}$ and $\nu\mathfrak{v}^{\dagger} \not\sim \operatorname{Re}\phi_j$ for $j = 1, \dots, r$, and then increase a_0 to satisfy all assumptions for Lemma 6.1. Proposition 1.9 gives $v(\mathfrak{v}^{\nu}) \in v(\widehat{h} - H)$, so $\phi_j - \nu\mathfrak{v}^{\dagger} = \phi_j - (\mathfrak{v}^{\nu})^{\dagger}$ ($j = 1, \dots, r$) is $\widehat{h}$-repulsive by [8, Lemma 4.5.13(iv)], so γ-repulsive for $\gamma := v\mathfrak{m} > 0$. Now A splits over K, and K is 1-linearly surjective if $r \geqslant 3$, hence $\dim_{\mathbb{C}} \ker_{\mathrm{U}} A = r$ by (1.2) and Lemma 1.4. Thus by Corollary 4.13 we have $y, y', \dots, y^{(r)} \prec \mathfrak{m}$ for all $y \in \mathcal{C}^r[i]$ with $A(y) = 0$, $y \prec 1$. In particular, $\mathfrak{m}^{-1}h_a, \mathfrak{m}^{-1}h_a', \dots, \mathfrak{m}^{-1}h_a^{(r)} \prec 1$ for all $a \geqslant a_0$. Thus the assumptions on $\mathfrak{m}$ and the h_a made just before Lemma 9.12 are satisfied for a suitable choice of a_2, so we can appeal to Corollary 9.13. $\square$

The assumption that K is 1-linearly surjective for $r \geqslant 3$ was only used in the proof above to obtain $\dim_{\mathbb{C}} \ker_{\mathrm{U}} A = r$. So if A as in this proof satisfies $\dim_{\mathbb{C}} \ker_{\mathrm{U}} A = r$, then we can drop this assumption about K, also in the next corollary.

Corollary 9.15. *Suppose $(P, 1, \widehat{h})$, f, g, $\mathfrak{m}$ are as in Proposition 9.14. Then*

$$f - g \in \mathfrak{m}\,\mathcal{C}^r[\mathrm{i}]^{\preccurlyeq}.$$

Proof. If $\mathfrak{m} \succcurlyeq 1$, then Lemma 8.1(ii) applied with $y = (f-g)/\mathfrak{m}$ and $1/\mathfrak{m}$ in place of $\mathfrak{m}$ gives what we want. Now assume $\mathfrak{m} \prec 1$. Since $\widehat{h}$ is special over H, Proposition 9.14 applies to $\mathfrak{m}^{r+1}$ in place of $\mathfrak{m}$, so $(f-g)^{(j)} \preccurlyeq \mathfrak{m}^{r+1}$ for $j = 0, \dots, r$. Now apply Lemma 9.3 to suitable representatives of $f - g$ and $\mathfrak{m}$. □

Later in this section we use Proposition 9.14 and its Corollary 9.15 to strengthen some results from Section 8.

Weighted refinements of results in Section 8. We now adopt the setting of the subsection *Reformulations* of Section 8. Thus $H \supseteq \mathbb{R}$ is a real closed Hardy field with asymptotic integration, and $K := H[\mathrm{i}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$ is its algebraic closure, with value group $\Gamma := v(H^\times) = v(K^\times)$. The next lemma and its corollary refine Lemma 8.5. Let P, Q, R, L_Q, w be as introduced before that lemma, set $\mathfrak{v} := |\mathfrak{v}(L_Q)| \in H^>$, and, in case $\mathfrak{v} \prec 1$, $\Delta := \Delta(\mathfrak{v})$.

Lemma 9.16. *Let $f \in K^\times$ and $\phi_1, \dots, \phi_r \in K$ be such that*

$$L_Q \;=\; f(\partial - \phi_1) \cdots (\partial - \phi_r), \qquad \operatorname{Re}\phi_1, \dots, \operatorname{Re}\phi_r \;\succcurlyeq\; 1.$$

Assume $\mathfrak{v} \prec 1$ and $R \prec_\Delta \mathfrak{v}^{w+1} Q$. Let $\mathfrak{m} \in H^\times$, $\mathfrak{m} \prec 1$, $P(0) \preccurlyeq \mathfrak{v}^{w+2}\mathfrak{m} Q$. Suppose that for $j = 1, \dots, r$ and all $\nu \in \mathbb{Q}$ with $w < \nu < w+1$, $\phi_j - (\mathfrak{v}^\nu)^\dagger$ and $\phi_j - (\mathfrak{v}^\nu \mathfrak{m})^\dagger$ are alike. Then $P(y) = 0$ and $y, y', \dots, y^{(r)} \preccurlyeq \mathfrak{m}$ for some $y \prec \mathfrak{v}^w$ in $\mathcal{C}^{<\infty}[\mathrm{i}]$. If $P, Q \in H\{Y\}$, then there is such y in $\mathcal{C}^{<\infty}$.

Proof. Note that $\phi_1, \dots, \phi_r \preccurlyeq \mathfrak{v}^{-1}$ by (1.3), and $R \prec_\Delta \mathfrak{v}^{w+1} Q$ gives $f^{-1} R \prec_\Delta \mathfrak{v}^w$. Take $\nu \in \mathbb{Q}$ such that $w < \nu < w+1$, $f^{-1}R \prec_\Delta \mathfrak{v}^\nu$ and $\nu \mathfrak{v}^\dagger \not\sim \operatorname{Re}\phi_j$ for $j = 1, \dots, r$. Set $A \;:=\; f^{-1} L_Q$. From $\nu < w+1$ and

$$R(0) \;=\; -P(0) \;\prec_\Delta\; \mathfrak{v}^{w+2}\mathfrak{m} Q$$

we obtain $f^{-1}R(0) \prec_\Delta \mathfrak{v}^\nu \mathfrak{m}$. Now apply successively Corollary 9.9, Lemma 6.1, and Corollary 7.5 to the equation $A(y) = f^{-1}R(y)$, $y \prec 1$ in the role of $(*)$ in Section 6 to obtain the first part. For the real variant, use instead Corollary 9.11 and Lemma 6.5. □

Lemma 9.16 with $\mathfrak{m}^{r+1}$ for $\mathfrak{m}$ has the following consequence, using Lemma 9.3:

Corollary 9.17. *Let $f \in K^\times$ and $\phi_1, \dots, \phi_r \in K$ be such that*

$$L_Q \;=\; f(\partial - \phi_1) \cdots (\partial - \phi_r), \qquad \operatorname{Re}\phi_1, \dots, \operatorname{Re}\phi_r \;\succcurlyeq\; 1.$$

Assume $\mathfrak{v} \prec 1$ and $R \prec_\Delta \mathfrak{v}^{w+1} Q$. Let $\mathfrak{m} \in H^\times$, $\mathfrak{m} \prec 1$, $P(0) \preccurlyeq \mathfrak{v}^{w+2}\mathfrak{m}^{r+1} Q$. Suppose that for $j = 1, \dots, r$ and all $\nu \in \mathbb{Q}$ with $w < \nu < w+1$, $\phi_j - (\mathfrak{v}^\nu)^\dagger$ and $\phi_j - (\mathfrak{v}^\nu \mathfrak{m}^{r+1})^\dagger$ are alike. Then for some $y \prec \mathfrak{v}^w$ in $\mathcal{C}^{<\infty}[\mathrm{i}]$ we have $P(y) = 0$ and $y \in \mathfrak{m}\,\mathcal{C}^r[\mathrm{i}]^{\preccurlyeq}$. If $P, Q \in H\{Y\}$, then there is such y in $\mathcal{C}^{<\infty}$.

Remark. If H is a $\mathcal{C}^\infty$-Hardy field, then Lemma 9.16 and Corollary 9.17 go through with $\mathcal{C}^{<\infty}[\mathrm{i}]$, $\mathcal{C}^{<\infty}$ replaced by $\mathcal{C}^\infty[\mathrm{i}]$, $\mathcal{C}^\infty$, respectively. Likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$. (Use Corollary 7.5.)

Next a variant of Lemma 8.6. *In the rest of this subsection $(P,\mathfrak{n},\widehat{h})$ is a deep, strongly repulsive-normal, Z-minimal slot in H of order $r \geqslant 1$ and weight $w := \operatorname{wt}(P)$. We assume also that $(P,\mathfrak{n},\widehat{h})$ is special (as will be the case if H is r-linearly newtonian, and ω-free if $r > 1$, by Lemma 1.7).*

Lemma 9.18. *Let $\mathfrak{m} \in H^\times$ be such that $v\mathfrak{m} \in v(\widehat{h} - H)$, $\mathfrak{m} \prec \mathfrak{n}$, and $P(0) \preccurlyeq \mathfrak{v}(L_{P_{\times\mathfrak{n}}})^{w+2}\,(\mathfrak{m}/\mathfrak{n})^{r+1}\,P_{\times\mathfrak{n}}$. Then for some $y \in \mathcal{C}^{<\infty}$,*

$$P(y) \ = \ 0, \quad y \in \mathfrak{m}\,(\mathcal{C}^r)^{\preccurlyeq}.$$

If $H \subseteq \mathcal{C}^\infty$, then there is such y in $\mathcal{C}^\infty$; likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.

Proof. Replace $(P,\mathfrak{n},\widehat{h})$, $\mathfrak{m}$ by $(P_{\times\mathfrak{n}},1,\widehat{h}/\mathfrak{n})$, $\mathfrak{m}/\mathfrak{n}$, respectively, to arrange $\mathfrak{n} = 1$. Then L_P has order r, $\mathfrak{v}(L_P) \prec^\flat 1$, and $P = Q - R$ where $Q, R \in H\{Y\}$, Q is homogeneous of degree 1 and order r, $L_Q \in H[\partial]$ has a strong $\widehat{h}$-repulsive splitting $(\phi_1,\dots,\phi_r) \in K^r$ over $K = H[\mathrm{i}]$, and $R \prec_{\Delta^*} \mathfrak{v}(L_P)^{w+1}P_1$ with $\Delta^* := \Delta\big(\mathfrak{v}(L_P)\big)$. By Lemma 1.5 we have $\mathfrak{v}(L_P) \sim \mathfrak{v}(L_Q) \asymp \mathfrak{v}$, so $\operatorname{Re}\phi_j \succcurlyeq \mathfrak{v}^\dagger \succcurlyeq 1$ for $j = 1,\dots,r$, and $\Delta = \Delta^*$. Moreover, $P(0) \preccurlyeq \mathfrak{v}^{w+2}\mathfrak{m}^{r+1}Q$. Let $\nu \in \mathbb{Q}$, $\nu > w$, and $j \in \{1,\dots,r\}$. Then $0 < v(\mathfrak{v}^\nu) \in v(\widehat{h} - H)$ by Proposition 1.9, so ϕ_j is γ-repulsive for $\gamma = v(\mathfrak{v}^\nu)$, thus ϕ_j and $\phi_j - (\mathfrak{v}^\nu)^\dagger$ are alike by Lemma 1.15. Likewise, $0 < v(\mathfrak{v}^\nu\mathfrak{m}^{r+1}) \in v(\widehat{h} - H)$ since $\widehat{h}$ is special over H, so ϕ_j and $\phi_j - (\mathfrak{v}^\nu\mathfrak{m}^{r+1})^\dagger$ are alike. Therefore $\phi_j - (\mathfrak{v}^\nu)^\dagger$ and $\phi_j - (\mathfrak{v}^\nu\mathfrak{m}^{r+1})^\dagger$ are alike as well. Corollary 9.17 now gives $y \prec \mathfrak{v}^w$ in $\mathcal{C}^{<\infty}$ with $P(y) = 0$ and $y \in \mathfrak{m}\,(\mathcal{C}^r)^{\preccurlyeq}$. For the rest use the remark following that corollary. □

Corollary 9.19. *Suppose $\mathfrak{n} = 1$, and let $\mathfrak{m} \in H^\times$ be such that $v\mathfrak{m} \in v(\widehat{h} - H)$. Then there are $h \in H$ and $y \in \mathcal{C}^{<\infty}$ such that:*

$$\widehat{h} - h \ \preccurlyeq \ \mathfrak{m}, \qquad P(y) \ = \ 0, \qquad y \ \prec \ 1, \ y \in (\mathcal{C}^r)^{\preccurlyeq}, \qquad y - h \in \mathfrak{m}\,(\mathcal{C}^r)^{\preccurlyeq}.$$

If $H \subseteq \mathcal{C}^\infty$, then we have such $y \in \mathcal{C}^\infty$; likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.

Proof. Suppose first that $\mathfrak{m} \succcurlyeq 1$, and let $h := 0$ and y be as in Lemma 8.6 for $\phi = \mathfrak{n} = 1$. Then $y \prec 1$, $y \in (\mathcal{C}^r)^{\preccurlyeq}$, so $y\mathfrak{m} \prec 1$, $y/\mathfrak{m} \prec 1$, $y/\mathfrak{m} \in (\mathcal{C}^r)^{\preccurlyeq}$ by the Product Rule. Next assume $\mathfrak{m} \prec 1$ and set $\mathfrak{v} := |\mathfrak{v}(L_P)| \in H^>$. By Proposition 1.9 we can take $h \in H$ such that $\widehat{h} - h \prec (\mathfrak{v}\mathfrak{m})^{(w+3)(r+1)}$, and then by Proposition 1.8 we have

$$P_{+h}(0) \ = \ P(h) \ \prec \ (\mathfrak{v}\mathfrak{m})^{w+3}P \ \preccurlyeq \ \mathfrak{v}^{w+3}\mathfrak{m}^{r+1}P_{+h}.$$

By [8, Lemma 4.5.35], $(P_{+h},1,\widehat{h}-h)$ is strongly repulsive-normal, and by [8, Corollary 3.3.8] it is deep with $\mathfrak{v}(L_{P_{+h}}) \asymp_{\Delta(\mathfrak{v})} \mathfrak{v}$. Hence Lemma 9.18 applies to the slot $(P_{+h},1,\widehat{h}-h)$ in place of $(P,1,\widehat{h})$ to yield a $z \in \mathcal{C}^{<\infty}$ with $P_{+h}(z) = 0$ and $(z/\mathfrak{m})^{(j)} \preccurlyeq 1$ for $j = 0,\dots,r$. Lemma 8.1 gives $z^{(j)} \prec 1$ for $j = 0,\dots,r$. Set $y := h + z$; then $P(y) = 0$, $y^{(j)} \prec 1$ and $\big((y-h)/\mathfrak{m}\big)^{(j)} \preccurlyeq 1$ for $j = 0,\dots,r$. □

We now use the results above to approximate zeros of P in $\mathcal{C}^{<\infty}$ by elements of H:

Corollary 9.20. *Suppose H is Liouville closed, $\mathrm{I}(K) \subseteq K^\dagger$, $\mathfrak{n} = 1$, and our slot $(P,1,\widehat{h})$ in H is ultimate. Assume also that K is 1-linearly surjective if $r \geqslant 3$. Let $y \in \mathcal{C}^{<\infty}$ and $h \in H$, $\mathfrak{m} \in H^\times$ be such that $P(y) = 0$, $y \prec 1$, and $\widehat{h} - h \preccurlyeq \mathfrak{m}$. Then $y - h \in \mathfrak{m}\,(\mathcal{C}^r)^{\preccurlyeq}$.*

Proof. Corollary 9.19 gives $h_1 \in H$, $z \in \mathcal{C}^{<\infty}$ with $\widehat{h} - h_1 \preccurlyeq \mathfrak{m}$, $P(z) = 0$, $z \prec 1$, and $\big((z - h_1)/\mathfrak{m}\big)^{(j)} \preccurlyeq 1$ for $j = 0, \dots, r$. Now

$$\frac{y-h}{\mathfrak{m}} \;=\; \frac{y-z}{\mathfrak{m}} + \frac{z-h_1}{\mathfrak{m}} + \frac{h_1-h}{\mathfrak{m}}$$

with $\big((y-z)/\mathfrak{m}\big)^{(j)} \preccurlyeq 1$ for $j = 0, \dots, r$ by Corollary 9.15. Also $(h_1 - h)/\mathfrak{m} \in H$ and $(h_1 - h)/\mathfrak{m} \preccurlyeq 1$, so $\big((h_1 - h)/\mathfrak{m}\big)^{(j)} \preccurlyeq 1$ for all $j \in \mathbb{N}$. □

10. Asymptotic Similarity

Let H be a Hausdorff field and $\widehat{H}$ an immediate valued field extension of H. Equip $\widehat{H}$ with the unique field ordering making it an ordered field extension of H such that $\mathcal{O}_{\widehat{H}}$ is convex [ADH, 3.5.12]. Let $f \in \mathcal{C}$ and $\widehat{f} \in \widehat{H}$ be given.

Definition 10.1. Call f **asymptotically similar** to $\widehat{f}$ over H (notation: $f \sim_H \widehat{f}$) if $f \sim \phi$ in $\mathcal{C}$ and $\phi \sim \widehat{f}$ in $\widehat{H}$ for some $\phi \in H$. (Note that then $f \in \mathcal{C}^\times$ and $\widehat{f} \neq 0$.)

Recall that the binary relations $\sim$ on $\mathcal{C}^\times$ and $\sim$ on $\widehat{H}^\times$ are equivalence relations which restrict to the same equivalence relation on $H^\times$. As a consequence, if $f \sim_H \widehat{f}$, then $f \sim \phi$ in $\mathcal{C}$ for any $\phi \in H$ with $\phi \sim \widehat{f}$ in $\widehat{H}$, and $\phi \sim \widehat{f}$ in $\widehat{H}$ for any $\phi \in H$ with $f \sim \phi$ in $\mathcal{C}$. Moreover, if $f \in H$, then $f \sim_H \widehat{f} \Leftrightarrow f \sim \widehat{f}$ in $\widehat{H}$, and if $\widehat{f} \in H$, then $f \sim_H \widehat{f} \Leftrightarrow f \sim \widehat{f}$ in $\mathcal{C}$.

Lemma 10.2. *Let $f_1 \in \mathcal{C}$, $f_1 \sim f$, let $\widehat{f_1} \in \widehat{H}_1$ for an immediate valued field extension $\widehat{H}_1$ of H, and suppose $\widehat{f} \sim \theta$ in $\widehat{H}$ and $\widehat{f_1} \sim \theta$ in $\widehat{H}_1$ for some $\theta \in H$. Then: $f \sim_H \widehat{f} \Leftrightarrow f_1 \sim_H \widehat{f_1}$.*

For $\mathfrak{n} \in H^\times$ we have $f \sim_H \widehat{f} \Leftrightarrow \mathfrak{n} f \sim_H \mathfrak{n}\widehat{f}$. Moreover, by [9, Lemma 2.1]:

Lemma 10.3. *Let $g \in \mathcal{C}$, $\widehat{g} \in \widehat{H}$, and suppose $f \sim_H \widehat{f}$ and $g \sim_H \widehat{g}$. Then $1/f \sim_H 1/\widehat{f}$ and $fg \sim_H \widehat{f}\,\widehat{g}$. Moreover,*

$$f \preccurlyeq g \text{ in } \mathcal{C} \quad\Longleftrightarrow\quad \widehat{f} \preccurlyeq \widehat{g} \text{ in } \widehat{H},$$

and likewise with $\prec$, $\asymp$, or $\sim$ in place of $\preccurlyeq$.

Lemma 10.3 readily yields:

Corollary 10.4. *Suppose $\widehat{f}$ is transcendental over H and $Q(f) \sim_H Q(\widehat{f})$ for all $Q \in H[Y]^{\neq}$. Then we have:*

- (i) *a subfield $H(f) \supseteq H$ of $\mathcal{C}$ generated by f over H;*
- (ii) *a field isomorphism $\iota\colon H(f) \to H(\widehat{f})$ over H with $\iota(f) = \widehat{f}$;*
- (iii) *with $H(f)$ and ι as in* (i) *and* (ii) *we have $g \sim_H \iota(g)$ for all $g \in H(f)^\times$, hence for all $g_1, g_2 \in H(f)$: $g_1 \preccurlyeq g_2$ in $\mathcal{C} \Leftrightarrow \iota(g_1) \preccurlyeq \iota(g_2)$ in $\widehat{H}$.*

Also, ι in (ii) *is unique and is an ordered field isomorphism, where the ordering on $H(f)$ is its ordering as a Hausdorff field.*

Proof. To see that ι is order preserving, use that ι is a valued field isomorphism by (iii), and apply [ADH, 3.5.12]. □

Here is the analogue when $\widehat{f}$ is algebraic over H:

Corollary 10.5. *Suppose $\widehat{f}$ is algebraic over H with minimum polynomial P over H of degree d, and $P(f) = 0$, $Q(f) \sim_H Q(\widehat{f})$ for all $Q \in H[Y]^{\neq}$ of degree $< d$. Then we have:*

(i) *a subfield $H[f] \supseteq H$ of $\mathcal{C}$ generated by f over H;*
(ii) *a field isomorphism $\iota \colon H[f] \to H[\widehat{f}]$ over H with $\iota(f) = \widehat{f}$;*
(iii) *with $H[f]$ and ι as in* (i) *and* (ii) *we have $g \sim_H \iota(g)$ for all $g \in H[f]^{\times}$, hence for all $g_1, g_2 \in H[f]$: $g_1 \preccurlyeq g_2$ in $\mathcal{C} \Leftrightarrow \iota(g_1) \preccurlyeq \iota(g_2)$ in $\widehat{H}$.*

Also, $H[f]$ and ι in (i) *and* (ii) *are unique and ι is an ordered field isomorphism, where the ordering on $H(f)$ is its ordering as a Hausdorff field.*

If $\widehat{f} \notin H$, then to show that $f - \phi \sim_H \widehat{f} - \phi$ for all $\phi \in H$ it is enough to do this for ϕ arbitrarily close to $\widehat{f}$:

Lemma 10.6. *Let $\phi_0 \in H$ be such that $f - \phi_0 \sim_H \widehat{f} - \phi_0$. Then $f - \phi \sim_H \widehat{f} - \phi$ for all $\phi \in H$ with $\widehat{f} - \phi_0 \prec \widehat{f} - \phi$.*

Proof. Let $\phi \in H$ with $\widehat{f} - \phi_0 \prec \widehat{f} - \phi$. Then $\phi_0 - \phi \succ \widehat{f} - \phi_0$, so $\widehat{f} - \phi = (\widehat{f} - \phi_0) + (\phi_0 - \phi) \sim \phi_0 - \phi$. By Lemma 10.3 we also have $\phi_0 - \phi \succ f - \phi_0$, and hence likewise $f - \phi \sim \phi_0 - \phi$. □

We define: $f \approx_H \widehat{f} :\Leftrightarrow f - \phi \sim_H \widehat{f} - \phi$ for all $\phi \in H$. If $f \approx_H \widehat{f}$, then $f \sim_H \widehat{f}$ as well as $f, \widehat{f} \notin H$, and $\mathfrak{n} f \approx_H \mathfrak{n} \widehat{f}$ for all $\mathfrak{n} \in H^{\times}$. Hence $f \approx_H \widehat{f}$ iff $f, \widehat{f} \notin H$ and the isomorphism $\iota \colon H + Hf \to H + H\widehat{f}$ of H-linear spaces that is the identity on H and sends f to $\widehat{f}$ satisfies $g \sim_H \iota(g)$ for all nonzero $g \in H + Hf$.

Here is an easy consequence of Lemma 10.6:

Corollary 10.7. *Suppose $\widehat{f} \notin H$ and $f - \phi_0 \sim_H \widehat{f} - \phi_0$ for all $\phi_0 \in H$ such that $\phi_0 \sim \widehat{f}$. Then $f \approx_H \widehat{f}$.*

Proof. Take $\phi_0 \in H$ with $\phi_0 \sim \widehat{f}$, and let $\phi \in H$ be given. If $\widehat{f} - \phi \prec \widehat{f}$, then $f - \phi \sim_H \widehat{f} - \phi$ by hypothesis; otherwise we have $\widehat{f} - \phi \succcurlyeq \widehat{f} \succ \widehat{f} - \phi_0$, and then $f - \phi \sim_H \widehat{f} - \phi$ by Lemma 10.6. □

Lemma 10.2 yields an analogue for $\approx_H$:

Lemma 10.8. *Let $f_1 \in \mathcal{C}$ be such that $f_1 - \phi \sim f - \phi$ for all $\phi \in H$, and let $\widehat{f_1}$ be an element of an immediate valued field extension of H such that $v(\widehat{f} - \phi) = v(\widehat{f_1} - \phi)$ for all $\phi \in H$. Then $f \approx_H \widehat{f}$ iff $f_1 \approx_H \widehat{f_1}$.*

Let $g \in \mathcal{C}$ be eventually strictly increasing with $g(t) \to +\infty$ as $t \to +\infty$; we then have the Hausdorff field $H \circ g = \{h \circ g : h \in H\}$, with ordered valued field isomorphism $h \mapsto h \circ g \colon H \to H \circ g$. (See [9, Section 2].) Suppose

$$\widehat{h} \mapsto \widehat{h} \circ g \; : \; \widehat{H} \to \widehat{H} \circ g$$

extends this isomorphism to a valued field isomorphism, where $\widehat{H} \circ g$ is an immediate valued field extension of the Hausdorff field $H \circ g$. Then

$$f \sim_H \widehat{f} \quad \Longleftrightarrow \quad f \circ g \sim_{H \circ g} \widehat{f} \circ g, \qquad f \approx_H \widehat{f} \quad \Longleftrightarrow \quad f \circ g \approx_{H \circ g} \widehat{f} \circ g.$$

The complex version. We now assume in addition that H is real closed, with algebraic closure $K := H[i] \subseteq \mathcal{C}[i]$. We take i with $i^2 = -1$ also as an element of a field $\widehat{K} := \widehat{H}[i]$ extending both $\widehat{H}$ and K, and equip $\widehat{K}$ with the unique valuation ring of $\widehat{K}$ lying over $\mathcal{O}_{\widehat{H}}$. (See [8, remarks following Lemma 4.1.2].) Then $\widehat{K}$ is an immediate valued field extension of K. Let $f \in \mathcal{C}[i]$ and $\widehat{f} \in \widehat{K}$ below.

Call f **asymptotically similar** to $\widehat{f}$ over K (notation: $f \sim_K \widehat{f}$) if for some $\phi \in K$ we have $f \sim \phi$ in $\mathcal{C}[i]$ and $\phi \sim \widehat{f}$ in $\widehat{K}$. Then $f \in \mathcal{C}[i]^\times$ and $\widehat{f} \neq 0$. As before, if $f \sim_K \widehat{f}$, then $f \sim \phi$ in $\mathcal{C}[i]$ for any $\phi \in K$ for which $\phi \sim \widehat{f}$ in $\widehat{K}$, and $\phi \sim \widehat{f}$ in $\widehat{K}$ for any $\phi \in K$ for which $f \sim \phi$ in $\mathcal{C}[i]$. Moreover, if $f \in K$, then $f \sim_K \widehat{f}$ reduces to $f \sim \widehat{f}$ in $\widehat{K}^\times$. Likewise, if $\widehat{f} \in K$, then $f \sim_K \widehat{f}$ reduces to $f \sim \widehat{f}$ in $\mathcal{C}[i]^\times$.

Lemma 10.9. *Let $f_1 \in \mathcal{C}[i]$ with $f_1 \sim f$. Let $\widehat{H}_1$ be an immediate valued field extension of H, let $\widehat{K}_1 := \widehat{H}_1[i]$ be the corresponding immediate valued field extension of K obtained from $\widehat{H}_1$ as $\widehat{K}$ was obtained from $\widehat{H}$. Let $\widehat{f}_1 \in \widehat{K}_1$, and $\theta \in K$ be such that $\widehat{f} \sim \theta$ in $\widehat{K}$ and $\widehat{f}_1 \sim \theta$ in $\widehat{K}_1$. Then $f \sim_K \widehat{f}$ iff $f_1 \sim_K \widehat{f}_1$.*

For $\mathfrak{n} \in K^\times$ we have $f \sim_K \widehat{f} \Leftrightarrow \mathfrak{n} f \sim_K \mathfrak{n}\widehat{f}$, and $f \sim_K \widehat{f} \Leftrightarrow \overline{f} \sim_K \overline{\widehat{f}}$ (complex conjugation). Here is a useful observation relating $\sim_K$ and $\sim_H$:

Lemma 10.10. *Suppose $f \sim_K \widehat{f}$ and $\operatorname{Re} \widehat{f} \succcurlyeq \operatorname{Im} \widehat{f}$; then*

$$\operatorname{Re} f \ \succcurlyeq\ \operatorname{Im} f, \qquad \operatorname{Re} f \ \sim_H \ \operatorname{Re} \widehat{f}.$$

Proof. Let $\phi \in K$ be such that $f \sim \phi$ in $\mathcal{C}[i]$ and $\phi \sim \widehat{f}$ in $\widehat{K}$. The latter yields $\operatorname{Re}\phi \succcurlyeq \operatorname{Im}\phi$ in H and $\operatorname{Re}\phi \sim \operatorname{Re}\widehat{f}$ in $\widehat{H}$. Using that $f = (1+\varepsilon)\phi$ with $\varepsilon \prec 1$ in $\mathcal{C}[i]$ it follows easily that $\operatorname{Re} f \succcurlyeq \operatorname{Im} f$ and $\operatorname{Re} f \sim \operatorname{Re}\phi$ in $\mathcal{C}$. □

Corollary 10.11. *Suppose $f \in \mathcal{C}$ and $\widehat{f} \in \widehat{H}$. Then $f \sim_H \widehat{f}$ iff $f \sim_K \widehat{f}$.*

Lemmas 10.3 and 10.6 go through with $\mathcal{C}[i]$, K, $\widehat{K}$, and $\sim_K$ in place of $\mathcal{C}$, H, $\widehat{H}$, and $\sim_H$. We define: $f \approx_K \widehat{f} :\Leftrightarrow f - \phi \sim_K \widehat{f} - \phi$ for all $\phi \in K$. Now Corollary 10.7 goes through with K, $\sim_K$, $\approx_K$ in place of H, $\sim_H$, $\approx_H$.

Lemma 10.12. *Suppose $f \in \mathcal{C}$, $\widehat{f} \in \widehat{H}$, and $f \sim_H \widehat{f}$. Then $f + gi \sim_K \widehat{f} + gi$ for all $g \in H$.*

Proof. Let $g \in H$, and take $\phi \in H$ with $f \sim \phi$ in $\mathcal{C}$ and $\phi \sim \widehat{f}$ in $\widehat{H}$. Suppose first that $g \prec \phi$. Then $gi \prec \phi$, and together with $f - \phi \prec \phi$ this yields $(f + gi) - \phi \prec \phi$, that is, $f + gi \sim \phi$ in $\mathcal{C}[i]$. Using likewise analogous properties of $\prec$ on $\widehat{K}$ we obtain $\phi \sim \widehat{f} + gi$ in $\widehat{K}$. If $\phi \prec g$, then $f \preccurlyeq \phi \prec gi$ and thus $f + gi \sim gi$ in $\mathcal{C}[i]$, and likewise $\widehat{f} + gi \sim gi$ in $\widehat{K}$. Finally, suppose $g \asymp \phi$. Take $c \in \mathbb{R}^\times$ and $\varepsilon \in H$ with $g = c\phi(1+\varepsilon)$ and $\varepsilon \prec 1$. We have $f = \phi(1+\delta)$ where $\delta \in \mathcal{C}$, $\delta \prec 1$, so $f + gi = \phi(1+ci)(1+\rho)$ where $\rho = (1+ci)^{-1}(\delta + ci\varepsilon) \prec 1$ in $\mathcal{C}[i]$, so $f + gi \sim \phi(1+ci)$ in $\mathcal{C}[i]$. Likewise, $\widehat{f} + gi \sim \phi(1+ci)$ in $\widehat{K}$. □

Corollary 10.13. *Suppose $f \in \mathcal{C}$ and $\widehat{f} \in \widehat{H}$. Then $f \approx_H \widehat{f}$ iff $f \approx_K \widehat{f}$.*

Proof. If $f \approx_K \widehat{f}$, then for all $\phi \in H$ we have $f - \phi \sim_K \widehat{f} - \phi$, so $f - \phi \sim_H \widehat{f} - \phi$ by Corollary 10.11, hence $f \approx_H \widehat{f}$. Conversely, suppose $f \approx_H \widehat{f}$. Then for all $\phi \in K$ we have $f - \operatorname{Re}\phi \sim_H \widehat{f} - \operatorname{Re}\phi$, so $f - \phi \sim_K \widehat{f} - \phi$ by Lemma 10.12. □

Next we exploit that K is algebraically closed:

Lemma 10.14. *$f \approx_K \widehat{f} \implies Q(f) \sim_K Q(\widehat{f})$ for all $Q \in K[Y]^{\neq}$.*

Proof. Factor $Q \in K[Y]^{\neq}$ as

$$Q = a(Y - \phi_1)\cdots(Y - \phi_n) \qquad \text{where } a \in K^{\times} \text{ and } \phi_1, \dots, \phi_n \in K,$$

and use $f - \phi_j \sim_K \widehat{f} - \phi_j$ ($j = 1, \dots, n$) and the complex version of Lemma 10.3. □

This yields a more useful "complex" version of Corollary 10.4:

Corollary 10.15. *Suppose $f \approx_K \widehat{f}$. Then $\widehat{f}$ is transcendental over K, and:*

(i) *f generates over K a subfield $K(f)$ of $\mathcal{C}[i]$;*
(ii) *we have a field isomorphism $\iota\colon K(f) \to K(\widehat{f})$ over K with $\iota(f) = \widehat{f}$;*
(iii) *$g \sim_K \iota(g)$ for all $g \in K(f)^{\times}$, hence for all $g_1, g_2 \in K(f)$:*

$$g_1 \preccurlyeq g_2 \text{ in } \mathcal{C}[i] \iff \iota(g_1) \preccurlyeq \iota(g_2) \text{ in } \widehat{K}.$$

(*Thus the restriction of the binary relation $\preccurlyeq$ on $\mathcal{C}[i]$ to $K(f)$ is a dominance relation on the field $K(f)$ in the sense of* [ADH, 3.1.1].)

In the next lemma $f = g + hi$, $g, h \in \mathcal{C}$, and $\widehat{f} = \widehat{g} + \widehat{h}i$, $\widehat{g}, \widehat{h} \in \widehat{H}$. Recall from [8, Lemma 4.1.3] that if $\widehat{f} \notin K$, then $v(\widehat{g} - H) \subseteq v(\widehat{h} - H)$ or $v(\widehat{h} - H) \subseteq v(\widehat{g} - H)$.

Lemma 10.16. *Suppose $f \approx_K \widehat{f}$ and $v(\widehat{g} - H) \subseteq v(\widehat{h} - H)$. Then $g \approx_H \widehat{g}$.*

Proof. Let $\rho \in H$ with $\rho \sim \widehat{g}$; by Corollary 10.7 it is enough to show that $g - \rho \sim_H \widehat{g} - \rho$. Take $\sigma \in H$ with $\widehat{g} - \rho \succcurlyeq \widehat{h} - \sigma$, and set $\phi := \rho + \sigma i \in K$. Then

$$\operatorname{Re}(f - \phi) = g - \rho \quad \text{and} \quad \operatorname{Re}(\widehat{f} - \phi) = \widehat{g} - \rho \succcurlyeq \widehat{h} - \sigma = \operatorname{Im}(\widehat{f} - \phi),$$

and so by $f - \phi \sim_H \widehat{f} - \phi$ and Lemma 10.10 we have $g - \rho \sim_H \widehat{g} - \rho$. □

Corollary 10.17. *If $f \approx_K \widehat{f}$, then $\operatorname{Re} f \approx_H \operatorname{Re} \widehat{f}$ or $\operatorname{Im} f \approx_H \operatorname{Im} \widehat{f}$.*

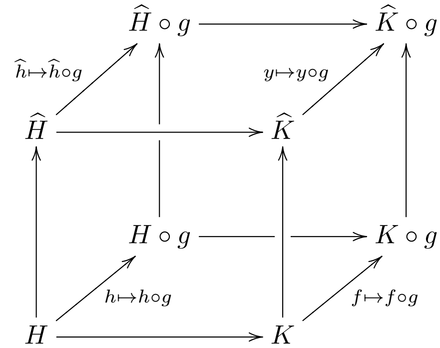
Let $g \in \mathcal{C}$ be eventually strictly increasing with $g(t) \to +\infty$ as $t \to +\infty$; we then have the subfield $K \circ g = (H \circ g)[i]$ of $\mathcal{C}[i]$. Suppose the valued field isomorphism

$$h \mapsto h \circ g\colon H \to H \circ g$$

is extended to a valued field isomorphism

$$\widehat{h} \mapsto \widehat{h} \circ g\colon \widehat{H} \to \widehat{H} \circ g,$$

where $\widehat{H} \circ g$ is an immediate valued field extension of the Hausdorff field $H \circ g$. In the same way we took a common valued field extension $\widehat{K} = \widehat{H}[i]$ of $\widehat{H}$ and $K = H[i]$ we now take a common valued field extension $\widehat{K} \circ g = (\widehat{H} \circ g)[i]$ of $\widehat{H} \circ g$ and $K \circ g = (H \circ g)[i]$. This makes $\widehat{K} \circ g$ an immediate extension of $K \circ g$, and we have a unique valued field isomorphism $y \mapsto y \circ g\colon \widehat{K} \to \widehat{K} \circ g$ extending the above map $\widehat{h} \mapsto \widehat{h} \circ g\colon \widehat{H} \to \widehat{H} \circ g$ and sending $i \in \widehat{K}$ to $i \in \widehat{K} \circ g$. This map $\widehat{K} \to \widehat{K} \circ g$ also extends $f \mapsto f \circ g\colon K \to K \circ g$ and is the identity on $\mathbb{C}$. See the commutative diagram below, where the labeled arrows are valued field isomorphisms and all unlabeled arrows are natural inclusions.



Now we have

$$f \sim_K \widehat{f} \quad \Longleftrightarrow \quad f \circ g \sim_{K\circ g} \widehat{f} \circ g, \qquad f \asymp_K \widehat{f} \quad \Longleftrightarrow \quad f \circ g \asymp_{K\circ g} \widehat{f} \circ g.$$

At various places in the next section we use this for a Hardy field H and active $\phi > 0$ in H, with $g = \ell^{\text{inv}}$, $\ell \in \mathcal{C}^1$, $\ell' = \phi$. In that situation, $H^\circ := H \circ g$, $\widehat{H}^\circ := \widehat{H} \circ g$, and $h^\circ := h \circ g$, $\widehat{h}^\circ := \widehat{h} \circ g$ for $h \in H$ and $\widehat{h} \in \widehat{H}$, and likewise with K and $\widehat{K}$ and their elements instead of H and $\widehat{H}$.

11. Differentially Algebraic Hardy Field Extensions

In this section we are finally able to generate under reasonable conditions Hardy field extensions by solutions in $\mathcal{C}^{<\infty}$ of algebraic differential equations, culminating in the proof of our main theorem. We begin with a generality about enlarging differential fields within an ambient differential ring. Here, a *differential subfield* of a differential ring E is a differential subring of E whose underlying ring is a field.

Lemma 11.1. *Let K be a differential field with irreducible $P \in K\{Y\}^{\neq}$ of order $r \geqslant 1$, and E a differential ring extension of K with $y \in E$ such that $P(y) = 0$ and $Q(y) \in E^\times$ for all $Q \in K\{Y\}^{\neq}$ of order $< r$. Then y generates over K a differential subfield $K\langle y\rangle \supseteq K$ of E. Moreover, y has P as a minimal annihilator over K and $K\langle y\rangle$ equals*

$$\left\{ \frac{A(y)}{B(y)} :\ A, B \in K\{Y\}, \text{order}\, A \leqslant r,\, \deg_{Y^{(r)}} A < \deg_{Y^{(r)}} P,\, B \neq 0,\, \text{order}\, B < r \right\}.$$

Proof. Let $p \in K[Y_0, \dots, Y_r]$ with distinct indeterminates $Y_0, \dots, Y_r$ be such that $P(Y) = p(Y, Y', \dots, Y^{(r)})$. The K-algebra morphism $K[Y_0, \dots, Y_r] \to E$ sending Y_i to $y^{(i)}$ for $i = 0, \dots, r$ extends to a K-algebra morphism $K(Y_0, \dots, Y_{r-1})[Y_r] \to E$ with p in its kernel, and so induces a K-algebra morphism

$$\iota \ :\ K(Y_0, \dots, Y_{r-1})[Y_r]/(p) \to E, \qquad (p) \ :=\ pK(Y_0, \dots, Y_{r-1})[Y_r].$$

Now p as an element of $K(Y_0, \dots, Y_{r-1})[Y_r]$ remains irreducible [60, Chapter IV, §2]. Thus $K(Y_0, \dots, Y_{r-1})[Y_r]/(p)$ is a field, so ι is injective, and it is routine to check that the image of ι is $K\langle y\rangle$ as described; see also [ADH, 4.1.6]. □

In the rest of this section H is a real closed Hardy field with $H \supseteq \mathbb{R}$, and $\widehat{H}$ is an immediate H-field extension of H.

Application to Hardy fields. Let $f \in \mathcal{C}^{<\infty}$ and $\widehat{f} \in \widehat{H}$. Note that if $Q \in H\{Y\}$ and $Q(f) \sim_H Q(\widehat{f})$, then $Q(f) \in \mathcal{C}^\times$. So by Lemma 11.1 with $E = \mathcal{C}^{<\infty}$, $K = H$:

Lemma 11.2. *Suppose $\widehat{f}$ is* d*-algebraic over H with minimal annihilator P over H of order $r \geqslant 1$, and $P(f) = 0$ and $Q(f) \sim_H Q(\widehat{f})$ for all $Q \in H\{Y\} \setminus H$ with* order $Q < r$. *Then $f \notin H$ and:*

(i) *f is hardian over H;*
(ii) *we have a (necessarily unique) isomorphism $\iota\colon H\langle f\rangle \to H\langle \widehat{f}\rangle$ of differential fields over H such that $\iota(f) = \widehat{f}$.*

With an extra assumption ι in Lemma 11.2 is an isomorphism of H-fields:

Corollary 11.3. *Let $\widehat{f}$, f, P, r, ι be as in Lemma 11.2, and suppose also that $Q(f) \sim_H Q(\widehat{f})$ for all $Q \in H\{Y\}$ with* order $Q = r$ *and* $\deg_{Y^{(r)}} Q < \deg_{Y^{(r)}} P$. *Then $g \sim_H \iota(g)$ for all $g \in H\langle f\rangle^\times$, hence for $g_1, g_2 \in H\langle f\rangle$ we have*

$$g_1 \preccurlyeq g_2 \text{ in } \mathcal{C} \iff \iota(g_1) \preccurlyeq \iota(g_2) \text{ in } \widehat{H}.$$

Moreover, ι is an isomorphism of H-fields.

Proof. Most of this follows from Lemmas 10.3 and 11.2 and the description of $H\langle f\rangle$ in Lemma 11.1. For the last statement, use [ADH, 10.5.8]. □

Analogues for K. We have the d-valued extension $K := H[\mathrm{i}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$ of H. As before we arrange that $\widehat{K} = \widehat{H}[\mathrm{i}]$ is a d-valued extension of $\widehat{H}$ as well as an an immediate extension of K. Let $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$ and $\widehat{f} \in \widehat{K}$. We now have the obvious "complex" analogues of Lemma 11.2 and Corollary 11.3:

Lemma 11.4. *Suppose $\widehat{f}$ is* d*-algebraic over K with minimal annihilator P over K of order $r \geqslant 1$, and $P(f) = 0$ and $Q(f) \sim_K Q(\widehat{f})$ for all $Q \in K\{Y\} \setminus K$ with* order $Q < r$. *Then*

(i) *f generates over K a differential subfield $K\langle f\rangle$ of $\mathcal{C}^{<\infty}[\mathrm{i}]$;*
(ii) *we have a (necessarily unique) isomorphism $\iota\colon K\langle f\rangle \to K\langle \widehat{f}\rangle$ of differential fields over K such that $\iota(f) = \widehat{f}$.*

Corollary 11.5. *Let $\widehat{f}$, f, P, r, ι be as in Lemma 11.4, and suppose also that $Q(f) \sim_K Q(\widehat{f})$ for all $Q \in K\{Y\}$ with* order $Q = r$ *and* $\deg_{Y^{(r)}} Q < \deg_{Y^{(r)}} P$. *Then $g \sim_K \iota(g)$ for all $g \in K\langle f\rangle^\times$, so for all $g_1, g_2 \in K\langle f\rangle$ we have:*

$$g_1 \preccurlyeq g_2 \text{ in } \mathcal{C}[\mathrm{i}] \iff \iota(g_1) \preccurlyeq \iota(g_2) \text{ in } \widehat{K}.$$

Thus the relation $\preccurlyeq$ on $\mathcal{C}[\mathrm{i}]$ restricts to a dominance relation on the field $K\langle f\rangle$.

From K being algebraically closed we obtain a useful variant of Corollary 11.5:

Corollary 11.6. *Suppose $f \approx_K \widehat{f}$, and $P \in K\{Y\}$ is irreducible with*

$$\operatorname{order} P = \deg_{Y'} P = 1, \quad P(f) = 0 \text{ in } \mathcal{C}^{<\infty}[\mathrm{i}], \quad P(\widehat{f}) = 0 \text{ in } \widehat{K}.$$

Then P is a minimal annihilator of $\widehat{f}$ over K, f generates over K a differential subfield $K\langle f\rangle = K(f)$ of $\mathcal{C}^{<\infty}[\mathrm{i}]$, and we have an isomorphism $\iota\colon K\langle f\rangle \to K\langle \widehat{f}\rangle$ of differential fields over K such that $\iota(f) = \widehat{f}$ and $g \sim_K \iota(g)$ for all $g \in K\langle f\rangle^\times$. Thus for all $g_1, g_2 \in K\langle f\rangle$: $g_1 \preccurlyeq g_2$ in $\mathcal{C}[\mathrm{i}] \iff \iota(g_1) \preccurlyeq \iota(g_2)$ in $\widehat{K}$.

Proof. By Corollary 10.15, $\widehat{f}$ is transcendental over K, so P is a minimal annihilator of $\widehat{f}$ over K by [ADH, 4.1.6]. Now use Lemma 11.1 and Corollary 10.15. ☐

This corollary leaves open whether $\operatorname{Re} f$ or $\operatorname{Im} f$ is hardian over H. This issue is critical for us and we treat a special case in Proposition 11.15 below. The example following Corollary 5.24 in [9] shows that that there is a differential subfield of $\mathcal{C}^{<\infty}[i]$ such that the binary relation $\preccurlyeq$ on $\mathcal{C}[i]$ restricts to a dominance relation on it, but which is not contained in $F[i]$ for any Hardy field F.

Sufficient conditions for asymptotic similarity. Let $\widehat{h}$ be an element of our immediate H-field extension $\widehat{H}$ of H. Note that in the next variant of [ADH, 11.4.3] we use ddeg instead of ndeg.

Lemma 11.7. *Let $Q \in H\{Y\}^{\neq}$, $r := \operatorname{order} Q$, $h \in H$, and $\mathfrak{v} \in H^{\times}$ be such that $\widehat{h} - h \prec \mathfrak{v}$ and $\operatorname{ddeg}_{\prec\mathfrak{v}} Q_{+h} = 0$. Assume $y \in \mathcal{C}^{<\infty}$ and $\mathfrak{m} \in H^{\times}$ satisfy*

$$y - h \preccurlyeq \mathfrak{m} \prec \mathfrak{v}, \qquad \left(\frac{y-h}{\mathfrak{m}}\right)', \dots, \left(\frac{y-h}{\mathfrak{m}}\right)^{(r)} \preccurlyeq 1.$$

Then $Q(y) \sim Q(h)$ in $\mathcal{C}^{<\infty}$ and $Q(h) \sim Q(\widehat{h})$ in $\widehat{H}$; in particular, $Q(y) \sim_H Q(\widehat{h})$.

Proof. We have $y = h + \mathfrak{m}u$ with $u = \frac{y-h}{\mathfrak{m}} \in \mathcal{C}^{<\infty}$ and $u, u', \dots, u^{(r)} \preccurlyeq 1$. Now

$$Q_{+h,\times\mathfrak{m}} \;=\; Q(h) + R \qquad \text{with } R \in H\{Y\},\ R(0) = 0,$$

which in view of $\operatorname{ddeg} Q_{+h,\times\mathfrak{m}} = 0$ gives $R \prec Q(h)$. Thus

$$Q(y) \;=\; Q_{+h,\times\mathfrak{m}}(u) \;=\; Q(h) + R(u), \qquad R(u) \;\preccurlyeq\; R \;\prec\; Q(h),$$

so $Q(y) \sim Q(h)$ in $\mathcal{C}^{<\infty}$. Increasing $|\mathfrak{m}|$ if necessary we arrange $\widehat{h} - h \preccurlyeq \mathfrak{m}$, and then a similar computation with $\widehat{h}$ instead of y gives $Q(h) \sim Q(\widehat{h})$ in $\widehat{H}$. ☐

In the remainder of this subsection we assume that H is ungrounded and $H \neq \mathbb{R}$.

Corollary 11.8. *Suppose $\widehat{h}$ is* d*-algebraic over H with minimal annihilator P over H of order $r \geqslant 1$, and let $y \in \mathcal{C}^{<\infty}$ satisfy $P(y) = 0$. Suppose for all Q in $H\{Y\} \setminus H$ of order $< r$ there are $h \in H$, $\mathfrak{m}, \mathfrak{v} \in H^{\times}$, and an active $\phi > 0$ in H such that $\widehat{h} - h \preccurlyeq \mathfrak{m} \prec \mathfrak{v}$, $\operatorname{ddeg}_{\prec\mathfrak{v}} Q^{\phi}_{+h} = 0$, and*

$$\mathfrak{d}^j\left(\frac{y-h}{\mathfrak{m}}\right) \preccurlyeq 1 \qquad \textit{for } j = 0, \dots, r-1 \textit{ and } \mathfrak{d} := \phi^{-1}\partial.$$

Then $y \notin H$ and y is hardian over H.

Proof. Let $Q \in H\{Y\} \setminus H$ have order $< r$, and take h, $\mathfrak{m}$, $\mathfrak{v}$, ϕ as in the statement of the corollary. By Lemma 11.2 it is enough to show that then $Q(y) \sim_H Q(\widehat{h})$. We use $(\ \ )^{\circ}$ as explained at the beginning of Section 8. Thus we have the Hardy field H° and the H-field isomorphism $h \mapsto h^{\circ} \colon H^{\phi} \to H^{\circ}$, extended to an H-field isomorphism $\widehat{f} \mapsto \widehat{f}^{\circ} \colon \widehat{H}^{\phi} \to \widehat{H}^{\circ}$, for an immediate H-field extension $\widehat{H}^{\circ}$ of H°. Set $u := (y-h)/\mathfrak{m} \in \mathcal{C}^{<\infty}$. We have $\operatorname{ddeg}_{\prec\mathfrak{v}^{\circ}} Q^{\phi\circ}_{+h^{\circ}} = 0$ and $(u^{\circ})^{(j)} \preccurlyeq 1$ for $j = 0, \dots, r-1$; hence $Q^{\phi\circ}(y^{\circ}) \sim_{H^{\circ}} Q^{\phi\circ}(\widehat{h}^{\circ})$ by Lemma 11.7. Now $Q^{\phi\circ}(y^{\circ}) = Q(y)^{\circ}$ in $\mathcal{C}^{<\infty}$ and $Q^{\phi\circ}(\widehat{h}^{\circ}) = Q(\widehat{h})^{\circ}$ in $\widehat{H}^{\circ}$, hence $Q(y) \sim_H Q(\widehat{h})$. ☐

Using Corollary 11.3 instead of Lemma 11.2 we show likewise:

Corollary 11.9. *Suppose $\widehat{h}$ is* d*-algebraic over H with minimal annihilator P over H of order $r \geqslant 1$, and let $y \in \mathcal{C}^{<\infty}$ satisfy $P(y) = 0$. Suppose that for all Q in $H\{Y\} \setminus H$ with* $\operatorname{order} Q \leqslant r$ *and* $\deg_{Y^{(r)}} Q < \deg_{Y^{(r)}} P$ *there are $h \in H$, $\mathfrak{m}, \mathfrak{v} \in H^\times$, and an active $\phi > 0$ in H with $\widehat{h} - h \preccurlyeq \mathfrak{m} \prec \mathfrak{v}$,* $\operatorname{ddeg}_{\prec\mathfrak{v}} Q^{\phi}_{+h} = 0$*, and*

$$\delta^j\left(\frac{y-h}{\mathfrak{m}}\right) \preccurlyeq 1 \qquad \text{for } j = 0,\dots,r \text{ and } \delta := \phi^{-1}\partial.$$

Then y is hardian over H and there is an isomorphism $H\langle y\rangle \to H\langle\widehat{h}\rangle$ of H-fields over H sending y to $\widehat{h}$.

In the next subsection we use Corollary 11.9 to fill in certain kinds of holes in Hardy fields.

Generating immediate d**-algebraic Hardy field extensions.** *In this subsection H is Liouville closed, $(P, \mathfrak{n}, \widehat{h})$ is a special Z-minimal slot in H of order $r \geqslant 1$, $K := H[\mathrm{i}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$, $\mathrm{I}(K) \subseteq K^\dagger$, and K is 1-linearly surjective if $r \geqslant 3$.* We first treat the case where $(P, \mathfrak{n}, \widehat{h})$ is a hole in H (not just a slot):

Theorem 11.10. *Assume $(P, \mathfrak{n}, \widehat{h})$ is a deep, ultimate, and strongly repulsive-normal hole in H, and $y \in \mathcal{C}^{<\infty}$, $P(y) = 0$, $y \prec \mathfrak{n}$. Then y is hardian over H, and there is an isomorphism $H\langle y\rangle \to H\langle\widehat{h}\rangle$ of H-fields over H sending y to $\widehat{h}$.*

Proof. Replacing $(P, \mathfrak{n}, \widehat{h})$, y by $(P_{\times\mathfrak{n}}, 1, \widehat{h}/\mathfrak{n})$, $y/\mathfrak{n}$ we arrange $\mathfrak{n} = 1$. Let Q in $H\{Y\} \setminus H$, $\operatorname{order} Q \leqslant r$, and $\deg_{Y^{(r)}} Q < \deg_{Y^{(r)}} P$. Then $Q \notin Z(H, \widehat{h})$, so we have $h \in H$ and $\mathfrak{v} \in H^\times$ such that $h - \widehat{h} \prec \mathfrak{v}$ and $\operatorname{ndeg}_{\prec\mathfrak{v}} Q_{+h} = 0$. Take any $\mathfrak{m} \in H^\times$ with $\widehat{h} - h \preccurlyeq \mathfrak{m} \prec \mathfrak{v}$. Take $\mathfrak{w} \in H^\times$ with $\mathfrak{m} \prec \mathfrak{w} \prec \mathfrak{v}$. Then $\operatorname{ndeg} Q_{+h,\times\mathfrak{w}} = 0$, so we have active ϕ in H, $0 < \phi \prec 1$, with $\operatorname{ddeg} Q^{\phi}_{+h,\times\mathfrak{w}} = 0$, and hence $\operatorname{ddeg}_{\prec\mathfrak{w}} Q^{\phi}_{+h} = 0$. Thus renaming $\mathfrak{w}$ as $\mathfrak{v}$ we have arranged $\operatorname{ddeg}_{\prec\mathfrak{v}} Q^{\phi}_{+h} = 0$.

Set $\delta := \phi^{-1}\partial$; by Corollary 11.9 it is enough to show that $\delta^j\big(\frac{y-h}{\mathfrak{m}}\big) \preccurlyeq 1$ for $j = 0,\dots,r$. Now using $(\ \ )^\circ$ as before, the hole $(P^{\phi\circ}, 1, \widehat{h}^\circ)$ in H° is special, Z-minimal, deep, ultimate, and strongly repulsive-normal, by Lemmas 8.3 and 8.4. It remains to apply Corollary 9.20 to this hole in H° with h°, $\mathfrak{m}^\circ$, y° in place of h, $\mathfrak{m}$, y. □

Corollary 11.11. *Let ϕ be active in H, $0 < \phi \preccurlyeq 1$, and suppose the slot $(P^\phi, \mathfrak{n}, \widehat{h})$ in H^ϕ is deep, ultimate, and strongly split-normal. Then $P(y) = 0$ and $y \prec \mathfrak{n}$ for some $y \in \mathcal{C}^{<\infty}$. If $(P^\phi, \mathfrak{n}, \widehat{h})$ is strongly repulsive-normal, then any such y is hardian over H with $y \notin H$.*

Proof. Lemma 8.6 gives $y \in \mathcal{C}^{<\infty}$ with $P(y) = 0$, $y \prec \mathfrak{n}$. Now suppose $(P^\phi, \mathfrak{n}, \widehat{h})$ is strongly repulsive-normal, and $y \in \mathcal{C}^{<\infty}$, $P(y) = 0$, $y \prec \mathfrak{n}$. Use [8, Lemma 3.2.14] to arrange that $(P, \mathfrak{n}, \widehat{h})$ is a hole in H. Then the hole $(P^{\phi\circ}, \mathfrak{n}^\circ, \widehat{h}^\circ)$ in H° is special, Z-minimal, deep, ultimate, and strongly repulsive-normal. Theorem 11.10 with H°, $(P^{\phi\circ}, \mathfrak{n}^\circ, \widehat{h}^\circ)$, y° in place of H, $(P, \mathfrak{n}, \widehat{h})$, y now shows that y° is hardian over H° with $y^\circ \notin H^\circ$. Hence y is hardian over H and $y \notin H$. □

Achieving 1-linear newtonianity. For the proof of our main theorem we need to show first that for any d-maximal Hardy field H the corresponding $K = H[\mathrm{i}]$ is 1-linearly newtonian, the latter being a key hypothesis in Lemma 11.18 below. In this subsection we take this vital step: Corollary 11.17.

Lemma 11.12. *Every* d*-maximal Hardy field is* 1*-newtonian.*

Proof. Let H be a d-maximal Hardy field. Then H is ω-free by Theorem 2.3, and H is Liouville closed with $\mathrm{I}(K) \subseteq K^\dagger$ for $K := H[\mathrm{i}]$ by [9, Corollary 6.12]. Hence H is 1-linearly newtonian by [8, Corollary 1.7.29]. Towards a contradiction, assume that H is not 1-newtonian. Then [8, Lemma 3.2.1] yields a minimal hole $(P, \mathfrak{n}, \widehat{h})$ in H of order $r = 1$. Using [8, Lemma 3.2.26], replace $(P, \mathfrak{n}, \widehat{h})$ by a refinement to arrange that $(P, \mathfrak{n}, \widehat{h})$ is quasilinear. Then $(P, \mathfrak{n}, \widehat{h})$ is special, by Lemma 1.7. Using [8, Corollary 4.5.42], further refine $(P, \mathfrak{n}, \widehat{h})$ to arrange that $(P^\phi, \mathfrak{n}, \widehat{h})$ is eventually deep, ultimate, and strongly repulsive-normal. Now Corollary 11.11 gives a proper d-algebraic Hardy field extension of H, contradicting d-maximality of H. $\square$

In the rest of this subsection H has asymptotic integration. We have the d-valued extension $K := H[\mathrm{i}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$ of H and as before we arrange that $\widehat{K} = \widehat{H}[\mathrm{i}]$ is a d-valued extension of $\widehat{H}$ as well as an immediate d-valued extension of K.

Lemma 11.13. *Suppose H is Liouville closed and $\mathrm{I}(K) \subseteq K^\dagger$. Let $(P, \mathfrak{n}, \widehat{f})$ be an ultimate minimal hole in K of order $r \geqslant 1$ with $\deg P = 1$, where $\widehat{f} \in \widehat{K}$, such that $\dim_{\mathbb{C}} \ker_{\mathrm{U}} L_P = r$. Assume also that K is ω-free if $r \geqslant 2$. Let $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$ be such that $P(f) = 0$, $f \prec \mathfrak{n}$. Then $f \approx_K \widehat{f}$.*

Proof. Replacing $(P, \mathfrak{n}, \widehat{f})$, f by $(P_{\times\mathfrak{n}}, 1, \widehat{f}/\mathfrak{n})$, $f/\mathfrak{n}$ we arrange $\mathfrak{n} = 1$. Let $\theta \in K^\times$ be such that $\theta \sim \widehat{f}$; we claim that $f \sim \theta$ in $\mathcal{C}[\mathrm{i}]$ (and so $f \sim_K \widehat{f}$). Applying Proposition 8.9 and Remark 8.11 to the linear minimal hole $(P_{+\theta}, \theta, \widehat{f} - \theta)$ in K gives $g \in \mathcal{C}^{<\infty}[\mathrm{i}]$ such that $P_{+\theta}(g) = 0$ and $g \prec \theta$. Then $P(\theta + g) = 0$ and $\theta + g \prec 1$, thus $L_P(y) = 0$ and $y \prec 1$ for $y := f - (\theta + g) \in \mathcal{C}^{<\infty}[\mathrm{i}]$. Hence $y \prec \theta$ by the version of Lemma 4.10 for slots in K; see the remark following Corollary 4.13. Therefore $f - \theta = y + g \prec \theta$ and so $f \sim \theta$, as claimed.

The refinement $(P_{+\theta}, 1, \widehat{f} - \theta)$ of $(P, 1, \widehat{f})$ is ultimate thanks to the remarks before Proposition 1.13, and $L_{P_{+\theta}} = L_P$, so we can apply the claim to $(P_{+\theta}, 1, \widehat{f} - \theta)$ instead of $(P, 1, \widehat{f})$ and $f - \theta$ instead of f to give $f - \theta \sim_K \widehat{f} - \theta$. Since this holds for all $\theta \in K$ with $\theta \sim \widehat{f}$, the sentence preceding Lemma 10.12 then yields $f \approx_K \widehat{f}$. $\square$

Corollary 11.14. *Let $(P, \mathfrak{n}, \widehat{f})$ be a hole of order* 1 *in K with $\deg P = 1$.* (*We do not assume here that $\widehat{f} \in \widehat{K}$.*) *Then there is an embedding $\iota \colon K\langle \widehat{f} \rangle \to \mathcal{C}^{<\infty}[\mathrm{i}]$ of differential K-algebras such that $\iota(g) \sim_K g$ for all $g \in K\langle \widehat{f} \rangle^\times$.*

Proof. Note that $(P, \mathfrak{n}, \widehat{f})$ is minimal. We first show how to arrange that H is Liouville closed and ω-free with $\mathrm{I}(K) \subseteq K^\dagger$ and $\widehat{f} \in \widehat{K}$. Let H_1 be a maximal Hardy field extension of H. Then H_1 is Liouville closed and ω-free, with $\mathrm{I}(K_1) \subseteq K_1^\dagger$ for $K_1 := H_1[\mathrm{i}] \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$. Let $\widehat{H}_1$ be the newtonization of H_1; then $\widehat{K}_1 := \widehat{H}_1[\mathrm{i}]$ is newtonian [ADH, 14.5.7]. Lemma 1.6 gives an embedding $K\langle \widehat{f} \rangle \to \widehat{K}_1$ of valued differential fields over K; let $\widehat{f}_1$ be the image of $\widehat{f}$ under this embedding. If $\widehat{f}_1 \in K_1 \subseteq \mathcal{C}^{<\infty}[\mathrm{i}]$, then we are done, so assume $\widehat{f}_1 \notin K_1$. Then $(P, \mathfrak{n}, \widehat{f}_1)$ is a hole in K_1, and we replace H, K, $(P, \mathfrak{n}, \widehat{f})$ by H_1, K_1, $(P, \mathfrak{n}, \widehat{f}_1)$, and $\widehat{K}$ by $\widehat{K}_1$, to arrange that H is Liouville closed and ω-free with $\mathrm{I}(K) \subseteq K^\dagger$ and $\widehat{f} \in \widehat{K}$.

Replacing $(P, \mathfrak{n}, \widehat{f})$ by a refinement we also arrange that $(P, \mathfrak{n}, \widehat{f})$ is ultimate and $\mathfrak{n} \in H^\times$, by Proposition 1.13. Then Proposition 8.9 yield an $f \in \mathcal{C}^{<\infty}[\mathrm{i}]$

with $P(f)=0$, $f\prec\mathfrak{n}$. Now Lemma 11.13 gives $f\approx_K \widehat{f}$, and it remains to appeal to Corollary 11.6. $\square$

Proposition 11.15. *Suppose H is ω-free and 1-newtonian. Let $(P,\mathfrak{n},\widehat{f})$ be a hole in K of order 1 with $\deg P=1$, $\widehat{f}\in\widehat{K}$. Let $f\in\mathcal{C}^{<\infty}[i]$, $P(f)=0$, and $f\approx_K\widehat{f}$. Then $\operatorname{Re} f$ or $\operatorname{Im} f$ generates a proper* d*-algebraic Hardy field extension of H.*

Proof. Let $\widehat{g}:=\operatorname{Re}\widehat{f}$ and $\widehat{h}:=\operatorname{Im}\widehat{f}$. By [8, Lemma 4.1.3] we have $v(\widehat{g}-H)\subseteq v(\widehat{h}-H)$ or $v(\widehat{h}-H)\subseteq v(\widehat{g}-H)$. Below we assume $v(\widehat{g}-H)\subseteq v(\widehat{h}-H)$ (so $\widehat{g}\in\widehat{H}\setminus H$) and show that then $g:=\operatorname{Re} f$ generates a proper d-algebraic Hardy field extension of H. (If $v(\widehat{h}-H)\subseteq v(\widehat{g}-H)$ one shows likewise that $\operatorname{Im} f$ generates a proper d-algebraic Hardy field extension of H.) The hole $(P,\mathfrak{n},\widehat{f})$ in K is minimal, and by arranging $\mathfrak{n}\in H^\times$ we see that $\widehat{g}$ is d-algebraic over H, by a remark preceding [8, Lemma 4.3.7]. Every element of $Z(H,\widehat{g})$ has order $\geqslant 2$, by [8, Corollary 3.2.16] and 1-newtonianity of H. We arrange that the linear part A of P is monic, so $A=\partial-a$ with $a\in K$, $A(\widehat{f})=-P(0)$ and $A(f)=-P(0)$. Then [8, Example 1.1.7 and Remark 1.1.9] applied to $F=\mathcal{C}^{<\infty}$ yields $Q\in H\{Y\}$ with $1\leqslant\operatorname{order} Q\leqslant 2$ and $\deg Q=1$ such that $Q(\widehat{g})=0$ and $Q(g)=0$. Hence $\operatorname{order} Q=2$ and Q is a minimal annihilator of $\widehat{g}$ over H.

Towards applying Corollary 11.8 to Q, $\widehat{g}$, g in the role of P, $\widehat{h}$, y there, let R in $H\{Y\}\setminus H$ have order <2. Then $R\notin Z(H,\widehat{g})$, so we have $h\in H$ and $\mathfrak{v}\in H^\times$ such that $\widehat{g}-h\prec\mathfrak{v}$ and $\operatorname{ndeg}_{\prec\mathfrak{v}} R_{+h}=0$. Take any $\mathfrak{m}\in H^\times$ with $\widehat{g}-h\preccurlyeq\mathfrak{m}\prec\mathfrak{v}$. By Lemma 10.16 we have $g\approx_H\widehat{g}$ and thus $g-h\preccurlyeq\mathfrak{m}$. After changing $\mathfrak{v}$ as in the proof of Theorem 11.10 we obtain an active ϕ in H, $0<\phi\preccurlyeq 1$, such that $\operatorname{ddeg}_{\prec\mathfrak{v}} R^{\phi}_{+h}=0$. Set $\delta:=\phi^{-1}\partial$; by Corollary 11.8 it is now enough to show that $\delta\big((g-h)/\mathfrak{m}\big)\preccurlyeq 1$.

Towards this and using $(\ \ )^\circ$ as before, we have $f^\circ\approx_{K^\circ}\widehat{f}^\circ$, and $g^\circ\approx_{H^\circ}\widehat{g}^\circ$ by the facts about composition in Section 10. Moreover, $(g-h)^\circ\preccurlyeq\mathfrak{m}^\circ$, and H° is ω-free and 1-newtonian, hence closed under integration by [ADH, 14.2.2]. We now apply Corollary 11.6 with H°, K°, $P^{\phi\circ}$, f°, $\widehat{f}^\circ$ in the role of H, K, P, f, $\widehat{f}$ to give $\big(f^\circ/\mathfrak{m}^\circ\big)'\approx_{K^\circ}\big(\widehat{f}^\circ/\mathfrak{m}^\circ\big)'$, hence $(g^\circ/\mathfrak{m}^\circ)'\approx_{H^\circ}(\widehat{g}^\circ/\mathfrak{m}^\circ)'$ by [8, Lemma 4.1.4] and Lemma 10.16. Therefore,

$$\big((g-h)^\circ/\mathfrak{m}^\circ\big)' \;=\; (g^\circ/\mathfrak{m}^\circ)'-(h^\circ/\mathfrak{m}^\circ)' \;\sim_H\; (\widehat{g}^\circ/\mathfrak{m}^\circ)'-(h^\circ/\mathfrak{m}^\circ)' \;=\; \big((\widehat{g}-h)^\circ/\mathfrak{m}^\circ\big)'.$$

Now $(\widehat{g}-h)^\circ/\mathfrak{m}^\circ\preccurlyeq 1$, so $\big((\widehat{g}-h)^\circ/\mathfrak{m}^\circ\big)'\prec 1$, hence $\big((g-h)^\circ/\mathfrak{m}^\circ\big)'\prec 1$ by the last display, and thus $\delta\big((g-h)/\mathfrak{m}\big)\prec 1$, which is more than enough. $\square$

If K has a hole of order and degree 1, then K has a proper d-algebraic differential field extension inside $\mathcal{C}^{<\infty}[i]$, by Corollary 11.14. Here is a Hardy version:

Lemma 11.16. *Suppose K has a hole of order and degree 1. Then H has a proper* d*-algebraic Hardy field extension.*

Proof. If H is not d-maximal, then H has indeed a proper d-algebraic Hardy field extension, and if H is d-maximal, then H is Liouville closed, ω-free, 1-newtonian, and $\operatorname{I}(K)\subseteq K^\dagger$, by Proposition 2.1, [9, Corollary 6.12], Theorem 2.3, and Lemma 11.12. So assume below that H is Liouville closed, ω-free, 1-newtonian, and $\operatorname{I}(K)\subseteq K^\dagger$, and $(P,\mathfrak{n},\widehat{f})$ is a hole of order and degree 1 in K. Using a remark preceding Lemma 1.10 we arrange that $\widehat{f}\in\widehat{K}:=\widehat{H}[i]$, $\widehat{H}$ an immediate ω-free newtonian H-field extension of H. Then $\widehat{K}$ is also newtonian by [ADH, 14.5.7]. By

Proposition 1.13 we can replace $(P, \mathfrak{n}, \widehat{f})$ by a refinement to arrange that $(P, \mathfrak{n}, \widehat{f})$ is ultimate and $\mathfrak{n} \in H^\times$. Proposition 8.9 now yields $f \in \mathcal{C}^{<\infty}[i]$ with $P(f) = 0$ and $f \prec \mathfrak{n}$. Then $f \approx_K \widehat{f}$ by Lemma 11.13, and so $\operatorname{Re} f$ or $\operatorname{Im} f$ generates a proper d-algebraic Hardy field extension of H, by Proposition 11.15. □

Corollary 11.17. *If H is* d*-maximal, then K is* 1*-linearly newtonian.*

Proof. Assume H is d-maximal. Then K is ω-free by Theorem 2.3 and [ADH, 11.7.23]. If K is not 1-linearly newtonian, then it has a hole of order and degree 1, by [8, Lemma 3.2.5], and so H has a proper d-algebraic Hardy field extension, by Lemma 11.16, contradicting d-maximality of H. □

Finishing the story. With one more lemma we will be done.

Lemma 11.18. *Suppose H is Liouville closed, ω-free, not newtonian, and $K := H[i]$ is* 1*-linearly newtonian. Then H has a proper* d*-algebraic Hardy field extension.*

Proof. Theorem 1.16 yields a Z-minimal special hole $(Q, 1, \widehat{b})$ in H of order $r \geqslant 1$ and an active ϕ in H with $0 < \phi \preccurlyeq 1$ such that the hole $(Q^\phi, 1, \widehat{b})$ in H^ϕ is deep, strongly repulsive-normal, and ultimate. By [8, Proposition 1.7.28], K is 1-linearly surjective and $\mathrm{I}(K) \subseteq K^\dagger$. Then Corollary 11.11 applied to $(Q, 1, \widehat{b})$ in the role of $(P, \mathfrak{n}, \widehat{h})$ gives a $y \in \mathcal{C}^{<\infty} \setminus H$ that is hardian over H with $Q(y) = 0$. Hence $H\langle y\rangle$ is a proper d-algebraic Hardy field extension of H. □

Recall from the introduction that an *H-closed field* is an ω-free newtonian Liouville closed H-field. Recall also that Hardy fields containing $\mathbb{R}$ are H-fields. The main result of this paper now follows in a few lines:

Theorem 11.19. *A Hardy field is* d*-maximal iff it contains $\mathbb{R}$ and is H-closed.*

Proof. The "if" part is a special case of [ADH, 16.0.3]. Suppose H is d-maximal. By Proposition 2.1 and Theorem 2.3, $H \supseteq \mathbb{R}$ and H is Liouville closed and ω-free. By Corollary 11.17, K is 1-linearly newtonian, so H is newtonian by Lemma 11.18. □

Theorem 11.19 and Corollary 7.8 yield Theorem B in a refined form:

Corollary 11.20. *Any Hardy field F has a* d*-algebraic H-closed Hardy field extension. If F is a $\mathcal{C}^\infty$-Hardy field, then so is any such extension, and likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.*

12. Transfer Theorems

From [ADH, 16.3] we recall the notion of a *pre-$\Lambda\Omega$-field* $\boldsymbol{H} = (H, \mathrm{I}, \Lambda, \Omega)$: this is a pre-$H$-field H equipped with a *$\Lambda\Omega$-cut* $(\mathrm{I}, \Lambda, \Omega)$ of H. A *$\Lambda\Omega$-field* is a pre-$\Lambda\Omega$-field $\boldsymbol{H} = (H; \dots)$ where H is an H-field. If $\boldsymbol{M} = (M; \dots)$ is a pre-$\Lambda\Omega$-field and H is a pre-H-subfield of M, then H has a unique expansion to a pre-$\Lambda\Omega$-field $\boldsymbol{H}$ such that $\boldsymbol{H} \subseteq \boldsymbol{M}$. By [ADH, 16.3.19 and remarks before it], a pre-H-field H has exactly one or exactly two $\Lambda\Omega$-cuts, and H has a unique $\Lambda\Omega$-cut iff

(1) H is grounded; or
(2) there exists $b \asymp 1$ in H such that $v(b')$ is a gap in H; or
(3) H is ω-free.

In particular, each d-maximal Hardy field M (being ω-free) has a unique expansion to a pre-$\Lambda\Omega$-field $\boldsymbol{M}$, namely $\boldsymbol{M} = \big(M; \mathrm{I}(M), \Lambda(M), \omega(M)\big)$, and then $\boldsymbol{M}$ is a $\Lambda\Omega$-field with constant field $\mathbb{R}$. Below we always view any d-maximal Hardy field as an $\Lambda\Omega$-field in this way.

Lemma 12.1. *Let H be a Hardy field. Then H has an expansion to a pre-$\Lambda\Omega$-field $\boldsymbol{H}$ such that $\boldsymbol{H} \subseteq \boldsymbol{M}$ for every* d*-maximal Hardy field $M \supseteq H$.*

Proof. Since every d-maximal Hardy field containing H also contains $\mathrm{D}(H)$, it suffices to show this for $\mathrm{D}(H)$ in place of H. So we assume H is d-perfect, and thus a Liouville closed H-field. For each d-maximal Hardy field $M \supseteq H$ we now have $\mathrm{I}(H) = \mathrm{I}(M) \cap H$ by [ADH, 11.8.2], $\Lambda(H) = \Lambda(M) \cap H$ by [ADH, 11.8.6], and $\omega(H) = \overline{\omega}(H) = \overline{\omega}(M) \cap H = \omega(M) \cap H$ by [9, Corollary 6.2], as required. $\square$

Given a Hardy field H, we call the unique expansion $\boldsymbol{H}$ of H to a pre-$\Lambda\Omega$-field with the property stated in the previous lemma the **canonical $\Lambda\Omega$-expansion** of H.

Corollary 12.2. *Let H, H^* be Hardy fields, with their canonical $\Lambda\Omega$-expansions $\boldsymbol{H}$ and $\boldsymbol{H}^*$, respectively, such that $H \subseteq H^*$. Then $\boldsymbol{H} \subseteq \boldsymbol{H}^*$.*

Proof. Let M^* be any d-maximal Hardy field extension of H^*. Then $\boldsymbol{H} \subseteq \boldsymbol{M}^*$ as well as $\boldsymbol{H}^* \subseteq \boldsymbol{M}^*$, hence $\boldsymbol{H} \subseteq \boldsymbol{H}^*$. $\square$

In the rest of this section $\mathcal{L} = \{0, 1, -, +, \cdot, \partial, \leqslant, \preccurlyeq\}$ is the language of ordered valued differential rings [ADH, p. 678]. We view each ordered valued differential field as an $\mathcal{L}$-structure in the natural way. Given an ordered valued differential field H and a subset A of H we let $\mathcal{L}_A$ be $\mathcal{L}$ augmented by names for the elements of A, and expand the $\mathcal{L}$-structure H to an $\mathcal{L}_A$-structure by interpreting the name of any $a \in A$ as the element a of H; cf. [ADH, B.3]. Let H be a Hardy field and σ be an $\mathcal{L}_H$-sentence. We now have our Hardy field analogue of the "Tarski principle" [ADH, B.12.14] in real algebraic geometry promised in the introduction:

Theorem 12.3. *The following are equivalent:*

(i) *$M \models \sigma$ for some* d*-maximal Hardy field $M \supseteq H$;*
(ii) *$M \models \sigma$ for every* d*-maximal Hardy field $M \supseteq H$;*
(iii) *$M \models \sigma$ for every maximal Hardy field $M \supseteq H$;*
(iv) *$M \models \sigma$ for some maximal Hardy field $M \supseteq H$.*

Proof. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are obvious, since "maximal $\Rightarrow$ d-maximal"; it remains to show (i) $\Rightarrow$ (ii). Let M, M^* be d-maximal Hardy field extensions of H. By Lemma 12.1 and Corollary 12.2 expand M, M^*, H to pre-$\Lambda\Omega$-fields $\boldsymbol{M}$, $\boldsymbol{M}^*$, $\boldsymbol{H}$, such that $\boldsymbol{H} \subseteq \boldsymbol{M}$ and $\boldsymbol{H} \subseteq \boldsymbol{M}^*$. In [ADH, introduction to Chapter 16] we extended $\mathcal{L}$ to a language $\mathcal{L}^{\iota}_{\Lambda\Omega}$, and explained in [ADH, 16.5] how each pre-$\Lambda\Omega$-field $\boldsymbol{K}$ is construed as an $\mathcal{L}^{\iota}_{\Lambda\Omega}$-structure in such a way that every extension $\boldsymbol{K} \subseteq \boldsymbol{L}$ of pre-$\Lambda\Omega$-fields corresponds to an extension of the associated $\mathcal{L}^{\iota}_{\Lambda\Omega}$-structures. By [ADH, 16.0.1], the $\mathcal{L}^{\iota}_{\Lambda\Omega}$-theory $T^{\mathrm{nl},\iota}_{\Lambda\Omega}$ of H-closed $\Lambda\Omega$-fields eliminates quantifiers, and $\boldsymbol{M}, \boldsymbol{M}^* \models T^{\mathrm{nl},\iota}_{\Lambda\Omega}$ by Theorem 11.19. Hence $\boldsymbol{M} \equiv_H \boldsymbol{M}^*$ [ADH, B.11.6], so if $\boldsymbol{M} \models \sigma$, then $\boldsymbol{M}^* \models \sigma$. $\square$

Corollaries 1 and 2 from the introduction are special cases of Theorem 12.3. By Corollary 7.7, $\mathcal{C}^{\infty}$-maximal and $\mathcal{C}^{\omega}$-maximal Hardy fields are d-maximal, so Theorem 12.3 also yields Corollary 5 from the introduction in a stronger form:

Corollary 12.4. *If $H \subseteq \mathcal{C}^\infty$ and $M \models \sigma$ for some* d*-maximal Hardy field extension M of H, then $M \models \sigma$ for every $\mathcal{C}^\infty$-maximal Hardy field $M \supseteq H$. Likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.*

The structure induced on $\mathbb{R}$. In the next corollary H is a Hardy field and $\varphi(x)$ is an $\mathcal{L}_H$-formula where $x = (x_1, \dots, x_n)$ and $x_1, \dots, x_n$ are distinct variables. Also, $\mathcal{L}_{\mathrm{OR}} = \{0, 1, -, +, \cdot, \leqslant\}$ is the language of ordered rings, and the ordered field $\mathbb{R}$ of real numbers is interpreted as an $\mathcal{L}_{\mathrm{OR}}$-structure in the obvious way. By Theorem 11.19, d-maximal Hardy fields are H-closed fields, so from [ADH, 16.6.7, B.12.13] in combination with Theorem 12.3 we obtain:

Corollary 12.5. *There is a quantifier-free $\mathcal{L}_{\mathrm{OR}}$-formula $\varphi_{\mathrm{OR}}(x)$ such that for all* d*-maximal Hardy fields $M \supseteq H$ and $c \in \mathbb{R}^n$: $M \models \varphi(c) \;\Leftrightarrow\; \mathbb{R} \models \varphi_{\mathrm{OR}}(c)$.*

This yields Corollary 3 from the introduction. The first of the examples after that corollary is covered by [ADH, 5.1.18, 11.8.25, 11.8.26]; for the details of the second example we refer to [6, Section 7.1].

Uniform finiteness. We now let H be a Hardy field and $\varphi(x, y)$ and $\theta(x)$ be $\mathcal{L}_H$-formulas, where $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$.

Lemma 12.6. *There is a $B = B(\varphi) \in \mathbb{N}$ such that for all $f \in H^m$: if for some* d*-maximal Hardy field extension M of H there are more than B tuples $g \in M^n$ with $M \models \varphi(f, g)$, then for every* d*-maximal Hardy field extension M of H there are infinitely many $g \in M^n$ with $M \models \varphi(f, g)$.*

Proof. Fix a d-maximal Hardy field extension M^* of H. By [3, Proposition 6.4] we have $B = B(\varphi) \in \mathbb{N}$ such that for all $f \in (M^*)^m$: if $M^* \models \varphi(f, g)$ for more than B many $g \in (M^*)^n$, then $M^* \models \varphi(f, g)$ for infinitely many $g \in (M^*)^n$. Now use Theorem 12.3. □

In the proof of the next lemma we use that $\mathcal{C}$ has the cardinality $\mathfrak{c} = 2^{\aleph_0}$ of the continuum, hence $|H| = \mathfrak{c}$ if $H \supseteq \mathbb{R}$.

Lemma 12.7. *Suppose H is* d*-maximal and $S := \big\{ f \in H^m : H \models \theta(f) \big\}$ is infinite. Then $|S| = \mathfrak{c}$.*

Proof. Let $d := \dim(S)$ be the dimension of the definable set $S \subseteq H^m$ as introduced in [3]. If $d = 0$, then $|S| = |\mathbb{R}| = \mathfrak{c}$ by remarks following [3, Proposition 6.4]. Suppose $d > 0$, and for $g = (g_1, \dots, g_m) \in H^m$ and $i \in \{1, \dots, m\}$, let $\pi_i(g) := g_i$. Then for some $i \in \{1, \dots, m\}$, the subset $\pi_i(S)$ of H has nonempty interior, by [3, Corollary 3.2], and hence $|S| = |H| = \mathfrak{c}$. □

The two lemmas above together now yield Corollary 4 from the introduction.

Transfer between maximal Hardy fields and transseries. Let $\boldsymbol{T}$ be the unique expansion of $\mathbb{T}$ to a pre-$\Lambda\Omega$-field, so $\boldsymbol{T}$ is an H-closed $\Lambda\Omega$-field with small derivation and constant field $\mathbb{R}$.

Lemma 12.8. *Let H be a pre-H-subfield of $\mathbb{T}$ with $H \not\subseteq \mathbb{R}$. Then H has a unique expansion to a pre-$\Lambda\Omega$-field.*

Proof. If H is grounded, this follows from [ADH, 16.3.19]. Suppose H is not grounded. Then H has asymptotic integration by the proof of [ADH, 10.6.19] applied to $\Delta := v(H^\times)$. Starting with an $h_0 \succ 1$ in H with $h_0' \asymp 1$ we construct a logarithmic sequence (h_n) in H as in [ADH, 11.5], so $h_n \asymp \ell_n$ for all n. Hence $\Gamma^<$ is cofinal in $\Gamma_{\mathbb{T}}^<$, so H is ω-free by [ADH, remark before 11.7.20]. Now use [ADH, 16.3.19] again. $\square$

In the rest of this section H is a Hardy field with canonical $\Lambda\Omega$-expansion $\boldsymbol{H}$, and $\iota\colon H \to \mathbb{T}$ is an embedding of ordered differential fields, and thus of pre-H-fields.

Corollary 12.9. *The map ι is an embedding $\boldsymbol{H} \to \boldsymbol{T}$ of pre-$\Lambda\Omega$-fields.*

Proof. If $H \not\subseteq \mathbb{R}$, then this follows from Lemma 12.8. Suppose $H \subseteq \mathbb{R}$. Then ι is the identity on H, so extends to the embedding $\mathbb{R}(x) \to \mathbb{T}$ that is the identity on $\mathbb{R}$ and sends the germ x to $x \in \mathbb{T}$. Now use that $\mathbb{R}(x) \not\subseteq \mathbb{R}$ and Corollary 12.2. $\square$

Recall from [ADH, B.4] that for any $\mathcal{L}_H$-sentence σ we obtain an $\mathcal{L}_{\mathbb{T}}$-sentence $\iota(\sigma)$ by replacing the name of each $h \in H$ occurring in σ with the name of $\iota(h)$.

Corollary 12.10. *Let σ be an $\mathcal{L}_H$-sentence. Then* (i)–(iv) *in Theorem 12.3 are also equivalent to:*

(v) $\mathbb{T} \models \iota(\sigma)$.

Proof. Let M be a d-maximal Hardy field extension of H; it suffices to show that $M \models \sigma$ iff $\mathbb{T} \models \iota(\sigma)$. For this, mimick the proof of (i) $\Rightarrow$ (ii) in Theorem 12.3, using Corollary 12.9. $\square$

Corollary 12.10 yields the first part of Corollary 6 from the introduction, even in a stronger form. We prove the second part of that corollary in Section 13: Corollary 13.2. There we also use:

Lemma 12.11. *ι extends uniquely to an embedding $H(\mathbb{R}) \to \mathbb{T}$ of pre-H-fields.*

Proof. Let $\widehat{H}$ be the H-field hull of H in $H(\mathbb{R})$. Then ι extends uniquely to an H-field embedding $\widehat{\iota}\colon \widehat{H} \to \mathbb{T}$ by [ADH, 10.5.13]. By [ADH, remark before 4.6.21] and [ADH, 10.5.16] $\widehat{\iota}$ extends uniquely to an embedding $H(\mathbb{R}) \to \mathbb{T}$ of H-fields. $\square$

We now derive Theorem A from the introduction (in stronger form):

Corollary 12.12. *If $P \in H\{Y\}$, $f < g$ in H, and $P(f) < 0 < P(g)$, then each* d*-maximal Hardy field extension of H contains a y with $f < y < g$ and $P(y) = 0$.*

Proof. By [54], the ordered differential field $\mathbb{T}_{\text{g}}$ of grid-based transseries is H-closed with small derivation and has the differential intermediate value property (DIVP). Hence $\mathbb{T}$ also has DIVP, by completeness of T_H (see the introduction). Now use Corollary 12.10. $\square$

Corollary 12.13. *For every $P \in H\{Y\}$ of odd degree there is an H-hardian germ y with $P(y) = 0$.*

Proof. This follows from Theorem 11.20 and [ADH, 14.5.3]. Alternatively, use Corollary 12.12: with Proposition 2.1, arrange $H \supseteq \mathbb{R}$ and H is Liouville closed, and appeal to the example following Corollary 1.9 in [9]. $\square$

Note that if $H \subseteq \mathcal{C}^\infty$, then in the previous two corollaries we have $H\langle y\rangle \subseteq \mathcal{C}^\infty$, by Corollary 7.8; likewise with $\mathcal{C}^\omega$ in place if $\mathcal{C}^\infty$.

The following contains Corollaries 7 and 8 from the introduction. In [ADH] we defined a differential field F to be *weakly* d*-closed* if every $P \in F\{Y\} \setminus F$ has a zero in F. If F is weakly d-closed, then F is clearly linearly surjective, and also linearly closed by [ADH, 5.8.9].

Corollary 12.14. *If H is* d*-maximal, then $K := H[\mathrm{i}]$ is weakly* d*-closed.*

Proof. By our main Theorem 11.19, if H is d-maximal, then H is newtonian, and thus K is weakly d-closed by [ADH, 14.5.7, 14.5.3]. $\square$

The remarks after Corollary 8 in the introduction concerning fundamental systems of solutions to scalar linear differential equations over K follow from Lemma 4.3 and Corollary 12.14 in combination with the equivalence (1.2).

13. Embeddings into Transseries and Maximal Hardy Fields

We first derive a fact about "Newton-Liouville closure" (as defined in [ADH]). Let $\boldsymbol{H}$ be a $\Lambda\Omega$-field with underlying H-field H. Then $\boldsymbol{H}$ has an ω-free newtonian Liouville closed $\Lambda\Omega$-field extension, called a *Newton-Liouville closure* of $\boldsymbol{H}$, that embeds over $\boldsymbol{H}$ into any ω-free newtonian Liouville closed $\Lambda\Omega$-field extension of $\boldsymbol{H}$ [ADH, 16.4.8]. Any two Newton-Liouville closures of $\boldsymbol{H}$ are isomorphic over $\boldsymbol{H}$ [ADH, 16.4.9], and this permits us to speak of *the* Newton-Liouville closure of $\boldsymbol{H}$. By [ADH, 14.5.10, 16.4.1, 16.4.8], the constant field of the Newton-Liouville closure of $\boldsymbol{H}$ is the real closure of $C := C_H$. Let $\boldsymbol{M} = (M; \dots)$ be an H-closed $\Lambda\Omega$-field extension of $\boldsymbol{H}$ and $\boldsymbol{H}^{\mathrm{da}} := (H^{\mathrm{da}}; \dots)$ be the $\Lambda\Omega$-subfield of $\boldsymbol{M}$ with $H^{\mathrm{da}} := \{f \in M : f \text{ is d-algebraic over } H\}$.

Proposition 13.1. *Let $\boldsymbol{H}^*$ be a* d*-algebraic $\Lambda\Omega$-field extension of $\boldsymbol{H}$ such that the constant field of $\boldsymbol{H}^*$ is algebraic over C. Then there is an embedding $\boldsymbol{H}^* \to \boldsymbol{M}$ over $\boldsymbol{H}$, and the image of any such embedding is contained in $\boldsymbol{H}^{\mathrm{da}}$.*

Proof. The image of any embedding $\boldsymbol{H}^* \to \boldsymbol{M}$ over $\boldsymbol{H}$ is d-algebraic over H and thus contained in $\boldsymbol{H}^{\mathrm{da}}$. For existence, take a Newton-Liouville closure $\boldsymbol{M}^*$ of $\boldsymbol{H}^*$. Then $\boldsymbol{M}^*$ is also a Newton-Liouville closure of $\boldsymbol{H}$, by [ADH, 16.0.3], and thus embeds into $\boldsymbol{M}$ over $\boldsymbol{H}$. $\square$

Let $\mathcal{L}$ be the language of ordered valued differential rings, as in Section 12. The second part of Corollary 6 in the introduction now follows from the next result:

Corollary 13.2. *Let H be a Hardy field, $\iota\colon H \to \mathbb{T}$ an ordered differential field embedding, and H^* a* d*-maximal* d*-algebraic Hardy field extension of H. Then ι extends to an ordered valued differential field embedding $H^* \to \mathbb{T}$, and so for any $\mathcal{L}_H$-sentence σ, $H^* \models \sigma$ iff $\mathbb{T} \models \iota(\sigma)$.*

Proof. We have $H(\mathbb{R}) \subseteq H^*$, and so by Lemma 12.11 we arrange that $H \supseteq \mathbb{R}$. Let $\boldsymbol{H}$, $\boldsymbol{H}^*$ be the canonical $\Lambda\Omega$-expansions of H, H^*, respectively, and let $\boldsymbol{T}$ be the expansion of $\mathbb{T}$ to a $\Lambda\Omega$-field. Then $\boldsymbol{H} \subseteq \boldsymbol{H}^*$, and by Lemma 12.9, ι is an embedding $\boldsymbol{H} \to \boldsymbol{T}$. By Proposition 13.1, ι extends to an embedding $\boldsymbol{H}^* \to \boldsymbol{T}$. $\square$

Consider the Hardy field $H := \mathbb{R}(\ell_0, \ell_1, \ell_2, \dots) \subseteq \mathcal{C}^\omega$ where $\ell_0 = x$ and $\ell_{n+1} = \log \ell_n$ for each n, and mimick this in $\mathbb{T}$ by setting $\ell_0 := x$ and $\ell_{n+1} := \log \ell_n$ in $\mathbb{T}$. This yields the unique ordered differential field embedding $H \to \mathbb{T}$ over $\mathbb{R}$ sending $\ell_n \in H$ to $\ell_n \in \mathbb{T}$ for all n. Its image is the H-subfield $\mathbb{R}(\ell_0, \ell_1, \dots)$ of $\mathbb{T}$. Since the sequence (ℓ_n) in $\mathbb{T}$ is coinitial in $\mathbb{T}^{>\mathbb{R}}$, each ordered differential subfield of $\mathbb{T}$ containing $\mathbb{R}(\ell_0, \ell_1, \dots)$ is an ω-free H-field, by the remark preceding [ADH, 11.7.20]. From Lemma 12.8 and Proposition 13.1 we obtain:

Corollary 13.3. *If $H \supseteq \mathbb{R}$ is an ω-free H-subfield of $\mathbb{T}$ and H^* is a* d*-algebraic H-field extension of H with constant field $\mathbb{R}$, then there exists an H-field embedding $H^* \to \mathbb{T}$ over H.*

Corollary 13.3 goes through with $\mathbb{T}$ replaced by its H-subfield

$$\mathbb{T}^{\mathrm{da}} := \big\{f \in \mathbb{T} : f \text{ is d-algebraic (over } \mathbb{Q})\big\}.$$

We now apply this observation to o-minimal structures. The *Pfaffian closure* of an expansion of the ordered field of real numbers is its smallest expansion that is closed under taking Rolle leaves of definable 1-forms of class $\mathcal{C}^1$. See Speissegger [86] for complete definitions, and the proof that the Pfaffian closure of an o-minimal expansion of the ordered field of reals remains o-minimal.

Corollary 13.4. *The Hardy field H of the Pfaffian closure of the ordered field of real numbers embeds as an H-field over $\mathbb{R}$ into $\mathbb{T}^{\mathrm{da}}$.*

Proof. Let $f\colon \mathbb{R} \to \mathbb{R}$ be definable in the Pfaffian closure of the ordered field of real numbers. The proof of [61, Theorem 3] gives $r \in \mathbb{N}$, semialgebraic $g\colon \mathbb{R}^{r+2} \to \mathbb{R}$, and $a \in \mathbb{R}$ such that $f|_{(a,\infty)}$ is $\mathcal{C}^{r+1}$ and $f^{(r+1)}(t) = g\big(t, f(t), \dots, f^{(r)}(t)\big)$ for all $t > a$. Take $P \in \mathbb{R}[Y_1, \dots, Y_{r+3}]^{\neq}$ vanishing identically on the graph of g; see [ADH, B.12.18]. Then $P\big(t, f(t), \dots, f^{(r+1)}(t)\big) = 0$ for $t > a$. Hence the germ of f is d-algebraic over $\mathbb{R}$, and so H is d-algebraic over $\mathbb{R}$. As H contains the ω-free Hardy field $\mathbb{R}(\ell_0, \ell_1, \dots)$, we can use the remark following Corollary 13.3. □

Question. Let H be the Hardy field of an o-minimal expansion of the ordered field of reals, and let $H^* \supseteq H$ be the Hardy field of the Pfaffian closure of this expansion. Does every embedding $H \to \mathbb{T}$ extend to an embedding $H^* \to \mathbb{T}$?

We mentioned in the introduction that an embedding $H \to \mathbb{T}$ as in Corollaries 13.2 and 13.4 can be viewed as an *expansion operator* for the Hardy field H and its inverse as a *summation operator.* The corollaries above concern the existence of expansion operators; this relied on the H-closedness of $\mathbb{T}$. Likewise, Theorem 11.19 and Proposition 13.1 also give rise to summation operators:

Corollary 13.5. *Let H be an ω-free H-field and let H^* be a* d*-algebraic H-field extension of H with C_{H^*} algebraic over C_H. Then any H-field embedding $H \to M$ into a* d*-maximal Hardy field extends to an H-field embedding $H^* \to M$.*

Hence any $\mathcal{L}$-isomorphism between an ordered differential subfield $H \supseteq \mathbb{R}(\ell_0, \ell_1, \dots)$ of $\mathbb{T}$ and a Hardy field F extends to an $\mathcal{L}$-isomorphism between the ordered differential subfield $H^* := \{f \in \mathbb{T} : f \text{ is d-algebraic over } H\}$ of $\mathbb{T}$ and a Hardy field extension of F. For $H = \mathbb{R}(\ell_0, \ell_1, \dots) \subseteq \mathbb{T}$ (so $H^* = \mathbb{T}^{\mathrm{da}}$) we recover the main result of [55]:

Corollary 13.6. *The H-field $\mathbb{T}^{\mathrm{da}}$ is $\mathcal{L}$-isomorphic to a Hardy field $\supseteq \mathbb{R}(\ell_0, \ell_1, \dots)$.*

Any Hardy field that is $\mathcal{L}$-isomorphic to $\mathbb{T}^{\mathrm{da}}$ is H-closed by [ADH, 16.6] and thus d-maximal by Theorem 11.19, so contains the Hardy field $\mathrm{D}(\mathbb{Q})$. Thus we have an $\mathcal{L}$-embedding $e\colon \mathrm{D}(\mathbb{Q}) \to \mathbb{T}^{\mathrm{da}}$, which we can view as an expansion operator for $\mathrm{D}(\mathbb{Q})$. We suspect that $e(\mathrm{D}(\mathbb{Q}))$ is independent of the choice of e.

In the remainder of this section we use the results above to determine the universal theory of Hardy fields. First, some generalities on valued differential fields.

Valued differential fields with very small derivation. Let K be a valued differential field with derivation ∂. Recall that if K has small derivation (that is, $\partial\smallo \subseteq \smallo$), then also $\partial\mathcal{O} \subseteq \mathcal{O}$ by [ADH, 4.4.2], so we have a unique derivation on the residue field $\boldsymbol{k} := \mathcal{O}/\smallo$ that makes the residue morphism $\mathcal{O} \to \boldsymbol{k}$ into a morphism of differential rings (and we call $\boldsymbol{k}$ with this induced derivation the differential residue field of K). We say that ∂ is **very small** if $\partial\mathcal{O} \subseteq \smallo$. So K has very small derivation iff K has small derivation and the induced derivation on $\boldsymbol{k}$ is trivial. If K has small derivation and $\mathcal{O} = C + \smallo$, then K has very small derivation. If K has very small derivation, then so does every valued differential subfield of K, and if L is a valued differential field extension of K with small derivation and $\boldsymbol{k}_L = \boldsymbol{k}$, then L has very small derivation. Moreover:

Lemma 13.7. *Let L be a valued differential field extension of K, algebraic over K, and suppose K has very small derivation. Then L also has very small derivation.*

Proof. By [ADH, 6.2.1], L has small derivation. The derivation of $\boldsymbol{k}$ is trivial and $\boldsymbol{k}_L$ is algebraic over $\boldsymbol{k}$ [ADH, 3.1.9], so the derivation of $\boldsymbol{k}_L$ is also trivial. $\square$

Next we focus on pre-d-valued fields with very small derivation. First an easy observation about asymptotic couples:

Lemma 13.8. *Let (Γ, ψ) be an asymptotic couple; then*

$$(\Gamma,\psi) \text{ has gap } 0 \iff (\Gamma,\psi) \text{ has small derivation and } \Psi \subseteq \Gamma^{<}.$$

In particular, if (Γ, ψ) has small derivation and does not have gap 0, then each asymptotic couple extending (Γ, ψ) has small derivation.

Corollary 13.9. *Suppose K is pre-*d*-valued with small derivation, and suppose 0 is not a gap in K. Then K has very small derivation.*

Proof. The previous lemma gives $g \in K^{\times}$ with $g \not\preccurlyeq 1$ and $g^{\dagger} \preccurlyeq 1$. Then for each $f \in K$ with $f \preccurlyeq 1$ we have $f' \prec g^{\dagger} \preccurlyeq 1$. $\square$

Corollary 13.10. *Suppose K is pre-*d*-valued of H-type with very small derivation. Then the* d*-valued hull* $\operatorname{dv}(K)$ *of K has small derivation.*

Proof. By Lemma 13.8, if 0 is not a gap in K, then every pre-d-valued field extension of K has small derivation. If 0 is a gap in K, then no $b \asymp 1$ in K satisfies $b' \asymp 1$, since K has very small derivation. Thus $\Gamma_{\operatorname{dv}(K)} = \Gamma$ by [ADH, 10.3.2(ii)], so 0 remains a gap in $\operatorname{dv}(K)$. In both cases, $\operatorname{dv}(K)$ has small derivation. $\square$

If K is pre-d-valued and ungrounded, then for each $\phi \in K$ which is active in K, the pre-d-valued field K^{ϕ} (with derivation $\delta = \phi^{-1}\partial$) has very small derivation.

The universal theory of Hardy fields. Let $\mathcal{L}^{\iota}$ be the language $\mathcal{L}$ of ordered valued differential rings from above augmented by a new unary function symbol ι. We view each pre-H-field H as an $\mathcal{L}^{\iota}$-structure by interpreting the symbols from $\mathcal{L}$ in the natural way and ι by the function $\iota\colon H \to H$ given by $\iota(a) := a^{-1}$ for $a \in H^{\times}$ and $\iota(0) := 0$. Since every Hardy field extends to a maximal one, each universal $\mathcal{L}^{\iota}$-sentence which holds in every maximal Hardy field also holds in every Hardy field. From [8, Section 1.1] recall that a valued differential field K has *very small derivation* if for each $f \in K$: $f \preccurlyeq 1 \Rightarrow f' \prec 1$. We now use Corollary 13.6 to show:

Proposition 13.11. *Let Σ be the set of universal $\mathcal{L}^{\iota}$-sentences true in all Hardy fields. Then the models of Σ are the pre-H-fields with very small derivation.*

For this we need a refinement of [ADH, 14.5.11]:

Lemma 13.12. *Let H be a pre-H-field with very small derivation. Then H extends to an H-closed field with small derivation.*

Proof. By Corollary 13.10, replacing H by its H-field hull, we first arrange that H is an H-field. Let (Γ, ψ) be the asymptotic couple of H. Then $\Psi^{\geqslant 0} \neq \emptyset$ or (Γ, ψ) has gap 0. Suppose (Γ, ψ) has gap 0. Let $H(y)$ be the H-field extension from [ADH, 10.5.11] for $K := H$, $s := 1$. Then $y \succ 1$ and $y^\dagger = 1/y \prec 1$, so replacing $H(y)$ by H we can arrange that $\Psi^{\geqslant 0} \neq \emptyset$. Then every pre-$H$-field extension of H has small derivation, and so we are done by [ADH, 14.5.11]. □

Proof of Proposition 13.11. The natural axioms for pre-H-fields with very small derivation formulated in $\mathcal{L}^\iota$ are universal, so all models of Σ are pre-H-fields with very small derivation. Conversely, any pre-H-field H with very small derivation is a model of Σ: use Lemma 13.12 to extend H to an H-closed field with small derivation, and note that the $\mathcal{L}^\iota$-theory of H-closed fields with small derivation is complete by [ADH, 16.6.3] and has a Hardy field model by Corollary 13.6. □

Similar arguments allow us to settle a conjecture from [1], in slightly strengthened form. For this, let $\mathcal{L}^\iota_x$ be $\mathcal{L}^\iota$ augmented by a constant symbol x. We view each Hardy field containing the germ of the identity function on $\mathbb{R}$ as an $\mathcal{L}^\iota_x$-structure by interpreting the symbols from $\mathcal{L}^\iota$ as described at the beginning of this subsection and the symbol x by the germ of the identity function on $\mathbb{R}$, which we also denote by x as usual. Each universal $\mathcal{L}^\iota_x$-sentence which holds in every maximal Hardy field also holds in every Hardy field containing x.

Proposition 13.13. *Let Σ_x be the set of universal $\mathcal{L}^\iota_x$-sentences true in all Hardy fields that contain x. Then the models of Σ_x are the pre-H-fields with distinguished element x satisfying $x' = 1$ and $x \succ 1$.*

This follows from [ADH, 14.5.11] and the next lemma just like Proposition 13.11 followed from Lemma 13.12 and [ADH, 16.6.3].

Lemma 13.14. *The $\mathcal{L}^\iota_x$-theory of H-closed fields with distinguished element x satisfying $x' = 1$ and $x \succ 1$ is complete.*

Proof. Let K_1, K_2 be models of this theory, and let x_1, x_2 be the interpretations of x in K_1, K_2. Then [ADH, 10.2.2, 10.5.11] gives an isomorphism $\mathbb{Q}(x_1) \to \mathbb{Q}(x_2)$ of valued ordered differential fields sending x_1 to x_2. To show that $K_1 \equiv K_2$ as $\mathcal{L}^\iota_x$-structures, identify $\mathbb{Q}(x_1)$ with $\mathbb{Q}(x_2)$ via this isomorphism. View $\Lambda\Omega$-fields as $\mathcal{L}^\iota_{\Lambda\Omega}$-structures where $\mathcal{L}^\iota_{\Lambda\Omega}$ extends $\mathcal{L}^\iota$ as specified in [ADH, Chapter 16]. (See also the proof of Theorem 12.3.) The ω-free H-fields K_1, K_2 uniquely expand to $\Lambda\Omega$-fields $\boldsymbol{K}_1$, $\boldsymbol{K}_2$. The H-subfield $\mathbb{Q}(x_1)$ of K_1 is grounded, so expands also uniquely to an $\Lambda\Omega$-field, and this $\Lambda\Omega$-field is a common substructure of both $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$. Hence $\boldsymbol{K}_1 \equiv_{\mathbb{Q}(x_1)} \boldsymbol{K}_2$ by [ADH, 16.0.1, B.11.6]. This yields the claim. □

A. Robinson [69] raised the issue of axiomatizing Σ_x. Proposition 13.13 provides a finite axiomatization. The completeness of the $\mathcal{L}^\iota$-theory of H-closed fields with small derivation together with Lemma 13.14 and Theorem 11.19 yield:

Corollary 13.15. *The set Σ of universal $\mathcal{L}^\iota$-sentences true in all Hardy fields is decidable, and so is the set Σ_x of universal $\mathcal{L}^\iota_x$-sentences true in all Hardy fields containing x.*

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The citation [ADH] refers to our book

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